Characterizing cautious choice ⋆

Mosquera, M.A. [a,∗] Borm, P. [b] Fiestras-Janeiro, M.G. [c]

García-Jurado, I. [d] Voorneveld, M. [b,e]

[a] *Department of Statistics and Operations Research, Faculty of Business Administration and Tourism, Universidade de Vigo, 32004 Ourense, Spain.*

[b] *CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.*

[c] *Department of Statistics and Operations Research, Faculty of Economics, Universidade de Vigo, 36310 Vigo, Spain.*

[d] *Department of Statistics and Operations Research, Faculty of Mathematics, Santiago de Compostela University, 15782 Santiago de Compostela, Spain.*

[e] *Department of Economics, Stockholm School of Economics, Box 6501, 113 83 Stockholm, Sweden.*

Abstract

The set of maximin actions in general decision problems is characterized.

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1 Introduction

Choosing between alternatives according to the maximin criterion essentially involves associating with each alternative the worst possible consequence and then choosing the alternative(s) for which this worst-case scenario offers the best possible result. Different ways of modeling these actions, consequences (or states), and preferences/utilities over them yield an abundance of applications of this decision principle and its sibling, minimax behavior, in the social sciences:

- Game theory: The minimax theorem of von Neumann (cf. von Neumann, 1928) is one of the cornerstones of game theory. It establishes maximin behavior as an equilibrating device that assigns to every mixed extension of a finite two person zero-sum (or purely antagonistic) game a well-defined value.
- Experimental economics: Sarin and Vahid (1999, 2001) show that maximin behavior is the outcome of a natural and simple dynamic process of strategy adjustment and provides a good prediction of human behavior in several experimental settings.
- Statistical decision theory: Next to the Bayesian paradigm, the maximin approach is standard in statistical decision theory (cf. Blackwell and

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* Corresponding author. Tel.: +34-988-368765; fax.: +34-988-368923.
Email address: mamrguez@uvigo.es (Mosquera, M.A.).

Girshick, 1954; Ferguson, 1967).

- Social choice and welfare: Rawlsian welfare aims for the maximization of the utility of the least "happy" member of a society; see Moulin (1988) for a textbook treatment.
- Operations research: Problems like the optimal location of warehouses often involve the minimization of suitable distance functions. Among these distance functions, the Chebychev/supremum norm is a common one, transforming the problem in one of the minimax type (cf. Love et al., 1988).
- Constrained optimization: The Lagrangean dual of a constrained minimization problem is of the maximin type (cf. Bazaraa et al., 1993, Ch. 6).

Given the ubiquity of the maximin principle, it is hardly surprising that also its foundations have been the subject of study. These studies tend to focus on one of two aspects: *(a)* characterizing the *order* induced by the maximin criterion, like in the classical study Milnor (1954) and in Barberà and Jackson (1988), or *(b)* characterizing the solution that assigns to each decision problem its *set* of maximin actions, like in Maskin (1979). [1]

Both Milnor (1954) and Maskin (1979) deal with decision problems in which the set of actions and the set of states are finite; moreover, both authors remark that their results can be extended to an infinite setting. Indeed, Milnor's results

[1] The paper by Arrow and Hurwicz (1972) is a hybrid between the two different approaches: it shows that a *set* of solutions has certain properties if and only if there is an *order* with certain properties on the set of vectors

$$\left\{ \left(\min_{\omega \in \Omega} u(a,\omega), \max_{\omega \in \Omega} u(a,\omega) \right) \in \mathbb{R}^2 : a \in A \right\}$$

of minimal and maximal payoffs associated with each action.

can be extended in a more or less direct way. This, however, is not the case for Maskin's characterizations. In this paper, we clarify the problems underlying an extension of Maskin's results. See, in particular, Remarks 4 and 5.

Moreover, in this paper we provide an axiomatic characterization of the solution that assigns to each decision problem, *with arbitrary sets of actions and states*, its (possibly empty) set of maximin actions. An explicit comparison between the properties used in our characterization and those used by Milnor and Maskin is provided in Remark 7.

This general setting is required in a number of the applications mentioned above, where the sets of actions or states may be infinite. Think, for instance, of estimating an unknown probability in the area of statistical decision theory. The main challenge posed by such a general setting — apart from the possible emptiness of the set of maximin actions — is that (all) maximin actions can be strictly dominated. See Example 2. Consequently, the classical domination axioms upon which many characterizations rely no longer hold: they exclude dominated actions from the solution of the problem. We introduce a new axiom, Inclusion of Weak Dominators, that changes the negative focus of the classical axioms (excluding 'bad' actions) to a positive one (including 'good' ones): it requires that any action weakly dominating a selected action should also be selected.

That maximin actions may be strictly dominated should not be seen as something negative. It simply stresses the fundamental difference in choice theory between rational choice on the one hand and our setting of cautious choice on the other. Rational choice involves searching for a best action, cautious choice involves trying to avoid disastrous outcomes, and even if one action strictly dominates another, their worst-case scenarios may well be the same. This is elaborated upon in Remarks 4 and 7.

