A new cost allocation rule for inventory transportation systems

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Abstract

In this paper we propose a context-specific cost allocation rule for inventory transportation systems. We consider the setting defined in Fiestras-Janeiro et al. [1] and propose a new cost allocation rule, the so-called AMEF value, which is inspired in the Shapley value. We prove that, under suitable conditions, the AMEF value provides core allocations. Besides, we provide a characterization of the AMEF value based on properties of balanced contributions, solidarity, and transfer.

Keywords: inventory transportation systems, cost allocation, cooperative games.

1 Introduction

An inventory transportation system (cf. [1]) appears when there are two or more agents facing Economic Order Quantity (EOQ) problems and their fixed order costs can be written as the sum of two components, one due to common setup costs and other due to firm-dependent transportation costs (see [2], for more details on EOQ problems). It is also assumed that the agents are located on a line route in the sense that, if a group of agents places a joint order, the order cost is the sum of the common setup cost plus the transportation cost of one of the agents in the group with a highest transportation cost (extreme agents). A real situation that can be suitably modeled as an inventory transportation system is the distribution of the products in a franchising business. One can consult [3] for other applications that could be under the scope of this model.

The feature we are interested in is the allocation of the total cost that a group of agents has to pay if they decide to place joint orders. This feature, applied to other inventory systems, has also been studied in the literature (see, for instance, [4] and [5]), which shows that cooperative game theory is an appropriate tool for solving the problem. In fact, cooperative game theory has been widely used to tackle cost allocation problems (see, for instance, [6]). The cost allocation problem in inventory transportation systems was studied by Fiestras-Janeiro et al. [1] by using this tool. They introduce the class of inventory transportation games as the set of cooperative cost games that model this problem. Moreover, they

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define and characterize a cost allocation rule, the line rule, that provides core allocations for the games in this class (under certain conditions).

In this paper we provide new insights for inventory transportation systems. First, we show that every inventory transportation system can be decomposed into a sum of simpler inventory transportation systems. The main characteristic of these simpler systems is that each of them has only one extreme agent. This fact is helpful in the research on this class of systems. Second, we propose a new allocation rule, the AMEF value (AMEF stands for **A**verage of the **M**arginal vectors with an **E**xtreme agent **F**irst), that provides core allocations under suitable conditions. For its definition we use more information than in the definition of the line rule. Finally, we show a characterization of the new rule in terms of properties which have the same flavor as others already studied in the literature such as balanced contributions (cf. [7]), transfer (cf. [8]), and a new property of solidarity.

The paper is organized as follows. Section 2 is devoted to recall some preliminaries from inventory transportation systems and cooperative game theory. Moreover, we provide a new property of inventory transportation systems. In sections 3 and 4 we propose and characterize the AMEF value. Finally, the Appendix contains the proofs of the main results in this paper.

2 Preliminaries

Fiestras-Janeiro et al. [1] introduced an inventory transportation system as a multiple agent situation where each agent faces a basic EOQ problem and where the fixed order cost of each agent i is the sum of a first component, $a > 0$ (common to all agents), plus a second component, $a_i > 0$, which is proportional to the distance of the agent to the supplier. N denotes the finite set of agents and the other parameters associated to every $i \in N$ are the usual in inventory systems: the deterministic demand per time unit $d_i > 0$ and the holding cost per item and per time unit $h_i > 0$. To meet the demand in time, each agent i keeps stock in hand by placing orders of size $Q_i > 0$.

In one of those systems, the agents in every coalition $S \subset N$ can cooperate by placing joint orders (forming what is called an order coalition). Moreover, the following assertions are assumed.

1. All the agents are located on the same line route. By this we mean that if a group of agents S places a joint order, its fixed cost is the sum of the first component a plus the second component of an agent in S whose distance from the supplier is maximal (that we denote by a_S; i.e., $a_S = \max\{a_i \mid i \in S\}$).

2. The supplier accepts and even encourages agents to form order coalitions at the beginning of each term. But, because of organizational reasons, once an order coalition S has been formed, the fixed cost that the supplier charges to this coalition, for each order throughout the term, is $a + a_S$. This means that, if in a particular order an agent $i \in S$ does not buy units of the product, then the supplier even charges $a + a_S$ to S.

