A Note on Coalitional Manipulation and Centralized Inventory Management *

M.A. Mosquera[1,†], I. García-Jurado[2] and M.G. Fiestras-Janeiro[3]

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Abstract: In this note we deal with inventory games as defined in Meca et al. (2003). In that context we introduce the property of *immunity to coalitional manipulation*, and demonstrate that the SOC-rule (Share the Ordering Cost) is the unique allocation rule for inventory games which satisfies this property.

Key Words: centralized multi-agent inventory cost situations, inventory games, coalitional manipulation, SOC-rule.

1 Introduction

Inventory centralization is known to reduce costs in several models of multi-agent inventory cost optimization. An important problem which arises in these models is how the central management should allocate the costs among the agents.

This problem has been tackled by several authors in the last years. For instance, Hartman and Dror (1996) consider some properties that an allocation rule should satisfy in the context of multi-agent stochastic continuous review models, and propose a rule satisfying these properties. Hartman et

[1]Department of Statistics and Operations Research, Faculty of Business Administration and Tourism, Vigo University, 32004 Ourense, Spain.

[2]Department of Statistics and OR, Faculty of Mathematics, Santiago de Compostela University, 15782 Santiago de Compostela, Spain.

[3]Department of Statistics and Operations Research, Faculty of Economics, Vigo University, 36271 Vigo, Spain.

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†Corresponding author. *E-mail address:* mamrguez@uvigo.es.

al. (2000) and Müller et al. (2002) study the core of a class of games arising in multi-agent stochastic single-period models. Meca et al. (2003) and Meca et al. (2004) introduce the so-called *inventory games*, which model cost allocation problems in multi-agent deterministic continuous review models, and propose and characterize the *SOC-rule* (Share the Ordering Cost) for these games, which always proposes core allocations. More recently, van den Heuvel et al. (2007) and Guardiola et al. (2007) analyze new classes of games connected with cost allocation in deterministic periodic review models, more precisely in economic lot-sizing problems. The main difference with Meca et al.'s setting is that in economic lot-sizing the orders can only be made in a collection of fixed time instants, and that the demand and the holding costs depend on the time period considered.

In this note we consider the model studied in Meca et al. (2003). So, although there are several classes of games arising in inventory centralization, when we write *inventory games* throughout this paper, we mean the class of games described in Meca et al. (2003). In this context we look for an allocation rule which is immune to possible manipulations of the agents involved in the problem via artificial merging or splitting. We prove that the unique allocation rule for inventory games which is immune to coalitional manipulation is the SOC-rule. Although coalitional manipulation had never been studied in the context of centralized inventory models, it is an interesting property which has deserved a wide attention in the economical literature. Ju (2003), Bergantiños and Sánchez (2003) and de Frutos (1999) are three recent examples in which coalitional manipulation is considered in various allocation problems.

The organization of this note is as follows. In the next section we remind the main features concerning inventory games and introduce a property of immunity to coalitional manipulation for allocation rules in this context. In Section 3 we state and prove the main result.

2 Coalitional Manipulation in Inventory Games

In this paper, as in Meca et al. (2003), we deal with inventory games. An inventory game is a cost TU game arising from a centralized multi-agent inventory cost situation, in which every agent faces a deterministic continuous review inventory problem which can be modeled as an Economic Production Quantity (EPQ) with shortages problem[1]. In one of these situations, a finite group of agents N agrees to make jointly the orders of a certain good which

[1]The EPQ with shortages problem is a rather general deterministic inventory model; see, for instance, Tersine (1994) for details.

all them need, so that they spend a instead of $|N|a$ ($a > 0$ being the fixed cost of an order) every time an order is placed. We denote by m_i ($m_i > 0$) the optimal number of orders per time unit for agent $i \in N$ if ordering alone. In Meca et al. (2003) it is proved that the triplet (N, a, m), with $m = (m_1, \ldots, m_n)$, characterizes such a centralized multi-agent inventory cost situation in the sense that, for every coalition $S \subset N$, $m_S = \sqrt{\sum_{i \in S} m_i^2}$ is the *optimal number of orders per time unit* for the agents in S and $c(S) = 2am_S$ is the *optimal average inventory cost per time unit* if they place their orders jointly; (N, c) is the inventory game associated with (N, a, m). Note that, according to this expression of the costs, the agents of N make savings if ordering together.

We denote by I^N the class of inventory games with player set N, by I the class of all inventory games, and by G the class of all cost TU games. Clearly,

$$I = \left\{(N, c) \in G \mid c(S) > 0 \text{ for all non-empty } S \subset N \text{ and } c(S)^2 = \sum_{i \in S} c(i)^2\right\}.$$

From now on, we sometimes identify a cost TU game (N, c) with its characteristic function c.

