Centralized inventory in a farming community*

M.G. Fiestras-Janeiro[1] I. García-Jurado[2] A. Meca[3] M.A. Mosquera[1]

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Abstract

A centralized inventory problem is a situation in which several agents face individual inventory problems and make an agreement to coordinate their orders with the objective of reducing costs. In this paper we identify a centralized inventory problem arising in a farming community in northwestern Spain, model the problem using two alternative approaches, find the optimal inventory policies for both models, and propose allocation rules for sharing the optimal costs in this context.

Key words: centralized inventory, cooperative games, EOQ models, Shapley value, core.

1 Introduction

A centralized inventory problem is a situation in which several agents face individual inventory problems and make an agreement to coordinate their orders with the objective of reducing costs. The inventory management in a centralized inventory problem comprises the following steps:

- Formulate a mathematical model describing the behavior of the inventory system.
- Identify an optimal inventory policy for the group of cooperating agents with respect to the model.
- Decide how the optimal costs are shared by the agents.

[1] *Departamento de Estatística e Investigación Operativa, Universidade de Vigo, Vigo, Spain.*
[2] *Departamento de Matemáticas, Universidade da Coruña, A Coruña, Spain.*
[3] *Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, Elche, Spain.*
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The analysis of centralized inventory problems using cooperative game theory has proved to be very successful. Some early papers in this research stream for continuous review problems are Meca et al (2003) and Hartman and Dror (1996) in the deterministic and stochastic contexts, respectively. The first papers dealing with deterministic periodic review papers from a game theoretical perspective are van den Heuvel et al (2007) and Guardiola et al (2009). Hartman et al (2000) is a pioneering paper on the analysis of stochastic periodic review multi-agent problems. Nagarajan and Sošić (2008), Dror and Hartman (2011) and Fiestras-Janeiro et al. (2012a) are recent surveys of centralized inventory models; Fiestras-Janeiro et al. (2011) reviews the applications of cooperative game theory for sharing cost problems.

In this paper we identify a centralized inventory problem arising in a farming community in northwestern Spain. It has to do with the dry feed that a set of farms order regularly from the same supplier. Each farm faces a continuous review inventory problem with a deterministic and linear demand, with no holding costs, with a limited capacity warehouse and without shortages. The supplier charges the farms taking into account the order sizes only. Since the supplier would prefer that the farms place joint orders, we suggest that it incorporates an order fee. We deal with two types of order fees: one which is fixed and one which has a fixed part and a variable part depending on the distance of the farm to the supplier. For the corresponding two models we find the optimal inventory policies of the farms and propose allocation rules for sharing the optimal costs. Our analysis generalizes those of Meca et al (2003) and Fiestras-Janeiro et al. (2012b) for the case of no holding costs and limited capacity warehouses.

2 The problem

This problem was identified using the feedback obtained from dairy farmers and cattle feed suppliers in northwestern Spain. A standard dairy farm in northwestern Spain has between 40 and 150 dairy cows. Typically, every dairy farm has an agreement with a dairy firm which guarantees that the firm buys the dairy production of the farm. Usually, the agreements stipulate conditions for the prices payed by the firm; sometimes, they also stipulate conditions for the cow feeding.

In general, the cow feeding is varied and the feeding ration must have the necessary nutrients to maintaining a high production of milk (between 25 and 35 liters per cow and day). The feeding ration can be decomposed into two parts. On one hand, a part that has to be stored at the farm in warehouses, called *silos*. On the other hand, a part that must be daily obtained and that cannot be stored. We are interested in the management of the former part, the one that is stored. From now on, we refer to this part of the feeding ration as the *dry feed*. The silos, where the dry feed is stored, have a constant maintenance cost. Indeed, this cost is negligible and can be considered to be zero. Each cow consumes about 10 kg of dry feed for producing about 30 liters of milk per day. This consumption is quite stable, so the demand in this inventory problem can be considered to be deterministic.

The dry feed is ordered to an external supplier. Sometimes the supplier is the same dairy firm with which the farm has its agreement; sometimes the farm belongs to a cooperative which has its own factory for producing cattle feed. In any case, it seems to be quite common that:

1. The supplier charges the farms taking into account the order sizes only. This means that there is a price per ton of dry feed, which is independent on the number of orders per year placed by a particular farm and on its location (in spite of the fact that transportation is included in the price).

2. The supplier frequently faces organizational problems and prefers that the farms place joint orders. Two advantageous consequences of that would be a relevant saving in the transportation costs and a more efficient scheduling of the production plans in the factory.