An inventory transportation system is denoted by the 2-tuple $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i \in N})$. Fiestras-Janeiro et al. [1] showed that, if S forms, all agents in S will coordinate their cycles. Then, the optimal size of the order and the optimal number of orders per time unit for agent $i \in S$ are

$$\hat{Q}_i = \sqrt{\frac{2(a + a_S)d_i^2}{\sum_{j \in S} d_j h_j}} \qquad \text{and} \qquad \hat{m}_S = \frac{d_i}{\hat{Q}_i} = \sqrt{\frac{\sum_{j \in S} d_j h_j}{2(a + a_S)}},$$

and the optimal total average cost per time unit is

$$C(S,\hat{Q}_i) = \sqrt{2(a+a_S)\sum_{j\in S} d_j h_j} = 2(a+a_S)\hat{m}_S.$$

One of the main issues concerning an inventory transportation system is how to split the total cost among the involved agents. For this purpose, one can use the cooperative game theory by associating a (transferable utility) cost game to each inventory transportation system. A cost game is a pair (N,c), where N is the finite set of agents and $c : 2^N \longrightarrow \mathbb{R}$ is the so-called characteristic function of the game, which assigns to each subset $S \subset N$ the cost $c(S)$ of the common project of agents in S. By convention, $c(\emptyset) = 0$. For an inventory transportation system $(N,\mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i\in N})$, Fiestras-Janeiro et al. [1] proposed the cost game (N,c) where, for every S, $c(S)$ is the minimal total average cost per time unit for S. Formally,

$$c(S) := 2(a+a_S)\hat{m}_S. \tag{1}$$

We denote by $\mathcal{ITS}$ the class of inventory transportation systems and by $\mathcal{ITG}$ the class of inventory transportation games, i.e. the class of cost games associated to inventory transportation systems.

A cost game is said to be subadditive if it is not beneficial for any coalition to split into several smaller disjoint sub-coalitions. Formally, (N,c) is subadditive if for each $S,T \subset N$ such that $S\cap T = \emptyset$, it holds that $c(S) + c(T) \geq c(S\cup T)$. Fiestras-Janeiro et al. [1] obtained that the subadditivity of an inventory transportation game (N,c) associated to $(N,\mathcal{I})$ is equivalent to the following condition:

$$\hat{m}_T \geq \frac{1}{2}\frac{a_T - a_S}{a+a_T}\hat{m}_S$$

for all $S,T \subset N$ such that $S\cap T = \emptyset$ and $a_S \leq a_T$.

Next, we show a new property of the class $\mathcal{ITG}$ which will be useful in the rest of the paper. It says that every inventory transportation system can be decomposed into inventory transportation systems with only one extreme agent. An agent $i \in N$ is said to be an extreme agent of $(N,\mathcal{I})$ if $a_i = a_N$, i.e. if its distance to the supplier is greater than or equal to the distance to the supplier of the other agents. $E_{(N,\mathcal{I})}$ represents the set of extreme agents of $(N,\mathcal{I})$.

Let $(N,\mathcal{I}) \in \mathcal{ITS}$ and let $(N,c) \in \mathcal{ITG}$ be its associated cost game. For each $k \in E_{(N,\mathcal{I})}$ we can define an inventory transportation system $(N,\mathcal{I}^k)$ where $a^k = a$, $h_i^k = h_i$, and $d_i^k = d_i$ for all $i \in N$, and

$$a_i^k = \begin{cases} a_N - \epsilon & \text{if } i \in E_{(N,\mathcal{I})} \setminus \{k\}, \\ a_i & \text{otherwise} \end{cases}$$

for some fixed $\epsilon \in (0, a_N)$. The interpretation of $(N,\mathcal{I}^k)$ is that all agents in $E_{(N,\mathcal{I})}$ but k moves a bit closer to the supplier and the rest of the agents remain in their initial positions on the line route to the supplier. The holding costs and demands remain unchanged for all the agents. The cost game arising from $(N,\mathcal{I}^k)$ is denoted by (N,c^k).

Next, we define an operation on the class $\mathcal{ITG}$. Roughly speaking, this operation defines a new inventory transportation system where each parameter is given by the maximum of the corresponding parameters in the inventory transportation systems which are being operated. Let J be a finite set and, $\{(N,\mathcal{I}^{(j)})\}_{j\in J}$ be a family of inventory transportation systems. We define the new inventory transportation system $(N, \bigvee_{j\in J}\mathcal{I}^{(j)})$, denoted by $(N,\mathcal{I}^{\vee})$, where $a^{\vee} = \max_{j\in J}\{a^{(j)}\}$, $a_i^{\vee} = \max_{j\in J}\{a_i^{(j)}\}$, $h_i^{\vee} =$

$\max_{j\in J}\{h_i^{(j)}\}$, and $d_i^{\vee} = \max_{j\in J}\{d_i^{(j)}\}$ for all $i \in N$. The cost game arising from $(N, \mathcal{I}^{\vee})$ is denoted by $(N, c^{\vee})$. From this definition the next proposition can be proven.