An important issue for these inventory games is how to allocate the total costs when the agents in N cooperate. An allocation rule is a map ϕ which assigns for every inventory game $(N, c) \in I$ an allocation of the total cost, i.e. a vector $\phi(c) = (\phi_i(c))_{i \in N} \in [0, +\infty)^N$ such that $\sum_{i \in N} \phi_i(c) = c(N)$. Meca et al. (2003) propose the SOC-rule σ which is defined, for every $(N, c) \in I$, and every $i \in N$, in the following way:

$$\sigma_i(c) := \frac{c(i)^2}{c(N)}.$$

Note that

$$\frac{c(i)^2}{c(N)} = \frac{c(i)^2}{\sum_{j \in N} c(j)^2} c(N),$$

so the SOC-rule is a proportional allocation rule. In Meca et al. (2003) some good properties of this rule are proved: it provides core allocations (i.e. $\sigma(c)$ belongs to the core of c for every $c \in I$), it can be characterized using convenient sets of properties (for instance, using efficiency, the null player property and a monotonicity property), it can be easily implemented in practice (according to the SOC-rule each agent pays his holding costs and the fixed order costs are payed proportionally to the squared m_i parameters of the agents). In this paper we demonstrate that it also behaves in an excellent way from the point of view of immunity to coalitional manipulation.

Immunity to coalitional manipulation in this context means the following. We want to find allocation rules which are immune to manipulations whereby a group of agents artificially merges to represent a single agent, or a single agent artificially splits to represent several agents. Immunity to these manipulations is relevant in practice because in many inventory situations it is feasible for the agents to merge, simply forming an a priori centralized inventory unit, or to split, by presenting the different sections of a unique firm which can only manage its inventory in a centralized way as if they were different inventory units. Let us formally introduce this property.

Definition 1 *Take $c \in I^N$, $d \in I^M$ and a non-empty $S \subset N$. We say that d is the S-manipulation of c if:*

- $M = (N \setminus S) \cup \{i_S\}$,
- $d(T) = c(T)$ *for all* $T \subset M$ *with* $i_S \notin T$,
- $d(T) = c((T \setminus \{i_S\}) \cup S)$ *for all* $T \subset M$ *with* $i_S \in T$.

Definition 2 *An allocation rule ϕ is said to be* **immune to coalitional manipulation** *if, for every $c, d \in I$ such that d is the S-manipulation of c for a certain S, it holds that*

$$\phi_{i_S}(d) = \sum_{i \in S} \phi_i(c). \tag{1}$$

Let us make some comments in relation with this property. Note first that it is the aggregation of a property of *no advantageous splitting* (corresponding to the "less than or equal to" in (1)) and a property of *no advantageous merging* (corresponding to the "greater than or equal to" in (1)).

Observe that the effect of merging or splitting is well modeled in Definition 1. Let us discuss this a bit. When the group of agents S merge in a centralized multi-agent inventory cost situation (N, a, m) with corresponding inventory game c, they have a new optimal number of orders per time unit. We have already mentioned that Meca et al. (2003) prove that this number is $m_S = \sqrt{\sum_{i \in S} m_i^2}$. Hence, it is easy to check that the inventory game associated with the resulting centralized multi-agent inventory cost situation is precisely d as defined in Definition 1.

So, immunity to coalitional manipulation seems to be an interesting property in this context. The main result of this note is that the unique rule which satisfies this property for the class of inventory games is the SOC-rule. The next section includes a proof of this result. We finish this section with an example which illustrates that the Shapley value is not immune to coalitional manipulation.

Example 1 *Consider the inventory game c, associated with the multi-agent inventory cost situation (N, a, m) given by $N = \{1, 2, 3\}$, $a = 2$, $m = (1, 2, 2)$. So, $c(1) = 4$, $c(2) = c(3) = 8$, $c(12) = c(13) = 8.94$, $c(23) = 11.31$, $c(N) = 12$. If $S = \{2, 3\}$, then the S-manipulation of c is the inventory game d given by $d(1) = 4$, $d(i_S) = 11.31$, $d(M) = 12$. The Shapley value of c and d is, respectively, $\Phi(c) = (1.88, 5.06, 5.06)$ and $\Phi(d) = (2.34, 9.66)$. Notice that $\Phi_{i_S}(d) = 9.66 \neq \Phi_2(c) + \Phi_3(c) = 10.12$, so Φ is not immune to coalitional manipulation.*

3 The Main Result

Theorem 1 *The unique allocation rule for inventory games which satisfies immunity to coalitional manipulation is the SOC-rule.*