The pricing structure indicated in item 1 is not appropriate if we are to get that farms place joint orders. To get this, the supplier could charge a fee each time an order is placed, this fee being the same whether the order is placed by a single farm or by a collection of farms which place joint orders. There would be another ways to encourage farmers to place joint orders, like for instance introducing quantity discounts. However, in this work we focus on the case of charging a fee.

In this paper we introduce and analyze two models in which order fees are charged by the supplier. In the first model these fees are fixed while in the second they have a fixed component and a variable component which depends on the distances of the farms to the supplier.

3 The model with fixed order costs

A *basic EOQ system without holding costs* is a multiple agent situation where each agent faces a continuous review inventory problem with a fixed order cost, with a deterministic and linear demand, with no holding costs, with a limited capacity warehouse and without shortages. N denotes the finite set of agents. The parameters associated to every $i \in N$ in one of these systems are:

- $a > 0$, the fixed cost per order,
- $d_i > 0$, the deterministic demand per time unit,
- $K_i > 0$, the capacity of i's warehouse.

Notice that the problem in Section 2 fits this model if the supplier charges a fixed fee a each time an order is placed. Observe that the optimal policy of an individual agent in this model is very simple. Since the holding costs are zero, agent i must place orders of maximum size, i.e. of size K_i, and then wait until the stock level is zero to place the next order.[1] The length of each

[1] In this paper we assume that the waiting time since an order is posed until it is received is deterministic, in which case it can be assumed to be zero. The deterministic nature of waiting times in this problem is realistic. In principle, the supplier guarantees to serve the orders the next day after they are posed.

cycle for agent i is thus K_i/d_i, a cycle being the time period between two consecutive orders, and the optimal cost per time unit for agent i is given by:

$$C_i = \frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{a}{K_i/d_i} = a\frac{d_i}{K_i}.$$

To illustrate the model, let us consider an example. The data in the example are fictitious, although they are plausible, in the sense that they could have been taken from existing farms in northwestern Spain.

Example 3.1. *We consider an example with five dairy farms* $N = \{1, 2, 3, 4, 5\}$*. The farms have* 40, 140, 120, 130 *and* 120 *cows, respectively. The fixed cost per order is* $a = 200$ *(in euros) and the demand (in tons per day) and the capacity of silos (in tons) for each dairy farm are given in the next table, whose two last rows depict the ratio demand/capacity and the optimal individual cost per day of each farm, respectively.*

i	1	2	3	4	5
d_i	0.4	1.4	1.2	1.3	1.2
K_i	4	10	8	8	6
d_i/K_i	0.1	0.14	0.15	0.1625	0.2
C_i	20	28	30	32.5	40

Now, if the agents in N cooperate by placing joint orders, they must coordinate their cycles. Since shortages are not allowed, the length of the joint cycle must be equal to the length of the shortest individual optimal cycle. It is clear that if the agents in N pay the order cost only once when ordering jointly, the order policy that minimize the total cost per time unit in this context is the one considered above: they order jointly, adjusting their cycles to the shortest individual optimal cycle. Then, for every non-empty $S \subset N$, the optimal cost per time unit is given by:

$$C_S = \frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{a}{\min_{i\in S} K_i/d_i} = a \max_{i\in S} \frac{d_i}{K_i}.$$

Example 3.2. *In the system in Example 3.1 the optimal cost per time unit for N (in euros per day) is*

$$C_N = a \max_{i\in N} \frac{d_i}{K_i} = 200 \times 0.2 = 40,$$

which is significantly lower that the sum of the optimal individual costs per day if the agents in N do not cooperate; this sum is $20 + 28 + 30 + 32.5 + 40 = 150.5$*. The reader may think that these amounts are not relevant. However, dairy farms in northwestern Spain are usually small family businesses for which costs of that magnitude are important.*

Now that we have a model and an optimal policy for this model we consider the question of how the total joint costs are allocated to the agents. For that purpose we use cooperative game theory. Although we introduce all the game theoretic concepts that we use in this paper, the reader interested in more details on cooperative games can see González-Díaz et al (2010). To start with, we give some definitions.

Definition 3.1. *A cost game is a pair (N, c) where N is a finite set and c is a map from 2^N (the class of subsets of N) to $\mathbb{R}$ which satisfies that $c(\emptyset) = 0$. Typically, for every non-empty $S \subset N$, $c(S)$ is interpreted as the minimum cost that the agents of S must hold if they cooperate.*

For every basic EOQ system without holding costs $(N, a, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ we can associate a cost game (N, c) given by

$$c(S) = a \max_{i\in S} \frac{d_i}{K_i}$$

for every non-empty $S \subset N$. Here $c(S)$ is simply C_S, the optimal cost per time unit if the agents in S cooperate.

