A characterization of the proportional rule in multi-issue allocation situations

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Abstract

In multi-issue allocation situations we have to divide a resource among a group of agents. The claim of each agent is a vector specifying the amount claimed by each agent on each issue. We present an axiomatic characterization of the proportional rule.

Keywords: multi-issue allocation situations, proportional rule.

1 Introduction

In bankruptcy situations a resource must be divided among several claimants. The problem arises when the resource is not sufficient to cover all claims. A typical example is when a firm goes bankrupt. The objective is to identify well-behaved rules for dividing the resource among agents. The literature devoted to the formal analysis of bankruptcy problems originates in a paper by O'Neill [9]. See Thomson [10] for a survey.

In bankruptcy situations the claim of each agent is a number. However, there are many real-world situations where the resource must be divided not on the basis of a single claim for each agent, but several claims related to different issues. These kind of problems are called multi-issue allocation (MIA) situations and were introduced in Calleja *et al* [2].

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Lorenzo-Freire *et al* [7], Moreno-Ternero [8], González-Alcón *et al* [4], Lorenzo-Freire *et al* [6], Ju *et al* [5], and Bergantiños *et al* [1] also study MIA situations.

Here is a concrete example. In most Spanish Universities, once the total annual operating budget for issues is decided, it is the responsibility of the senior administrators of each department to decide how much is allocated to each issue, such as research, teaching, ... and submit a quantified request. Once the issue allocations are finalized, the university compiles the university's annual operating budget and each department is notified of the amount assigned to each issue. Other examples are: The European Community who distributes its budget among several issues (agriculture, roads, research, ...) and each member state (Spain, France, ...) makes a claim for each issue. The Spanish government divides the budget among several issues (health, education, ...) and each autonomous region (Galicia, Madrid, Catalonia, ...) makes a claim for each issue. The government of Galicia divides its budget among several issues (roads, education, ...) and each City Council (Vigo, Santiago de Compostela, ...) makes a claim for each issue.

All these situations can be modeled as a 4-tuple $(R, N, E, (c_{ki})_{k\in R, i\in N})$ where R is the set of issues (research, teaching, ... in the university example), N is the set of agents (the departments), E is the resource (the amount the university has decided to assign to all the departments), and c_{ki} is the claim of agent i on issue k.

In bankruptcy a rule is a vector $(f_i)_{i\in N}$ where f_i is the amount assigned to agent i. In multi-issue allocation situations, two approaches are possible. Approach 1: as in bankruptcy, a rule is a vector $(f_i)_{i\in N}$ where f_i is the amount assigned to agent i. It is followed in Calleja *et al* [2], González-Alcón *et al* [4], and Ju *et al* [5].

Approach 2: we first divide the budget among the issues. In a second step, the amount assigned to each issue is divided among the agents. Here a rule is a matrix $(f_{ki})_{k\in R, i\in N}$ where f_{ki} denotes the amount received by agent i on issue k. No agent can spend part of the amount he receives for an issue on another issue. This approach is more popular in many situations, for example, the ones mentioned above. It is followed in Lorenzo-Freire *et al* [7], Moreno-Ternero [8], and Bergantiños *et al* [1].

In this paper we follow Approach 2. We focus on the proportional rule (P). Moreno-Ternero [8] is the only study that follows Approach 2 devoted to P in MIA situations, although he does not provide a characterization of P in MIA situations. Ju *et al* [5] study P in MIA situations following Approach 1. They characterize P generalizing the result given by Chun [3] for P for bankruptcy. We provide the first characterization of P using properties adapted to MIA situations. Our result also generalizes the characterization of P for bankruptcy given by Chun [3].

The paper is organized as follows. In Section 2 we introduce MIA situations. In Section 3 we present our results.

2 Multi-issue allocation situations

In this section we introduce the model of bankruptcy and the proportional bankruptcy rule. We then introduce multi-issue allocation situations.

A *bankruptcy problem*, O'Neill [9], is a triple (N, E, c). $N = \{1, \ldots, n\}$ is the set of agents. The resource $E \geq 0$ represents the amount to be divided among the agents,

$c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ and for each $i \in N$, c_i denotes the claim of player i. It is assumed that $0 \leq E \leq \sum_{i \in N} c_i$. A *bankruptcy rule* is a function ψ which associates with each bankruptcy problem (N, E, c) a vector $\psi(N, E, c) \in \mathbb{R}^N$ such that $\sum_{i \in N} \psi_i(N, E, c) = E$ and $0 \leq \psi_i(N, E, c) \leq c_i$ for each $i \in N$.