Proposition 2.1. *Let $(N, \mathcal{I}) \in \mathcal{ITS}$ and let $(N, \mathcal{I}^k) \in \mathcal{ITS}$ be defined as above for each $k \in E_{(N,\mathcal{I})}$. Then,*

$$(N, \mathcal{I}) = (N, \bigvee_{k\in E_{(N,\mathcal{I})}} \mathcal{I}^k).$$

Proof. See Appendix A. □

The above proposition shows that every inventory transportation system can be written as the maximum of inventory transportation systems with exactly one extreme agent.

3 The AMEF value

An *allocation rule* for inventory transportation systems is a mapping ψ that associates to each $(N, \mathcal{I}) \in \mathcal{ITS}$ a vector $\psi(N, \mathcal{I}) = (\psi_i(N, \mathcal{I}))_{i\in N}$ satisfying that $\sum_{i\in N} \psi_i(N, \mathcal{I}) = c(N)$, being $(N, c) \in \mathcal{ITG}$ its associated cost game. We look for allocations in the core of (N, c), which is the set

$$\mathcal{C}(N, c) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i\in N} x_i = c(N), \sum_{i\in S} x_i \leq c(S) \text{ for each } S \subset N \right\}.$$

Fiestras-Janeiro et al. [1] showed that the core of an inventory transportation game is nonempty if the game is subadditive. In the proof of this result they found that some particular marginal vectors belong to the core whenever the game is subadditive. Recall that a marginal vector is formed by the marginal contribution of each agent i to its predecessors according to a particular ordering. Formally, let $\Pi(N)$ be the set of all orderings in N. Every $\sigma \in \Pi(N)$ is a one-to-one map which associates to every element of N a natural number in $\{1, 2, \ldots, n\}$ (n denotes the number of elements of N). $\sigma(i) = j$ means that i has the j-th position in the ordering given by σ. Denote by σ^{-1} the inverse of map σ. For every $i \in N$, the set of predecessors of i with respect to $\sigma \in \Pi(N)$ is $P_i^{\sigma} = \{j \in N \mid \sigma(j) < \sigma(i)\}$. Now take $\sigma \in \Pi(N)$; the marginal vector associated with σ is defined as $m^{\sigma}(N, c) = (m_i^{\sigma}(N, c))_{i\in N}$, where $m_i^{\sigma}(N, c) = c(P_i^{\sigma} \cup \{i\}) - c(P_i^{\sigma})$ for each $i \in N$.

Fiestras-Janeiro et al. [1] considered those marginal vectors associated with orderings $\sigma \in \Pi(N)$ which invert the ordering given by the distances from the agents to the supplier; then they define the line rule as the average of those marginal vectors. Formally, let $\Pi(N, \mathcal{I})$ be the set of those orderings in $(N, \mathcal{I})$ and let (N, c) be its associated cost game. The line rule for this system, $L(N, \mathcal{I}) = (L_i(N, \mathcal{I}))_{i\in N}$, is given by

$$L_i(N, \mathcal{I}) = \frac{1}{|\Pi(N, \mathcal{I})|} \sum_{\sigma\in\Pi(N,\mathcal{I})} m_i^{\sigma}(N, c), \quad \text{for every } i \in N.$$

The line rule has the same flavor as the Shapley value (cf. [9]), but requires less computational effort. Moreover, Fiestras-Janeiro et al. [1] showed that the line rule always provides core elements whenever the associated game is subadditive, and that the Shapley value could provide allocations outside the core even when the associated game is subadditive. Remember that the Shapley value is defined as the average of all the marginal vectors,

$$\Phi(N, c) = \frac{1}{|\Pi(N)|} \sum_{\sigma\in\Pi(N)} m^{\sigma}(N, c).$$

In fact, Fiestras-Janeiro et al. [1] proved that all orderings $\sigma \in \Pi(N)$ such that $\sigma^{-1}(1) \in E_{(N,\mathcal{I})}$ satisfy that their corresponding marginal vectors belong to the core of the game whenever it is subadditive. Taking advantage of this fact, here we propose a new allocation rule as the average of all these marginal vectors. We call it, the **A**verage of the **M**arginal vectors with an **E**xtreme agent **F**irst value, shortly, the AMEF value. The formal definition is the following one, where $\Pi_{E_{(N,\mathcal{I})}}$ is the set of all orderings $\sigma \in \Pi(N)$ such that $\sigma^{-1}(1) \in E_{(N,\mathcal{I})}$.