The games associated to basic EOQ systems without holding costs belong to a well-known class of cost games described in Littlechild and Owen (1973) to model a pricing problem in the context of air transportation: the class of airport games.

Definition 3.2. *An airport game is a cost game (N, c) such that:*

- *$c(i) > 0$ for every $i \in N$, and*
- *$c(S) = \max\{c(i) \mid i \in S\}$ for every non-empty $S \subset N$.*

A relevant problem in cooperative game theory is the following: if a cost game (N, c) is given, how should the total cost $c(N)$ be allocated to the players? This is precisely the problem we are facing now in this paper. To solve it, the concept of allocation rule is an important one.

Definition 3.3. *An allocation rule for cost games is a map ϕ which associates for every cost game (N, c) a vector $\phi(N, c) = (\phi_i(N, c))_{i\in N} \in \mathbb{R}^N$. For every $i \in N$, $\phi_i(N, c)$ is the allocation that ϕ proposes for the agent i in the cost game (N, c).*

It is very convenient that an allocation for a particular cost game belongs to its core. The core of a cost game is defined below.

Definition 3.4. *Let (N, c) be a cost game. The core of (N, c) is the following set:*

$$Core(N, c) = \{x = (x_i)_{i\in N} \in \mathbb{R}^N \mid \sum_{i\in N} x_i = c(N) \text{ and } \sum_{i\in S} x_i \leq c(S)\ \forall S \subset N\}.$$

If $x \in Core(N, c)$, then x is a stable allocation, in the sense that it does not disappoint any of the possible coalitions within N. Notice that the core of a cost game can be an empty set.

One of the most important allocation rules for cost games is the Shapley value (see Shapley (1953)). A recent survey on this allocation rule is Moretti and Patrone (2008). In particular, the Shapley value is specially convenient for airport games (see, for instance, González-Díaz et al (2010)). The following result was proved in Littlechild and Owen (1973).

Theorem 3.1. *Let (N, c) be an airport game and let Φ denote the Shapley value. Then,*

1. $\Phi(N,c) \in Core(N,c)$.

2. *Assume without loss of generality that the agents in N are arranged in non-decreasing order of their ratios d_i/K_i, i.e. that $d_1/K_1 \leq \cdots \leq d_n/K_n$ (n denotes the number of elements of N). Then,*

$$\Phi_i(N,c) = \frac{c(1)}{n} + \sum_{j=2}^{i} \frac{c(j) - c(j-1)}{n-j+1}. \tag{1}$$

In view of the comments and results above, it seems that the Shapley value is a good choice for allocating the joint costs in a basic EOQ system without holding costs. To finish this section we obtain the proposal of the Shapley value for the system in Example 3.1.

Example 3.3. *Take the system in Example 3.1 and denote by (N,c) its associated cost game. For every $i \in N$, $c(i)$ equals the amount C_i in the table of Example 3.1. Then, using (1), it is easily obtained that:*

$$\Phi(N,c) = (4, 6, 6.667, 7.917, 15.417).$$

4 The model with transportation costs

A *basic EOQ system without holding costs and with transportation costs* is a multiple agent situation where each agent faces a continuous review inventory problem with a variable order cost, with a deterministic and linear demand, with no holding costs, with a limited capacity warehouse and without shortages. In fact, the variable order cost of each agent has two components: a fixed component and a variable component which is due to transportation costs and depends on the distance of the agent to the supplier. N denotes the finite set of agents. The parameters associated to every $i \in N$ in one of these systems are:

- $a > 0$, the fixed cost per order,
- $a_i > 0$, the transportation cost per order,
- $d_i > 0$, the deterministic demand per time unit,
- $K_i > 0$, the capacity of i's warehouse.

We also make the following two assumptions.

A1 All the agents are located on the same line route. We mean that if a group of agents S places a joint order, the correponding fixed cost is the sum of the first component a plus the second component of an agent in S whose distance from the supplier is maximal.

A2 The supplier accepts and even encourages agents to order jointly. At the beginning of the term the supplier asks what order coalitions have formed and, because of organizational reasons, once an order coalition $S \subset N$ has been formed, the order fee that the supplier charges to this coalition, for each order throughout the term, is $a + \max_{i \in S} a_i$.