The *proportional rule* (P) is defined for each $i \in N$ as $P_i(N, E, c) = \lambda c_i$ where $\lambda = \dfrac{E}{\sum_{j \in N} c_j}$.

A *multi-issue allocation (MIA) situation*, Calleja *et al* [2], is a 4-tuple (R, N, E, C). Here, $R = \{1, \ldots, r\}$ is the set of issues; $N = \{1, \ldots, n\}$ is the set of agents; $E \geq 0$ is the amount of resource to be divided; for each $i \in N$ and $k \in R$, c_{ki} is the amount claimed by player $i \in N$ on issue $k \in R$ and $C = (c_{ki})_{k \in R, i \in N} \in \mathbb{R}^{R \times N}_+$. We assume $0 \leq E \leq \sum_{k \in R} \sum_{i \in N} c_{ki}$.

Note that a bankruptcy situation is a MIA situation with $|R| = 1$.

In order to define a rule, two approaches are possible. In the first approach, a rule assigns an amount to each agent (Calleja *et al* [2], González-Alcón *et al* [4], and Ju *et al* [5]). In the second approach, a rule assigns an amount to each agent and each issue (Lorenzo-Freire *et al* [7], Moreno-Ternero [8], and Bergantiños *et al* [1]). We follow the second approach.

A *multi-issue allocation (MIA) rule* f is a function that associates with each MIA situation (R, N, E, C) a matrix $f(R, N, E, C) \in \mathbb{R}^{R \times N}$ such that

- $0 \leq f_{ki}(R, N, E, C) \leq c_{ki}$ for each $k \in R$ and each $i \in N$.
- $\sum_{k \in R} \sum_{i \in N} f_{ki}(R, N, E, C) = E$.

3 The Proportional rule

Probably the most important rules in bankruptcy are the proportional (P) rule, the constrained equal awards (CEA) rule, and the constrained equal losses (CEL) rule. The three rules have been extended to MIA situations. Lorenzo-Freire *et al* [7] and Bergantiños *et al* [1] give several characterizations of CEA and CEL using properties adapted to MIA situations. In this section we provide the first characterization of P using properties adapted to MIA situations.

There exist two ways of extending P from bankruptcy situations to MIA situations. We can use a two-stage procedure as in Lorenzo-Freire *et al* [7]. We first divide the resource among the issues, following the proportional bankruptcy rule. Second, we divide the amount assigned to each issue among the agents, following also the proportional bankruptcy rule. The second way is a one-stage procedure. We also assign to each agent in each issue, an amount proportional to the claim of the agent in the issue. Moreno-Ternero [8] proves that the proportional rule is the unique bankruptcy rule such that the one-stage extension and the two-stage extension coincide. Thus, we define the *proportional rule* as follows.

For each (R,N,E,C), each $k\in R$, and each $i\in N$, $P_{ki}(R,N,E,C)=\lambda c_{ki}$ where $\lambda=\dfrac{E}{\sum\limits_{l\in R}\sum\limits_{j\in N}c_{lj}}$.

Chun [3] introduces the property of non-advantageous transfer (NAT) in bankruptcy situations. A rule ψ satisfies NAT if for each (N,E,c), (N,E,c') and $M\subset N$ such that $c_i=c_i'$ when $i\in N\setminus M$ and $\sum\limits_{i\in M}c_i=\sum\limits_{i\in M}c_i'$, then $\sum\limits_{i\in M}\psi_i(N,E,c)=\sum\limits_{i\in M}\psi_i(N,E,c')$. Chun [3] proves that P is the unique bankruptcy rule satisfying NAT. A stronger form of this property, under the same hypotheses, states that $\psi_i(N,E,c)=\psi_i(N,E,c')$ for all $i\in N\setminus M$. Thus, NAT can be interpreted as follows. No group of agents S can change the amount received by any agent of $N\setminus S$ by transferring claims among themselves.

Bergantiños *et al* [1] extend several properties from bankruptcy situations to MIA situations by invoking the properties twice. The property is firstly interpreted for the set of issues and then for the agents within each issue. We apply the same idea to NAT.

