Modified Gauss rules for approximate calculation of some strongly singular integrals

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Abstract

The approach we follow consists in transforming the numerical evaluation of hypersingular integrals into the calculation of a nearly singular integral whose mass is distributed according to a positive parameter ε. To evaluate the latter we apply a Gauss quadrature formula associated with a nearly singular weight function. It is estimated the error in terms of ε. Some numerical results are presented.

Keywords: Strongly singular integral,Cauchy principal value,Gauss quadrature formula.

1 Introduction

The (boundary) integral equation method consists in transforming partial differential equations with d spatial variables into an integral equation over a $(d-1)$-dimensional surface. This strategy often requires us to evaluate some integrals that are strongly singular (cf. [4]).

Let $f(t)$ be a function on $[a,b]$ and $a<x<b$. If the following limit exists

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0}\left(\int_a^{x-\varepsilon}\frac{f(t)dt}{(t-x)^2}+\int_{x+\varepsilon}^b\frac{f(t)dt}{(t-x)^2}-\frac{2f(x)}{\varepsilon}\right) \tag{1}$$

the integral on the left side of (1) is called *Hadamard finite-part integral.* Since 1966 they have appeared a number of papers related to the numerical calculation of (1) (cf. [2]). Some of these contributions are based on the following relation

$$\frac{d}{dx}-\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)} = \neq\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)^2}, \tag{2}$$

where

$$-\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)} := \lim_{\varepsilon\to 0}\left(\int_a^{x-\varepsilon}\frac{f(t)dt}{(t-x)}+\int_{x+\varepsilon}^b\frac{f(t)dt}{(t-x)}\right), \tag{3}$$

is the Cauchy principal value integral which can also be expressed as

$$-\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)} = \lim_{\varepsilon\to 0^+}\int_a^b \frac{(t-x)f(t)dt}{(t-x)^2+\varepsilon^2}. \tag{4}$$

Note that for every ε the corresponding integral on the right side of Eq. (4) is nearly singular when $\varepsilon>0$ is small.

The aim of this note is to report on a new approach for evaluating the integrals (1) and (3) independently of one another. The method we applied to evaluate (1) is inspired by the following formulation suggested by Ang and Clements [3].

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0^+}\left(\int_a^b \frac{(t-x)^2 f(t)dt}{((t-x)^2+\varepsilon^2)^2}-\frac{\pi f(x)}{2\varepsilon}\right). \tag{5}$$

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Our attention is focused on integrals of the type $\displaystyle⨍_{-1}^{1}(t-x)^{-2}g(t)dt/w(t)$, where $g(t)$ is smooth (well scaled) and $w(t)^{-1}$ is nearly singular. For more information see [5] and the bibliography therein.

2 An alternative formulation

Equation (5) is of the form $\infty - \infty$, which is not favorable for numerical treatment. Assuming all necessary hypothesis, the following expression can also be obtained.

$$⨍_a^b \frac{f(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0^+}\int_a^b \frac{(t-x)^4\phi(x,t)dt}{((t-x)^2+\varepsilon^2)^2} + Q(x), \tag{6}$$

where $\phi(x,t) = \dfrac{f(t)-f(x)-f'(x)(x-t)}{(t-x)^2}$, and

$$Q(x) = \frac{f(x)(b-a)}{(a-x)(b-x)} + f'(x)\log\left|\frac{b-x}{a-x}\right|.$$

The integral on the right side of (6) is nearly singular when ε is small.

In what follows we consider the above problem when the interval is $[-1,1]$, so the integral to be calculated is the following.

$$\int_{-1}^{1}\frac{\phi(x,t)G(x,t,\varepsilon)dt}{\sqrt{1-t^2}}. \tag{7}$$

Now, we may consider the use of the Gauss quadrature formula w.r.t. the weight $W(x,t,\varepsilon) = G(x,t,\varepsilon)/\sqrt{1-x^2}$ where $G(x,t,\varepsilon) = \dfrac{\sqrt{1-t^2}(t-x)^4}{((t-x)^2+\varepsilon^2)^2}$ satisfies $\lim_{\varepsilon\to 0} G(x,t,\varepsilon) = \sqrt{1-t^2}$ for all $t \neq x$.

For ε small we calculate nodes and weights using the connection between modified moments and the first coefficients of the Chebyshev series expansion of $G(x,t,\varepsilon)$ (cf. [1]). If $\varepsilon \neq 0$ then a drawback is that the quadrature formula depends on x and ε. Actually, most of the contributions to this issue have not been able to avoid dependence on x. See, for example, [5]. In general, the coefficients of the Chebyshev series of $G(x,t,\varepsilon)$ can be estimated by using some FFT algorithm or Chebyshev interpolation formula (see (12) below).