This model is inspired in the inventory transportation situations treated in Fiestras-Janeiro et al (2012b). However, it is completely different, because now holding costs are zero and the capacities of the warehouses are limited. Notice that the problem in Section 2 fits this model if the supplier charges a fixed order fee plus a transportation order fee each time an order is placed. Assumptions A1 and A2 seem to be acceptable in many cases for the problem in Section 2.

Like in the model with fixed costs, the optimal policy of an individual agent in this model is very simple. Since the holding costs are zero, agent i must place orders of maximum size, i.e. of size K_i, and then wait until the stock level is zero to place the next order. The length of each cycle for agent i is thus K_i/d_i and the optimal cost per time unit for agent i is given by:

$$C_i = \frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{a + a_i}{K_i/d_i} = (a + a_i)\frac{d_i}{K_i}.$$

To illustrate the results in this section we use the system in Example 3.1.

Example 4.1. *Suppose that in the system of Example 3.1, the supplier poses a fixed order fee $a = 200$ plus a transportation fee depending on the distances to the farms. Suppose also that assumptions A1 and A2 hold and take the transportation fees: $a_1 = 150$, $a_2 = 250$, $a_3 = 100$, $a_4 = 200$, and $a_5 = 100$. All the fees are given in euros. The optimal individual cost per day of each farm is given in the following table.*

i	1	2	3	4	5
d_i	0.4	1.4	1.2	1.3	1.2
K_i	4	10	8	8	6
a_i	150	250	100	200	100
d_i/K_i	0.1	0.14	0.15	0.1625	0.2
C_i	35	63	45	65	60

Now, if the agents of a non-empty coalition $S \subset N$ cooperate by placing joint orders, they must coordinate their cycles. Since shortages are not allowed, the length of the joint cycle must be equal to the length of the shortest individual optimal cycle. It is clear that this is the optimal policy for the agents in S if they agree to make joint orders and assumptions A1 and A2 are satisfied. Formally, for every non-empty $S \subset N$, the optimal cost per time unit if the members of S agree to place joint orders is given by:

$$C_S = \frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{\max_{i\in S}\{a + a_i\}}{\min_{j\in S}\{K_j/d_j\}} = \max_{i\in S}\{(a + a_i)\} \max_{j\in S}\{\frac{d_j}{K_j}\} = \max_{i\in S}\max_{j\in S}\{(a + a_i)\frac{d_j}{K_j}\}.$$

In this way we can associate a cost game to every basic EOQ system without holding costs and with transportation costs.

Definition 4.1. *For every basic EOQ system without holding costs and with transportation costs*

$(N, a, \{a_i\}_{i \in N}, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$ *we can associate a cost game* (N, c) *given by*

$$c(S) = \max_{i \in S} \max_{j \in S} \{(a + a_i)\frac{d_j}{K_j}\} \tag{2}$$

for every non-empty $S \subset N$. *Here* $c(S)$ *is simply* C_S, *the optimal cost per time unit if the agents in* S *agree to place joint orders.*

In the system in Example 4.1 the optimal cost per day for N is

$$c(N) = \max_{i \in N} \max_{j \in N} \{(a + a_i)\frac{d_j}{K_j}\} = 450 \times 0.2 = 90,$$

which is significantly lower that the sum of the optimal individual costs per day if the agents in N do not cooperate; this sum is $\sum_{i \in N} c(i) = 35 + 63 + 45 + 65 + 60 = 268$. Unfortunately, this is not true in general, as we illustrate in the following example.

Example 4.2. *Consider a system with two agents* $N = \{1, 2\}$. *The fixed cost per order (in euros) is* $a = 200$ *and the demand (in tons per day), the capacity of the warehouses (in tons) and the transportation cost per order (in euros) for each agent are given in the table below, whose two last rows depict the ratio demand/capacity and the optimal individual cost per day of each agent, respectively.*

i	1	2
d_i	0.2	0.8
K_i	8	6
a_i	700	300
d_i/K_i	0.025	0.133
C_i	22.5	66.667

The associated cost game is the following:

S	{1}	{2}	{1, 2}
$c(S)$	22.5	66.667	120

In this case

$$c(1) + c(2) = 22.5 + 66.667 = 89.167 < c(\{1, 2\}) = 120.$$

Example 4.2 shows that not every cost game associated to a basic EOQ system without holding costs and with transportation costs is *subadditive*, in the sense of the following definition.

Definition 4.2. *Let* (N, c) *a cost game.* (N, c) *is said to be subadditive if, for all non-empty* $S, T \subset N$ *with* $S \cap T = \emptyset$, *it holds that* $c(S \cup T) \leq c(S) + c(T)$.