Definition 3.1. *The AMEF value is the allocation rule which associates to every $(N,\mathcal{I}) \in \mathcal{ITS}$, the allocation* $\mathrm{AMEF}(N,\mathcal{I}) = (\mathrm{AMEF}_i(N,\mathcal{I}))_{i\in N}$ *given by:*

$$\mathrm{AMEF}_i(N,\mathcal{I}) = \frac{1}{|\Pi_{E_{(N,\mathcal{I})}}|} \sum_{\sigma\in\Pi_{E_{(N,\mathcal{I})}}} m_i^{\sigma}(N,c),$$

for all $i \in N$.

Let us note that when $E_{(N,\mathcal{I})} = N$, the AMEF value coincides with the line rule and with the Shapley value of the associated cost game. The AMEF value and the line rule also coincide when all of the non extreme agents are located at the same distance from the supplier.

Theorem 3.1. *Let $(N,\mathcal{I}) \in \mathcal{ITS}$ and $(N,c) \in \mathcal{ITG}$ its associated cost game. If (N,c) is subadditive, then* $\mathrm{AMEF}(N,\mathcal{I}) \in \mathcal{C}(N,c)$.

Proof. Recall that $m^{\sigma}(N,c) \in \mathcal{C}(N,c)$ for all $\sigma \in \Pi_{E_{(N,\mathcal{I})}}$ whenever (N,c) is a subadditive game (see [1]). Moreover, since $\mathcal{C}(N,c)$ is a convex set, the result follows in view of the definition of the AMEF value. □

Example 3.1. *Consider the inventory transportation system with 4 firms, $a = 300$ and*

i	a_i	d_i	h_i
1	200	60	0.08
2	400	70	0.08
3	900	60	0.03
4	900	50	0.1

The associated cost game is

S	1	2	3	4	12	13	14
$c(S)$	69.28	88.54	65.73	109.54	120.66	125.86	153.36

S	23	24	34	123	124	134	234	N
$c(S)$	133.27	159.5	127.75	171.11	192.25	166.85	172.51	203.17

It is readily proven that (N,c) is subadditive. Moreover, $E_{(N,\mathcal{I})} = \{3,4\}$ and

$$\Pi_{E_{(N,\mathcal{I})}} = \{\sigma \in \Pi(\{1,2,3,4\}) \mid \sigma^{-1}(1) = 3 \text{ or } \sigma^{-1}(1) = 4\}.$$

Then, the AMEF value is

$$\mathrm{AMEF}(N,\mathcal{I}) = (39.94, 46.16, 39.93, 77.14) \in \mathcal{C}(N,c).$$

The line rule can be easily computed and its given by

$$L(N,\mathcal{I}) = (30.66, 44.76, 41.97, 85.78) \in \mathcal{C}(N,c).$$

The Shapley value for this game is

$$\Phi(N,c) = (45.47, 56.03, 35.53, 66.14) \in \mathcal{C}(N,c).$$

Figure 1 shows the positions of the above allocations into the $\mathcal{C}(N,c)$. This graphic was drawn with the toolbox TUGlab of MATLAB® (cf. [10]). Notice that the toolbox TUGlab is designed for benefit games and, hence, to deal with a cost game (N,c) like ours, the negative benefit game $(N,-c)$ has to be consider. The web page of TUGlab can be found in `<http://eio.usc.es/pub/io/xogos/index.php>`.