Proof. Let us see first that the SOC-rule σ satisfies immunity to coalitional manipulation. Take c, d and S as in Definition 1. Then,

$$\sigma_{i_S}(d) = \frac{d(i_S)^2}{d(M)} = \frac{c(S)^2}{c(N)} = \sum_{i\in S} \frac{c(i)^2}{c(N)} = \sum_{i\in S} \sigma_i(c).$$

Take now ϕ an allocation rule satisfying immunity to coalitional manipulation. Let us check that, for any $c \in I^N$ and any $j \in N$, $\phi_j(c)$ only depends on $c(N)^2$ and on $c(j)^2$. This is obviously true if $|N| \leq 2$. In any other case take $\bar{d}$, the $N \setminus \{j\}$-manipulation of c. Since $\bar{d}$ is a two-player game, $\phi_j(\bar{d})$ only depends on $\bar{d}(M)^2$ and $\bar{d}(j)^2$. Note that $\bar{d}(M) = c(N)$, $\bar{d}(j) = c(j)$ and, since ϕ satisfies immunity to coalitional manipulation,

$$\phi_j(c) = c(N) - \sum_{k\in N\setminus\{j\}} \phi_k(c) = \bar{d}(M) - \phi_{i_{N\setminus\{j\}}}(\bar{d}) = \phi_j(\bar{d}).$$

Hence, $\phi_j(c)$ only depends on $c(N)^2$ and on $c(j)^2$, which means that $\phi_j(c) = f(c(N)^2, c(j)^2)$ for all $c \in I^N$ and all $j \in N$. Assume now that f is linear in the second component (we demonstrate that it is true at the end of this proof). Then, $\phi_j(c) = g(c(N)^2)c(j)^2$ for all $c \in I^N$ and all $j \in N$. Thus,

$$c(N) = \sum_{k\in N} \phi_k(c) = g(c(N)^2) \sum_{k\in N} c(k)^2 = g(c(N)^2)c(N)^2,$$

and so

$$g(c(N)^2) = \frac{1}{c(N)}$$

which means that

$$\phi_j(c) = \frac{c(j)^2}{c(N)} = \sigma_j(c).$$

So, to finish the proof we only need to check that f is linear in the second component. To prove it, take into account that we have a collection of functions

$$\{f(\alpha, \cdot) \mid \alpha \in (0, +\infty)\}$$

satisfying that $f(\alpha, \cdot) : (0, \alpha] \rightarrow [0, \alpha]$, for all $\alpha \in (0, +\infty)$. Take now $\alpha, x, y \in (0, +\infty)$ with $x + y \leq \alpha$. Then, there exists $(N, \hat{c}) \in I$ such that $\alpha = \hat{c}(N)^2$, $x = \hat{c}(1)^2$, $y = \hat{c}(2)^2$. Define $S = \{1, 2\}$ and take $\hat{d}$, the S-manipulation of $\hat{c}$. Then,

$$\begin{aligned} f(\alpha, x + y) &= f(\hat{c}(N)^2, \sum_{k \in S} \hat{c}(k)^2) = f(\hat{c}(N)^2, \hat{c}(S)^2) = f(\hat{d}(M)^2, \hat{d}(i_S)^2) \\ &= \phi_{i_S}(\hat{d}) = \sum_{k \in S} \phi_k(\hat{c}) = \sum_{k \in S} f(\hat{c}(N)^2, \hat{c}(k)^2) \\ &= f(\alpha, x) + f(\alpha, y). \end{aligned}$$

So, for every $\alpha \in (0, +\infty)$, $f(\alpha, \cdot)$ is additive. Then, since $f(\alpha, \cdot)$ is also non-negative, it is clear that it is moreover increasing[2]. It is an easy exercise to prove that every increasing additive function $h : (0, \alpha] \rightarrow [0, \alpha]$ is also linear. This completes the proof. □

We finish this note with a comment. The class of p-additive cost games A^p (for every non-zero real number p) can be defined in the following way:

$$A^p = \left\{(N, c) \in G \mid c(S) > 0 \text{ for all non-empty } S \subset N \text{ and } c(S)^p = \sum_{i \in S} c(i)^p\right\}.$$

Notice that $I = A^2$. If we define allocation rule and immunity to coalitional manipulation for A^p in an analogous way as we did for I, Theorem 1 can be immediately extended to A^p. So, the unique allocation rule for p-additive games which satisfies immunity to coalitional manipulation is the modified SOC-rule σ^p, which is defined, for every $(N, c) \in A^p$ and every $i \in N$, by:

$$\sigma_i^p(c) = \frac{c(i)^p}{\sum_{j \in N} c(j)^p} c(N) = \frac{c(i)^p}{c(N)^{p-1}}.$$

[2]We do not mean strictly increasing, but just increasing, i.e. $x \leq y \Rightarrow f(\alpha, x) \leq f(\alpha, y)$ for each $x, y \in (0, \alpha]$.

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