We now wonder under what conditions cooperation is profitable in a basic EOQ system without holding costs and with transportation costs or, equivalently, under what conditions the associated cost game of such a system is subadditive. The next result provides a reply to this question. It says that the associated cost game of a basic EOQ system without holding costs and with transportation costs is subadditive if and only if, for every non-empty disjoint coalitions $S, T \subset N$, with T farther that S, either the number of orders of T is greater than S, or the number of orders of S is increased, need not be excessively large relative to the number of orders of T.

Theorem 4.1. *Let $(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with transportation costs, and let (N, c) be its associated cost game. Then (N, c) is subadditive if and only if, for every non-empty $S, T \subset N$ with $S \cap T = \emptyset$ and $\max_{i\in S}\{a_i\} \leq \max_{i\in T}\{a_i\}$, at least one of the two following conditions holds:*

1. $\max_{i\in S}\{\frac{d_i}{K_i}\} \leq \max_{i\in T}\{\frac{d_i}{K_i}\}$.
2. $\max_{i\in S}\{\frac{d_i}{K_i}\} > \max_{i\in T}\{\frac{d_i}{K_i}\}$ *and* $\frac{\max_{i\in T}\{a_i\} - \max_{i\in S}\{a_i\}}{a + \max_{i\in T}\{a_i\}} \leq \frac{\max_{i\in T}\{\frac{d_i}{K_i}\}}{\max_{i\in S}\{\frac{d_i}{K_i}\}}$.

Proof. See Appendix. □

Now we have a model and an optimal policy for this model provided that a coalition of agents decides to place joint orders. We also know under what conditions cooperation is profitable for that coalition. Next we consider the question of how the total joint costs are allocated among the agents.

First we prove that if cooperation is profitable in a basic EOQ system without holding costs and with transportation costs, then we can find a stable allocation of the total joint costs. In game theoretical terms, we prove that if the cost game associated to a basic EOQ system without holding costs and with transportation costs is subadditive, then its core is non-empty. For this proof, we need some concepts and notations in relation with a cost game (N, c).

We denote by $\Pi(N)$ the set of all permutations in N. Formally, every $\sigma \in \Pi(N)$ is a one-to-one map which associates to every element of N a natural number in $\{1, 2, \ldots, n\}$ (n denotes the number of elements of N). $\sigma(i) = j$ means that i has the j-th position in the ordering given by σ. Denote by σ^{-1} the inverse of map σ. For every $i \in N$, the set of predecessors of i with respect to $\sigma \in \Pi(N)$ is $P_i^\sigma = \{j \in N \mid \sigma(j) < \sigma(i)\}$. Now take $\sigma \in \Pi(N)$; the marginal vector associated with σ is defined as $m^\sigma(N, c) = (m_i^\sigma(N, c))_{i\in N}$, where $m_i^\sigma(N, c) = c(P_i^\sigma \cup \{i\}) - c(P_i^\sigma)$ for each $i \in N$. Notice that for every marginal vector m^σ, it holds that $\sum_{i\in N} m_i^\sigma(N, c) = c(N)$. Hence, a marginal vector of (N, c) is an allocation of $c(N)$ which allocates to every i its contribution to its predecessors according to a particular permutation.

Theorem 4.2. *Let $(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with transportation costs, and let (N, c) be its associated cost game. Assume that (N, c) is subadditive. Then, $Core(N, c)$ is non-empty.*

Proof. See Appendix. □

As in the previous section, we would like to identify an allocation rule which has good properties for the class of games associated to basic EOQ systems without holding costs and with transportation costs. The following example shows that the Shapley value may provide allocations outside the core in this context even when the cost game is subadditive. Recall that the Shapley value of a cost game (N,c) is the vector $\Phi(N,c) = (\Phi_i(N,c))_{i\in N}$ given by

$$\Phi_i(N,c) = \frac{1}{|\Pi(N)|} \sum_{\sigma\in\Pi(N)} m_i^{\sigma}(N,c)$$

for all $i \in N$.

Example 4.3. *Take a basic EOQ system without holding costs and with transportation costs with $N = \{1,2,3\}$. The fixed cost per order (in euros) is $a = 400$ and the demand (in tons per day), the capacity of warehouses (in tons) and the variable cost per order (in euros) for each agent are given in the next table, whose two last rows depict the ratio demand/capacity and the optimal individual cost per day of each agent, respectively.*

i	1	2	3
d_i	2	2	5
K_i	9	8	7
a_i	300	500	200
d_i/K_i	0.222	0.25	0.714
C_i	155.556	225	428.571

The associated cost game is the following.