Non-advantageous transfer across issues ($NATA$). Let (R,N,E,C) and (R',N',E',C') be such that $(R,N,E)=(R',N',E')$ and $S\subset R$ such that $\sum\limits_{k\in S}\left(\sum\limits_{i\in N}c_{ki}\right)=\sum\limits_{k\in S}\left(\sum\limits_{i\in N}c_{ki}'\right)$ and $c_{ki}=c_{ki}'$ when $(k,i)\in R\setminus S\times N$. Then, for each $k\in R\setminus S$,

$$\sum_{i\in N}f_{ki}(R,N,E,C)=\sum_{i\in N}f_{ki}(R,N,E,C').$$

$NATA$ implies that if the agents redistribute their claims among a group of issues, then the amount assigned to each other issue does not change. For instance, the level of total resources allocated to health care depends only on the total claim in health care $\left(\sum\limits_{i\in N}c_{ki}\right)$ and the aggregate claim on the rest of the issues $\left(\sum\limits_{l\neq k}\sum\limits_{i\in N}c_{li}\right)$, but it does not depend on the way in which this aggregate claim is redistributed among the rest of the issues.

Non-advantageous transfer within issues ($NATI$). Let (R,N,E,C) and (R',N',E',C') be such that $(R,N,E)=(R',N',E')$, $k\in R$, and $M\subset N$, such that $\sum\limits_{i\in M}c_{ki}=\sum\limits_{i\in M}c_{ki}'$ and $c_{li}=c_{li}'$ when $l\in R\setminus\{k\}$ or $i\in N\setminus M$. Then, for each $i\in N\setminus M$,

$$f_{ki}(R,N,E,C)=f_{ki}(R,N,E,C').$$

$NATI$ says that if a group of agents redistribute their claims within an issue, then the amount assigned to the other agents in this issue does not change. For instance, the amount received by a local government i in health care, provided that the claims of all agents in the rest of issues are the same, depends only on the local government's claim in health care (c_{ki}) and the aggregate claim of the other agents in health care $\left(\sum_{j \neq i} c_{kj}\right)$, but it does not depend in the way in which this aggregate claim in health care is redistributed among the rest of the agents.

We now give a characterization of the proportional rule with $NATA$ and $NATI$.

Theorem 1. Let (R, N, E, C) be such that $|N| \geq 3$ and $|R| \geq 3$. Then, P is the unique rule satisfying $NATA$ and $NATI$.

Proof of Theorem 1. It is obvious that P satisfies $NATA$ and $NATI$.

The uniqueness is a consequence of the following claims. We give a sketch of the proof. Let f be a rule satisfying $NATI$ and $NATA$.

Claim 1. Let $q = (q_{li})_{l \in R, i \in N}$ be such that for each $l \in R$, $(q_{li})_{i \in N}$ belongs to the simplex in $\mathbb{R}^N$. For each $\left(R, E, (x_l)_{l \in R}\right)$ we define f^q in such a way that for each $k \in R$,

$$f_k^q \left(R, E, (x_l)_{l \in R}\right) = \sum_{i \in N} f_{ki} \left(R, N, E, (q_{li} x_l)_{l \in R, i \in N}\right).$$

Then, $f^q = P$.

Claim 2. For each $k \in R$, $\sum_{i \in N} f_{ki} \left(R, N, E, (c_{li})_{l \in R, i \in N}\right) = \sum_{i \in N} P_{ki} \left(R, N, E, (c_{li})_{l \in R, i \in N}\right)$.

Claim 3. Let $k \in R$ and $d^k = (d_{lj})_{l \in R \setminus \{k\}, j \in N} \in \mathbb{R}_+^{R \setminus \{k\} \times N}$. For each $\left(N, E, (y_j)_{j \in N}\right)$ we define f^{d^k} in such a way that for each $i \in N$

$$f_i^{d^k} \left(N, E, (y_j)_{j \in N}\right) = f_{ki} \left(R, N, \frac{\sum_{l \in R, j \in N} d_{lj}}{\sum_{j \in N} y_j} E, (d_{lj})_{l \in R, j \in N}\right)$$

where $d_{kj} = y_j$ for all $j \in N$. Then, $f^{d^k} = P$.

Claim 4. For each $k \in R$ and $i \in N$,

$$f_{ki} \left(R, N, E, (c_{lj})_{l \in R, j \in N}\right) = P_{ki} \left(R, N, E, (c_{lj})_{j \in R, j \in N}\right). \blacksquare$$

Remark 1. The properties used in Theorem 1 are independent. Lorenzo-Freire *et al* [7] define a two-stage procedure to define MIA rules from bankruptcy rules. They first apply a rule for dividing the resource among the issues. Later, the amount assigned to each issue is divided among the agents by applying another rule. Note that this bankruptcy rule can be different from the first one.

- The MIA rule obtained by dividing among the issues with CEA and within each issue with P satisfies $NATI$ but fails $NATA$.

- The MIA rule obtained by dividing among the issues with P and within each issue with CEA satisfies $NATA$ but fails $NATI$.

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