In the next section, we formally define the class of decision problems, list the properties used in our characterization, state the characterization theorem and make some comments on the relationship between our results and others already existing in the literature. The proof of our characterization is contained in Section 3.

2 A characterization of the set of maximin actions

A *decision problem* is a tuple (A, Ω, u), where A is a nonempty set of actions, Ω is a nonempty set of states, and $u : A \times \Omega \to \mathbb{R}$ is a bounded function which represents the decision-maker's payoff/utility function. The set of all decision problems is denoted by $\mathcal{D}$.

A *solution* on $\mathcal{D}$ is a correspondence φ that assigns to every $(A, \Omega, u) \in \mathcal{D}$ a set $\varphi(A, \Omega, u) \subset A$ of actions. Our aim is to characterize the solution M that assigns to every decision problem $(A, \Omega, u) \in \mathcal{D}$ its set of *maximin actions*

$$M(A, \Omega, u) := \left\{ a \in A : \inf_{\omega \in \Omega} u(a, \omega) = \sup_{a' \in A} \inf_{\omega \in \Omega} u(a', \omega) \right\}.$$

Let us make a few remarks on our domain $\mathcal{D}$ of decision problems.

Remark 1 *Since only the order of the payoffs matters, order-preserving transformations do not affect the solution and the assumption that our payoffs are bounded entails no loss of generality.*[2]

[2] Simply apply an order-preserving transformation making the decision-maker's

The next two examples illustrate the difficulties one faces when passing from finite decision problems to general decision problems. Recall that an action $a \in A$ in a decision problem $D = (A, \Omega, u) \in \mathcal{D}$ is *strictly dominated* if there is an action $a' \in A$ with $u(a', \omega) > u(a, \omega)$ for all $\omega \in \Omega$.

Example 2 *Consider a decision problem* (A, Ω, u) *with* $A = \Omega = \mathbb{Z}$ *and* $u(a, \omega) = \arctan(a - \omega)$ *for all* $(a, \omega) \in \mathbb{Z} \times \mathbb{Z}$*. Then* $\inf_{\omega \in \mathbb{Z}} u(a, \omega) = -\pi/2$ *for all* $a \in \mathbb{Z}$*. Hence, every* $a \in \mathbb{Z}$ *is a maximin action, but also strictly dominated, for instance by* $a + 1$*. Moreover, if one were to restrict the action space to only two elements, say* $\bar{A} = \{1, 2\}$*, then* $M(\bar{A}, \Omega, u) = \bar{A}$*, and the maximin action* 1 *is strictly dominated.*

Example 3 *Consider a decision problem* (A, Ω, u) *with* $A = \mathbb{N}$*,* Ω *is any non-empty set, and* $u(a, \omega) = a/(a+1)$ *for all* $(a, \omega) \in A \times \Omega$*. Then* $\inf_{\omega \in \Omega} u(a, \omega) = a/(a + 1)$*, a function which does not achieve a maximum:* $M(A, \Omega, u) = \emptyset$*.*

In our general setting, some properties of simpler, finite problems no longer hold: (all) maximin actions can be strictly dominated (Example 2) and the set of maximin actions may be empty (Example 3). The following table displays the difficulties one may find in infinite cases.

payoff function bounded, like the arc-tangent transformation: $M(A, \Omega, u) = M(A, \Omega, \arctan u)$.

	A finite *Ω infinite*	*A infinite* *Ω finite*	*A infinite* *Ω infinite*
A maximin action can be strictly dominated	Possible (Example 2)	Impossible	Possible (Example 2)
The set of maximin actions can be empty	Impossible	Possible (Example 3)	Possible (Example 3)

Remark 4 *Regarding Example 2, a referee observed that "...the maximin choice rule is not so appealing in this context...In particular, it violates the basic requirement that a strictly dominated action should not be chosen." It is a requirement in the standard 'rational' (payoff maximizing) approach to choice theory, but definitely not in behavioral models like the satisficing approach of Simon (1956), nor in our setting of cautious choice, where the idea is to avoid 'disastrous' outcomes, rather than achieving a best one. It will be obvious from Example 2 that in an infinite setting the property (5) of strong domination that Maskin (1979) uses in the characterization of the maximin solution is not applicable.*

Remark 5 *In the choice of our domain $\mathcal{D}$, our aim was to be as nonrestrictive as possible: the sets of actions A and states Ω are only required to be nonempty to avoid trivialities. The sets may be finite, infinite, and in the latter case countably or uncountably infinite. The payoff function u is assumed to be bounded, which by Remark 1 entails no loss of generality. We believe that this generality is more natural than explicitly restricting oneself to a domain where the set of maximin actions is nonempty: it leaves no ghosts in* *the closet. Consequently, a characterization of the set of maximin actions in an infinite setting calls for some additional properties related to the possible non-existence. This is another reason why the results of Maskin (1979) do not extend to the infinite setting: he explicitly assumes that solutions assign to each problem a nonempty set of actions.*

We introduce some properties for a solution φ on $\mathcal{D}$. They are relatively standard and adapted from properties in Milnor (1954), Barberà and Jackson (1988), and Maskin (1979). An explicit discussion of the relations and differences is provided in Remark 7 after all properties have been introduced. Anonymity requires that the solution does not depend on the way actions and states are labeled.