S	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	{1, 2,3}
$c(S)$	155.556	225	428.571	225	500	642.857	642.857

It is easy to check that (N,c) is subadditive using Theorem 4.1. Moreover, Theorem 4.2 implies that $Core(N,c) \neq \emptyset$. Nevertheless, the proposal of the Shapley value is not in the core of (N,c). In fact, after some algebra it can be obtained that

$$\Phi(N,c) = (63.7566, 169.9074, 409.1931).$$

Since $\Phi_1(N,c) + \Phi_2(N,c) = 63.757 + 169.907 = 233.664 > 225 = c(\{1,2\})$, then $\Phi(N,c) \notin Core(N,c)$.

Now we define an allocation rule which always proposes core allocations in this context. Let $(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with transportation costs, and let (N,c) be its associated cost game. Consider the following two sets of permutations:

- $\Pi_1(N,c) = \{\sigma \in \Pi(N) \mid a_i \geq a_j$ implies that $\sigma(i) \leq \sigma(j)$, for all $i,j \in N\}$. So, $\Pi_1(N,c)$ is the set of permutations which reverse the ordering given by the transportation costs.

- $\Pi_2(N,c) = \{\sigma \in \Pi(N) \mid \frac{d_i}{K_i} \geq \frac{d_j}{K_j}$ implies that $\sigma(i) \leq \sigma(j)$, for all $i,j \in N\}$. So, $\Pi_2(N,c)$ is the set of permutations which reverse the ordering given by the ratios demand/capacity.

Our allocation rule proposes for (N,c) the mean of the average of the marginal vectors associated with permutations in $\Pi_1(N,c)$ and the average of the marginal vectors associated with permutations in $\Pi_2(N,c)$. We call this rule the *two-lines rule* because it is a kind of generalization of the line rule introduced in Fiestras-Janeiro et al (2012b). Formally, for any $i \in N$,

$$TL_i(N,c) = \frac{1}{2|\Pi_1(N,c)|} \sum_{\sigma \in \Pi_1(N,c)} m_i^{\sigma}(N,c) + \frac{1}{2|\Pi_2(N,c)|} \sum_{\sigma \in \Pi_2(N,c)} m_i^{\sigma}(N,c),$$

where TL denotes the two-lines rule.

Notice that all permutations $\sigma \in \Pi_1(N,\mathcal{I})$ satisfy that $\sigma^{-1}(1)$ is an agent whose distance to the supplier is maximal. In the proof of Theorem 4.2 we showed that the marginal vectors associated with all such permutations belong to $Core(N,c)$ (if (N,c) is subadditive). It could have been demonstrated in an analogous way that all marginal vectors associated with permutations σ satisfying that $\sigma^{-1}(1)$ is an agent with a maximal ratio demand/capacity also belong to $Core(N,c)$ if (N,c) is subadditive. Then, taking into account that $Core(N,c)$ is a convex set, it is clear that the following result holds.

Theorem 4.3. *Let $(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with transportation costs, and let (N,c) be its associated cost game. If (N,c) is subadditive, then $TL(N,c) \in Core(N,c)$.*

To finish this section we include an example in which we compute the proposal of the two-lines rule for a system. Notice that the computation of this rule is not too hard. For instance, it is typically much simpler computing it than computing the Shapley value.

Example 4.4. *Consider now the basic EOQ system without holding costs and with transportation costs given in Example 4.1. The associated cost game is given by*

S	{1}	{2}	{3}	{4}	{5}	{1, 2}	{1, 3}	{1, 4}	{1, 5}	{2, 3}	{2, 4}	{2, 5}
$c(S)$	35.0	63.0	45.0	65	60	63	52.5	65	70	67.5	73.125	90

S	{3, 4}	{3, 5}	{4, 5}	{1, 2, 3}	{1, 2, 4}	{1, 2, 5}	{1, 3, 4}	{1, 3, 5}	{1, 4, 5}
$c(S)$	65	60	80	67.5	73.125	90	65	70	80

S	{2, 3, 4}	{2, 3, 5}	{2, 4, 5}	{3, 4, 5}	{1, 2, 3, 4}	{1, 2, 3, 5}	{1, 2, 4, 5}	{1, 3, 4, 5}
$c(S)$	73.125	90	90	80	73.125	90	90	80

S	{2, 3, 4, 5}	N
$c(S)$	90	90

It can be easily checked that (N,c) is subadditive using Theorem 4.1. It is clear that

- $\Pi_1(N,c) = \{(2,4,1,3,5),(2,4,1,5,3)\}$, *and*

- $\Pi_2(N,c) = \{(5,4,3,2,1)\}$.