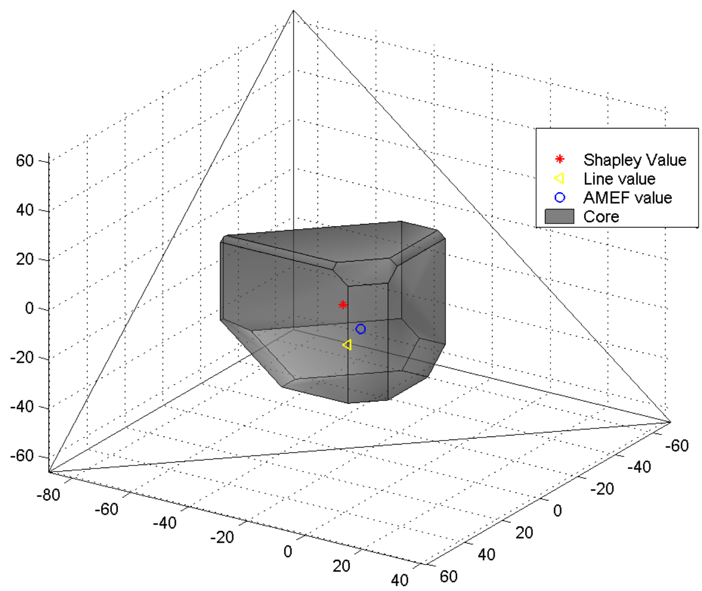

Figure 1: Core, Shapley value, AMEF value, and line rule of the game in Example 3.1

◇

4 Characterization of the AMEF value

Although the AMEF value seems hard to compute when there are many agents (at least harder than the line rule, but less than the Shapley value), it satisfies very good properties. In this section we provide a characterization of the AMEF value on $\mathcal{ITS}$.

In our characterization the inventory transportation systems with only one extreme agent play a special role. In fact, it is easy to prove the following property of the AMEF value. Let $(N,\mathcal{I}) \in \mathcal{ITS}$. Given the inventory transportation systems $(N,\mathcal{I}^k)$, for each $k \in E_{(N,\mathcal{I})}$, the AMEF value of $(N,\mathcal{I})$ is the average of the AMEF values of the systems $(N,\mathcal{I}^k)$, i.e.

$$\text{AMEF}(N,\mathcal{I}) = \frac{1}{|E_{(N,\mathcal{I})}|} \sum_{k \in E_{(N,\mathcal{I})}} \text{AMEF}(N,\mathcal{I}^k) \tag{2}$$

Next step is to define the properties we use in the characterization of the AMEF value. Let us denote by $\mathcal{ITS}^1$ the class of inventory transportation situations with only one extreme agent. Take $(N,\mathcal{I}) \in \mathcal{ITS}^1$ and let us denote by $e(N,\mathcal{I})$ the unique extreme agent of $(N,\mathcal{I})$ and by $(N,c) \in \mathcal{ITG}$ its associated cost game. Let ϕ be an allocation rule on $\mathcal{ITS}$.

One extreme agent balanced contributions (OBC). Take $(N,\mathcal{I}) \in \mathcal{ITS}^1$. For all pair of agents $i,j \in N \setminus \{e(N,\mathcal{I})\}$ it holds

$$\phi_i(N,\mathcal{I}) - \phi_i(N \setminus \{j\},\mathcal{I}_{-j}) = \phi_j(N,\mathcal{I}) - \phi_j(N \setminus \{i\},\mathcal{I}_{-i})$$

with $\mathcal{I}_{-j} = \mathcal{I}_{|_{N\setminus\{j\}}}$ and $\mathcal{I}_{-i} = \mathcal{I}_{|_{N\setminus\{i\}}}$.

One extreme agent solidarity (EAS). Take $(N,\mathcal{I}) \in \mathcal{ITS}^1$. It holds $\phi_{e(N,\mathcal{I})}(N,\mathcal{I}) = c(e(N,\mathcal{I}))$.

The OBC property is based on the Balanced Contribution property introduced by Myerson [7]. Roughly speaking, OBC property requires that if i and j are not extreme agents, the effect on j of i leaving the system is the same as the effect on i of j leaving the system. The EAS property states that the extreme agent is cooperating unselfishly, because he is joining the grand coalition providing some savings in the total cost (since $c(N \setminus \{n\}) + c(\{n\}) \geq c(N)$), but he is paying the same amount as if he is not cooperating with the others (we could say that he is resigning to savings). In some sense, this property implies that the extreme agent is altruistic. However, maybe this agent is altruistic in the present with the hope that other even more distant altruistic agents will join the game and pay in some time in the future.

Next theorem provides a characterization of the AMEF value on $\mathcal{ITS}^1$.

Theorem 4.1. *The AMEF value is the unique allocation rule defined on $\mathcal{ITS}^1$ which satisfies **OBC** and **EAS**.*

Proof. See Appendix A. □

This theorem says that if we are interested in sharing the cost arising from an inventory transportation system with only one extreme agent and, moreover we want that OBC and EAS are fulfilled, then the AMEF value should be selected. The properties in the above theorem are logically independent, i.e., all of them are needed to characterize the AMEF value. The proof can be found in the Appendix A.