Actually, it happens that the derivative is rarely available. In such case we may approximate $f'(x)$ by using one of the simplest formulas, i.e. the symmetric difference

$$\Delta(f,x,\eta) = \frac{f(x+\eta) - f(x-\eta)}{2\eta} \approx f'(x). \tag{8}$$

Thereby, Eq. (6) is now written as follows

$$⨍_{-1}^{1} \frac{f(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0^+} \int_{-1}^{1} \frac{(t-x)^4 \phi(x,t,\eta)dt}{((t-x)^2+\varepsilon^2)^2} + Q(x,\eta), \tag{9}$$

where $\phi(x,t,\eta) = \dfrac{f(t) - f(x) - \Delta(f,x,\eta)(x-t)}{(t-x)^2}$, and

$$Q(x,\eta) = \frac{-2f(x)}{(1+x)(1-x)} + \Delta(f,x,\eta)\log\left|\frac{1-x}{1+x}\right|.$$

If $f(t) = g(t)/w(t)$, then (9) can be easily transformed into

$$⨍_{-1}^{1} \frac{f(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0^+} \int_{-1}^{1} \frac{(t-x)^4 \phi(x,t,\eta)dt}{w(t)((t-x)^2+\varepsilon^2)^2} + Q(x,\eta), \tag{10}$$

where $\phi(x,t,\eta) = \dfrac{g(t)w(x) - g(x)w(t) - w(x)w(t)\Delta(g/w,x,\eta)(t-x)}{w(x)(t-x)^2}$, so now $t = x$ is a singularity that we must bear in mind.

If $\varepsilon \neq 0$ then $G(x,t,\varepsilon)$ depends on x, as can be seen below

$$G(x,t,\varepsilon) = \frac{(t-x)^4\sqrt{1-t^2}}{w(t)[(t-x)^2+\varepsilon^2]^2}. \tag{11}$$

An alternative to the FFT is the following approximation formula.

$$A_j \approx \frac{2}{N+1} \sum_{k=1}^{N+1} G(x,t_k,\varepsilon)T_j(t_k),\ j = 0,\cdots,m, \tag{12}$$

where T_j is the jth Chebyshev polynomial of the first kind and the points t_k are the zeros of T_{N+1}. The parameter m satisfies $m < N$. In this work we set $90 \leq m \leq 100$ and $N \geq 2^{10}$. Obviously, when using formula (12) the integer N does not represent the overall number of coefficients that have been actually estimated so the corresponding algorithm is $\mathcal{O}(N)$.

3 Error analysis

Here it is examined the theoretical role played by ε in (6). For this purpose, suppose that f can be expressed as a power series expansion at x, which converges uniformly on $(\alpha, \beta) \supset [a, b]$, so that the following equality holds true.

$$\int_a^b \frac{(t-x)^2(f(t)-f(x))dt}{((t-x)^2+\varepsilon^2)^2} = \sum_{k=1}^{\infty} \frac{f^{k)}(x)}{k!} \int_a^b \frac{(t-x)^{k+2}dt}{((t-x)^2+\varepsilon^2)^2}$$

Furthermore,

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{(t-x)^2(f(t)-f(x))dt}{((t-x)^2+\varepsilon^2)^2} = \sum_{k=1}^{\infty} \frac{f^{k)}(x)}{k!} \int_a^b (t-x)^{k-2}dt,$$

where the integral for $k = 1$ is understood as the principal value integral.

An estimate of the rate of convergence is given by the following result.

Proposition 3.1 *For every $k \geq 1$ the following estimate holds true.*

$$E_\varepsilon(x, k) = \int_a^b \frac{(t-x)^{k+2}dt}{((t-x)^2+\varepsilon^2)^2} - \int_a^b (t-x)^{k-2}dt = \mathcal{O}(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

uniformly w.r.t. $x \in [\alpha, \beta]$, $c < \alpha < \beta < d$.

Proof.

Let $\delta_x(t) = t - x$. If $k \geq 4$, then

$$\delta_x(t)^{k-2} - \frac{\delta_x(t)^{k+2}}{(\delta_x(t)^2+\varepsilon^2)^2} = \frac{\delta_x(t)^{k-2}\varepsilon^2(2\delta_x(t)^2+\varepsilon^2)}{(\delta_x(t)^2+\varepsilon^2)^2} \leq 2\delta_x(t)^{k-4}\varepsilon^2. \qquad (13)$$

The estimates for $k = 1, 2, 3$ can be obtained by integrating by parts. □

4 An example

Let us consider the finite-part integral [5]

$$\oint_{-1}^{1} \frac{dt}{\sqrt{1.21 - x^2}(t-0.25)^2} = -0.895622582871559.$$

[5] An analytical expression for this integral can be seen in [6].

The table below depicts the absolute errors when the above integral is approximated by applying a modification of the n-point Chebyshev quadrature formula, it is used (12) and the exact derivative of $1/\sqrt{1-t^2}$. The results listed in column *Spline* were obtained by Palamara [5] using quadrature rules based on spline interpolation.

$N = 2^{15}$							
n	$\varepsilon = 10^{-7}$	$\varepsilon = 0$	*Spline*	n	$\varepsilon = 10^{-7}$	$\varepsilon = 0$	*Spline*
2	$2.7e-02$	$2.7e-02$		16	$5.8e-09$	$4.6e-09$	$3.8e-03$
4	$1.8e-03$	$1.8e-03$		32	$8.3e-11$	$1.1e-09$	$4.5e-04$
8	$1.9e-05$	$1.9e-05$	$7.9e-02$	64	$8.4e-11$	$1.1e-09$	$2.7e-05$
$N = 2^{20}$							
n	$\varepsilon = 10^{-10}$	$\varepsilon = 0$	*Spline*	n	$\varepsilon = 10^{-10}$	$\varepsilon = 0$	*Spline*
2	$2.7e-02$	$2.7e-02$		16	$5.7e-09$	$5.7e-09$	$3.8e-03$
4	$1.8e-03$	$1.8e-03$		32	$7.7e-13$	$1.3e-12$	$4.5e-04$
8	$1.9e-05$	$1.9e-05$	$7.9e-02$	64	$8.8e-13$	$1.1e-12$	$2.7e-05$

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