Finally, the proposal of the two-lines rule is

$$\begin{aligned} TL(N,c) &= \frac{m^{(2,4,1,3,5)}(N,c)+m^{(2,4,1,5,3)}(N,c)}{4}+\frac{m^{(5,4,3,2,1)}(N,c)}{2} \\ &= \frac{(0,63,0,10.125,16.875)+(0,63,0,10.125,16.875)}{4}+\frac{(0,10,0,20,60)}{2} \\ &= (0,36.50,0,15.062,38.438). \end{aligned}$$

It might seem unfair that farms 1 *and* 3 *pay nothing according to* TL*. However, this follows from the fact that these two farms are not far from the supplier and, moreover, have small ratios demand capacity. Favoring the proximity to the supplier and a capacity of the warehouse in accordance with the demand can only have positive effects on the cooperation and on the supplier's interests.*

5 Conclusions

In this paper we identify a centralized inventory problem arising in a farming community in northwestern Spain and we model it using two alternative approaches.

In the first approach we assume that the supplier charges a fixed fee every time that an agent or a set of agents places an order. In this context we show that it is profitable for the agents to place joint orders, we obtain an optimal order policy and use cooperative game theory, in particular the Shapley value, to indicate how the total cost can be shared by the agents. In this context the Shapley value provides stable allocations.

In the second approach we assume that the supplier charges a fixed fee plus a variable transportation fee every time that an agent or a set of agents places an order. In this context we give a necessary and sufficient condition under which it is profitable for the agents to place joint orders (making two extra assumptions about their location and the supplier pricing policy). We again use cooperative game theory to design a rule for allocating the total cost among the cooperating agents and to study its properties.

Appendix

Here, the reader can find the proofs of the main theorems stated in this paper.

Proof of Theorem 4.1.

Proof. Assume first that (N,c) is subadditive and take a pair of non-empty coalitions $S,T \subset N$ with $S \cap T = \emptyset$ and $\max_{i\in S}\{a_i\} \leq \max_{i\in T}\{a_i\}$. It holds that

$$c(S\cup T) = \max_{i\in S\cup T}\{a+a_i\} \max_{i\in S\cup T}\{\frac{d_i}{K_i}\} = \max_{i\in T}\{a+a_i\} \max_{i\in S\cup T}\{\frac{d_i}{K_i}\}.$$

Then, the subadditivity condition implies that

$$\max_{i\in T}\{a+a_i\}(\max_{i\in S\cup T}\{\frac{d_i}{K_i}\}-\max_{i\in T}\{\frac{d_i}{K_i}\})\leq \max_{i\in S}\{a+a_i\}\max_{i\in S}\{\frac{d_i}{K_i}\}.$$

If $\max_{i\in S}\{\frac{d_i}{K_i}\}>\max_{i\in T}\{\frac{d_i}{K_i}\}$, then $\max_{i\in S\cup T}\{\frac{d_i}{K_i}\}=\max_{i\in S}\{\frac{d_i}{K_i}\}$. Thus, dividing both sides by $\max_{i\in T}\{a+a_i\}\max_{i\in S}\{\frac{d_i}{K_i}\}$, the inequality above becomes

$$1-\frac{\max_{i\in T}\{\frac{d_i}{K_i}\}}{\max_{i\in S}\{\frac{d_i}{K_i}\}}\leq\frac{\max_{i\in S}\{a+a_i\}}{\max_{i\in T}\{a+a_i\}}$$

which is equivalent to

$$\frac{\max_{i\in T}\{a_i\}-\max_{i\in S}\{a_i\}}{a+\max_{i\in T}\{a_i\}}\leq\frac{\max_{i\in T}\{\frac{d_i}{K_i}\}}{\max_{i\in S}\{\frac{d_i}{K_i}\}}.$$

Conversely, take a pair of non-empty coalitions $S,T\subset N$ with $S\cap T=\emptyset$ and $\max_{i\in S}\{a_i\}\leq\max_{i\in T}\{a_i\}$. If condition 1 in the statement holds, then

$$c(S\cup T)=\max_{i\in S\cup T}\{a+a_i\}\max_{i\in S\cup T}\{\frac{d_i}{K_i}\}=\max_{i\in T}\{a+a_i\}\max_{i\in T}\{\frac{d_i}{K_i}\}=c(T)\leq c(S)+c(T).$$