For characterizing the AMEF value on $\mathcal{ITS}$ we need one more property. Let ϕ be an allocation rule on $\mathcal{ITS}$.

E-Transfer (ETR). Let $(N,\mathcal{I})$, $(N,\mathcal{I}') \in \mathcal{ITS}$ such that $a' = a$, $a'_N = a_N$, $h'_i = h_i$ and $d'_i = d_i$ for all $i \in N$, and $E_{(N,\mathcal{I})} \cap E_{(N,\mathcal{I}')} = \emptyset$. Then,

$$|E_{(N,\mathcal{I})} \cup E_{(N,\mathcal{I}')}|\phi(N,\mathcal{I} \vee \mathcal{I}') = |E_{(N,\mathcal{I})}|\phi(N,\mathcal{I}) + |E_{(N,\mathcal{I}')}|\phi(N,\mathcal{I}').$$

Let us note that it is readily proven that $E_{(N,\mathcal{I} \vee \mathcal{I}')} = E_{(N,\mathcal{I})} \cup E_{(N,\mathcal{I}')}$.

The ETR property is based on the Transfer property, which constitutes a special form of additivity, used by Meca et al. [8]. Obviously, the AMEF value is not additive. However, it satisfies this kind of weighted additivity which allows a characterization à la Shapley.

Theorem 4.2. *The AMEF value is the unique allocation rule defined on $\mathcal{ITS}$ which satisfies **OBC**, **EAS** and **ETR**.*

Proof. See Appendix A. □

Again, the properties in the above theorem are logically independent. The proof can be found in the Appendix A.

5 Conclusions

This paper re-examines the cost sharing problem studied in [1] and proposes the AMEF value, a new allocation rule for inventory transportation systems. Those systems assume that agents are located on a line route. An interesting generalization of this model follows if that assumption is dropped and a more general spatial assumption is made instead (like, for instance, that agents are located on a tree rooted in the supplier). We plan to deal with this kind of generalized models in the future.

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A Appendix

Here the reader can find the proofs of the results in the paper.

Proof of Proposition 2.1. Let us note that, for all $k \in E_{(N,\mathcal{I})}$, $a^k = a$, $h_i^k = h_i$ and $d_i^k = d_i$ for all $i \in N$. Then, $a^\vee = a$, $h_i^\vee = h_i$ and $d_i^\vee = d_i$ for all $i \in N$. Moreover,

$$a_i^\vee = \max_{k \in E_{(N,\mathcal{I})}} \{a_i^k\} = \begin{cases} a_i & \text{if } i \notin E_{(N,\mathcal{I})} \\ \max\{a_N, \max_{k \in E_{(N,\mathcal{I})} \setminus \{i\}} \{a_N - \epsilon_k\}\} & \text{if } i \in E_{(N,\mathcal{I})} \end{cases}$$

for all $i \in N$. Then, $a_i^\vee = a_i$ for all $i \in N$ and therefore, $(N, \bigvee_{k \in E_{(N,\mathcal{I})}} \mathcal{I}^k) = (N, \mathcal{I})$. □

Proof of Theorem 4.1. First of all we show the uniqueness. Let φ and μ be two efficient allocation rules on $\mathcal{ITS}^1$ satisfying OBC and EAS. The proof will be done by induction on the number of agents.

Let $N = \{i, j\}$ and $E_{(N,\mathcal{I})} = \{j\}$. Then, by EAS, $\mu_j(N, \mathcal{I}) = c(j) = \varphi_j(N, \mathcal{I})$. Moreover, by definition, $\mu_i(N, \mathcal{I}) = c(N) - \mu_j(N, \mathcal{I}) = c(N) - c(j) = c(N) - \varphi_j(N, \mathcal{I}) = \varphi_i(N, \mathcal{I})$.

Let $k \in \mathbb{N}$ such that $k > 2$. Assume that $\varphi = \mu$ for all inventory transportation systems in $\mathcal{ITS}^1$ such that $|N| < k$.

Let $(N, \mathcal{I}) \in \mathcal{ITS}^1$ be such that $|N| = k$ and $E_{(N,\mathcal{I})} = \{e(N, \mathcal{I})\}$. By EAS, we know that

$$\mu_{e(N,\mathcal{I})}(N, \mathcal{I}) = c(e(N, \mathcal{I})) = \varphi_{e(N,\mathcal{I})}(N, \mathcal{I}).$$

Then, we only have to check the equality of the allocations for all agents but $e(N, \mathcal{I})$.

By definition

$$\begin{aligned}\sum_{i\neq e(N,\mathcal{I})} \mu_i(N,\mathcal{I}) &= c(N) - \mu_{e(N,\mathcal{I})}(N,\mathcal{I}) = c(N) - c(e(N,\mathcal{I})) = c(N) - \varphi_{e(N,\mathcal{I})}(N,\mathcal{I}) \\ &= \sum_{i\neq e(N,\mathcal{I})} \varphi_i(N,\mathcal{I}).\end{aligned}$$

Then,

$$\sum_{i\neq e(N,\mathcal{I})} [\mu_i(N,\mathcal{I}) - \varphi_i(N,\mathcal{I})] = 0. \tag{3}$$

By OBC applied to μ, for all $i, j \in N \setminus \{e(N,\mathcal{I})\}$,

$$\mu_i(N,\mathcal{I}) - \mu_i(N \setminus \{j\}, \mathcal{I}_{-j}) = \mu_j(N,\mathcal{I}) - \mu_j(N \setminus \{i\}, \mathcal{I}_{-i}).$$

Then, by induction

$$\mu_i(N,\mathcal{I}) - \varphi_i(N \setminus \{j\}, \mathcal{I}_{-j}) = \mu_j(N,\mathcal{I}) - \varphi_j(N \setminus \{i\}, \mathcal{I}_{-i}). \tag{4}$$

Applying again OBC to φ, for all $i, j \in N \setminus \{e(N,\mathcal{I})\}$,

$$\varphi_i(N,\mathcal{I}) - \varphi_i(N \setminus \{j\}, \mathcal{I}_{-j}) = \varphi_j(N,\mathcal{I}) - \varphi_j(N \setminus \{i\}, \mathcal{I}_{-i}). \tag{5}$$

By substracting Eq. (5) to Eq. (4), we obtain that

$$\mu_i(N,\mathcal{I}) - \varphi_i(N,\mathcal{I}) = \mu_j(N,\mathcal{I}) - \varphi_j(N,\mathcal{I}) \tag{6}$$

for all $i, j \in N \setminus \{e(N,\mathcal{I})\}$.

Combining Eq. (6) with Eq. (3) it holds that, for all $i \in N \setminus \{e(N,\mathcal{I})\}$

$$0 = \sum_{j\neq e(N,\mathcal{I})} \left[\mu_j(N,\mathcal{I}) - \varphi_j(N,\mathcal{I})\right] = (n-1)\left[\mu_i(N,\mathcal{I}) - \varphi_i(N,\mathcal{I})\right].$$

Then,

$$\mu_i(N,\mathcal{I}) = \varphi_i(N,\mathcal{I}), \qquad \text{for all } i \in N \setminus \{e(N,\mathcal{I})\}.$$

The AMEF value satisfies EAS by definition. Moreover, it also satisfies OBC since, for all $i \neq e(N,\mathcal{I})$, the AMEF value is the Shapley value of a particular game, as we show next, and it is well known that the Shapley value satisfies the property of balanced contributions for all pair of agents.

To conclude, we show the relation between the AMEF value and the Shapley value indicated above. Let $(N,\mathcal{I}) \in \mathcal{ITS}^1$ and let $(N,c) \in \mathcal{ITG}$ be its associated game. Consider that the extreme agent $e(N,\mathcal{I})$ decides to leave the grand coalition by paying his individual cost. Then, we define the game reduced by $e(N,\mathcal{I})$ as the cost game $(N \setminus \{e(N,\mathcal{I})\}, c_{-e(N,\mathcal{I})})$ where $c_{-e(N,\mathcal{I})}(S) = c(S \cup \{e(N,\mathcal{I})\}) - c(\{e(N,\mathcal{I})\})$ for all $S \subset N \setminus \{e(N,\mathcal{I})\}$. Then, it is readily proven that the AMEF value can be written as

$$\text{AMEF}(N,\mathcal{I}) = \begin{cases} \Phi(N \setminus \{e(N,\mathcal{I})\}, c_{-e(N,\mathcal{I})}) & \text{if } i \neq e(N,\mathcal{I}) \\ c(e(N,\mathcal{I})) & \text{if } i = e(N,\mathcal{I}) \end{cases}$$

for each $i \in N$. □

The above theorem is tight, since

EAS. The allocation rule given by $\varphi^1(N,\mathcal{I}) = \Phi(N,c)$, for every $(N,\mathcal{I}) \in \mathcal{ITS}^1$ with (N,c) as its associated cost game, satisfies OBC but not EAS,

OBC. The following allocation rule defined by

$$\varphi_i^2(N,\mathcal{I}) = \begin{cases} \dfrac{c(i)}{\sum_{j\in N\setminus\{e(N,\mathcal{I})\}} c(j)}(c(N) - c(e(N,\mathcal{I}))) & \text{if } i \neq e(N,\mathcal{I}) \\ c(e(N,\mathcal{I})) & \text{if } i = e(N,\mathcal{I}) \end{cases}$$

for all $(N,\mathcal{I}) \in \mathcal{ITS}^1$ such that $E_{(N,\mathcal{I})} = \{e(N,\mathcal{I})\}$, satisfies EAS but not OBC.