If condition 2 in the statement holds, then

$$(\max_{i\in T}\{a_i\}-\max_{i\in S}\{a_i\})\max_{i\in S}\{\frac{d_i}{K_i}\}\leq\max_{i\in T}\{a+a_i\}\max_{i\in T}\{\frac{d_i}{K_i}\}$$

and

$$\begin{aligned}c(S)+c(T) &= \max_{i\in S}\{a+a_i\}\max_{i\in S}\{\tfrac{d_i}{K_i}\}+\max_{i\in T}\{a+a_i\}\max_{i\in T}\{\tfrac{d_i}{K_i}\}\\ &\geq \max_{i\in S}\{a+a_i\}\max_{i\in S}\{\tfrac{d_i}{K_i}\}+(\max_{i\in T}\{a_i\}-\max_{i\in S}\{a_i\})\max_{i\in S}\{\tfrac{d_i}{K_i}\}\\ &= \max_{i\in T}\{a+a_i\}\max_{i\in S}\{\tfrac{d_i}{K_i}\}=\max_{i\in S\cup T}\{a+a_i\}\max_{i\in S\cup T}\{\tfrac{d_i}{K_i}\}=c(S\cup T).\end{aligned}$$

□

Proof of Theorem 4.2.

Proof. Take $\sigma\in\Pi(N)$ satisfying that $\sigma^{-1}(1)\in N$ is an agent whose distance to the supplier is maximal. We prove that $m^\sigma(N,c)$ belongs to the core of (N,c). It suffices to show that for every non-empty coalition $S\subset N$, it holds that $\sum_{i\in S}m_i^\sigma(N,c)\leq c(S)$. We distinguish two cases.

a) S contains the agent $\sigma^{-1}(1)$. Then,

$$\begin{aligned}\sum_{i\in S}m_i^\sigma(N,c) &= c(\sigma^{-1}(1))+\sum_{j\in S\setminus\{\sigma^{-1}(1)\}}\Big(c(P_j^\sigma\cup\{j\})-c(P_j^\sigma)\Big)\\ &= c(\sigma^{-1}(1))+\sum_{j\in S\setminus\{\sigma^{-1}(1)\}}\left((a+\max_{i\in N}\{a_i\})\max_{i\in P_j^\sigma\cup\{j\}}\{\frac{d_i}{K_i}\}-(a+\max_{i\in N}\{a_i\})\max_{i\in P_j^\sigma}\{\frac{d_i}{K_i}\}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in S}(a+\max_{i\in N}\{a_i\})\left(\max_{i\in P_j^\sigma\cup\{j\}}\{\frac{d_i}{K_i}\}-\max_{i\in P_j^\sigma}\{\frac{d_i}{K_i}\}\right)\\
&\leq \sum_{j\in S}(a+\max_{i\in N}\{a_i\})\left(\max_{i\in (P_j^\sigma\cup\{j\})\cap S}\{\frac{d_i}{K_i}\}-\max_{i\in P_j^\sigma\cap S}\{\frac{d_i}{K_i}\}\right)\\
&= (a+\max_{i\in N}\{a_i\})\sum_{j\in S}\left(\max_{i\in (P_j^\sigma\cup\{j\})\cap S}\{\frac{d_i}{K_i}\}-\max_{i\in P_j^\sigma\cap S}\{\frac{d_i}{K_i}\}\right)\\
&= (a+\max_{i\in S}\{a_i\})\max_{i\in S}\{\frac{d_i}{K_i}\}\\
&= c(S),
\end{aligned}
$$

where the inequality follows from the fact that the function $f(x)=\max\{x,y\}-x$ is non-increasing in x for all $y\in[0,\infty)$ (and taking $y=\frac{d_j}{K_j}$).

b) S does not contain the agent $\sigma^{-1}(1)$. In this case denote $\bar{S}=S\cup\{\sigma^{-1}(1)\}$. Using the same proof above we conclude that

$$\sum_{i\in\bar{S}} m_i^\sigma(N,c)\leq c(\bar{S}).$$

Now, taking into account that $m^\sigma_{\sigma^{-1}(1)}(N,c)=c(\sigma^{-1}(1))$ and that (N,c) is subadditive, it holds that

$$c(\sigma^{-1}(1))+\sum_{i\in S} m_i^\sigma(N,c)=\sum_{i\in\bar{S}} m_i^\sigma(N,c)\leq c(\bar{S})\leq c(\sigma^{-1}(1))+c(S),$$

which implies that $\sum_{i\in S} m_i^\sigma(N,c)\leq c(S)$. □

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