Proof of Theorem 4.2. First of all we show the uniqueness. Let φ and μ be two efficient allocation rules on $\mathcal{ITS}$ satisfying OBC, EAS, and ETR. The proof will be done by induction on the number of extreme agents.

Let $(N,\mathcal{I}) \in \mathcal{ITS}$ be such that $|E_{(N,\mathcal{I})}| = 1$. Then, by the Theorem 4.1, $\varphi(N,\mathcal{I}) = \mu(N,\mathcal{I}) =$ AMEF$(N,\mathcal{I})$.

Take $\ell \in \mathbb{N}$ such that $\ell > 1$. Assume that $\varphi(N,\mathcal{I}) = \mu(N,\mathcal{I})$ for all systems $(N,\mathcal{I}) \in \mathcal{ITS}$ such that $|E_{(N,\mathcal{I})}| < \ell$.

Let $(N,\mathcal{I}) \in \mathcal{ITS}$ be such that $|E_{(N,\mathcal{I})}| = \ell$ and (N,c) be the associated cost game. W.l.o.g. assume that $E_{(N,\mathcal{I})} = \{n-\ell+1,\ldots,n\}$. By Proposition 2.1, $(N,\mathcal{I}) = \left(N, \bigvee_{k=n-\ell+1}^{n} \mathcal{I}^k\right) = \left(N, \left(\bigvee_{k=n-\ell+1}^{n-1} \mathcal{I}^k\right) \vee \mathcal{I}^n\right)$. Then,

$$\begin{aligned} \ell\varphi(N,\mathcal{I}) &= (\ell-1)\varphi\left(N, \bigvee_{k=n-\ell+1}^{n-1} \mathcal{I}^k\right) + \varphi(N,\mathcal{I}^n) \\ &= (\ell-1)\mu\left(N, \bigvee_{k=n-\ell+1}^{n-1} \mathcal{I}^k\right) + \mu(N,\mathcal{I}^n) \\ &= \ell\mu(N,\mathcal{I}) \end{aligned}$$

where the first equality follows from ETR, the second one is a consequence of both the induction hypothesis and Theorem 4.1, since $\left(N, \bigvee_{k=n-\ell+1}^{n-1} \mathcal{I}^k\right) \in \mathcal{ITS}$ has $\ell-1$ extreme agents and $(N,\mathcal{I}^n) \in \mathcal{ITS}^1$, and, finally, the last inequality again follows from ETR.

To conclude, the AMEF value satisfies EAS and OBC as we saw in the proof of Theorem 4.1. Moreover, it also satisfies ETR by Eq. (2). □

The above theorem is also tight; see the following allocation rules defined for all $(N,\mathcal{I}) \in \mathcal{ITG}$:

EAS. $\varphi^3(N,\mathcal{I}) =$ AMEF$(N,\bar{\mathcal{I}})$, with $(N,\bar{\mathcal{I}})$ being an inventory transportation system where $\bar{a} = a$, $\bar{a}_i = a_N$, $\bar{h}_i = h_i$, and $\bar{d}_i = d_i$, for all $i \in N$, satisfies OBC and ETR but not EAS.

OBC. $\varphi^4(N,\mathcal{I}) = \dfrac{1}{|E_{(N,\mathcal{I})}|} \sum_{k\in E_{(N,\mathcal{I})}} \varphi^2(N,\mathcal{I}^k)$, satisfies ETR and EAS but not OBC.

ETR. The allocation rule,

$$\varphi_i^5(N,\mathcal{I}) = \begin{cases} \text{AMEF}_i(N,\mathcal{I}) & \text{if } |E_{(N,\mathcal{I})}| = 1 \\ \dfrac{c(i)}{\sum_{j\in N} c(j)} c(N) & \text{otherwise} \end{cases}$$

satisfies EAS and OBC but not ETR.

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