Anonymity (ANO). Let $(A, \Omega, u), (A', \Omega', u') \in \mathcal{D}$. If there are bijections $f : A \to A'$ and $g : \Omega \to \Omega'$ such that $u(a, \omega) = u'(f(a), g(\omega))$ for all $(a, \omega) \in A \times \Omega$, then $\varphi(A', \Omega', u') = f(\varphi(A, \Omega, u))$.

Independence of irrelevant actions states that if the action set of a decision problem is reduced, but some elements in the solution set of the large problem remain feasible, then the solution set of the small problem consists of the feasible elements in the solution set of the original problem.

Independence of irrelevant actions (IIA). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subsetneq A'$ and $u'_{|A\times\Omega} = u$. If $\varphi(A', \Omega, u') \cap A \neq \emptyset$, then $\varphi(A', \Omega, u') \cap A = \varphi(A, \Omega, u)$.

Inheritance of nonemptiness states that adding finitely many actions to a decision problem with a nonempty solution set yields a new decision problem whose solution set is also nonempty.

Inheritance of nonemptiness (INH-NEM). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subsetneq A'$ and $u'_{|A\times\Omega} = u$. If $\varphi(A, \Omega, u) \neq \emptyset$ and $A' \setminus A$ is a finite set, then $\varphi(A', \Omega, u') \neq \emptyset$.

In a decision problem $(A, \Omega, u) \in \mathcal{D}$, action $a' \in A$ *weakly dominates* action $a \in A$ if $u(a', \omega) \geq u(a, \omega)$ for all $\omega \in \Omega$, with a strict inequality for some $\omega \in \Omega$. The inclusion of weak dominators property states that if an action weakly dominates an action in the solution set of the problem, then also the weakly dominating action belongs to the solution set.

Inclusion of weak dominators (IWD). Let $(A, \Omega, u) \in \mathcal{D}$ and $a^*, a' \in A$. If $a^* \in \varphi(A, \Omega, u)$ and a' weakly dominates a^*, then $a' \in \varphi(A, \Omega, u)$.

The next property requires that duplicating states does not affect the solution set.

Duplication of states (DOS). Let $(A, \Omega, u), (A, \Omega', u') \in \mathcal{D}$ with $\Omega \subsetneq \Omega'$. If there is a surjection $g : \Omega' \to \Omega$ such that $u'(a, \omega') = u(a, g(\omega'))$ for all $(a, \omega') \in A \times \Omega'$, then $\varphi(A, \Omega', u') = \varphi(A, \Omega, u)$.

Continuity states that if an action is always contained in the solution set of a sequence of decision problems in $\mathcal{D}$ with fixed action and state spaces and pointwise convergent utility functions, then this action is also contained in the solution set of the limiting problem.

Continuity (CONT). Let $(A, \Omega, u) \in \mathcal{D}$ and let $\{(A, \Omega, u_k)\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $\lim_{k\to\infty} u_k(a, \omega) = u(a, \omega)$ for all $(a, \omega) \in A \times \Omega$. If there is an $a^* \in A$ with $a^* \in \varphi(A, \Omega, u_k)$ for all $k \in \mathbb{N}$, then $a^* \in \varphi(A, \Omega, u)$.

Restricted nonemptiness states that, for a given decision problem, if there exists some maximin action, then there also exists some element of the solution set. In the literature, this type of property is used in both cooperative games (cf. Voorneveld and van den Nouweland, 1998) and noncooperative games (cf. Dufwenberg et al., 2001; Norde et al., 1996; Voorneveld et al., 1999). In our context, it is related with the possible emptiness of the set of maximin actions.

Restricted Nonemptiness (r-NEM). Let $(A, \Omega, u) \in \mathcal{D}$. If $M(A, \Omega, u)$ is nonempty, then $\varphi(A, \Omega, u)$ is also nonempty.

Convexity states that if two actions belong to the solution set of a decision problem and an action is added whose payoff is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the above actions' payoffs, then the new action belongs to the solution set of the new problem. This is a standard risk neutrality property already present in Milnor (1954): if two actions belong to the problem's solution set, the decision-maker does not mind tossing a coin to decide between them.

Convexity (CONV). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A' = A \cup \{a'\}$ for some $a' \notin A$ and $u'_{|A \times \Omega} = u$. If there are $a^*, \tilde{a} \in \varphi(A, \Omega, u)$ such that

$$u'(a', \omega) = \frac{1}{2}u(a^*, \omega) + \frac{1}{2}u(\tilde{a}, \omega)$$

for all $\omega \in \Omega$, then $a' \in \varphi(A', \Omega, u')$.

Finally, if there is only one state, then the solution chooses the actions that maximize the payoff.

One state rationality (OSR). Take $(A, \Omega, u) \in \mathcal{D}$ with $|\Omega| = 1$. Writing $\Omega = \{\omega\}$, $\varphi(A, \Omega, u) = \arg\max_{a \in A} u(a, \omega)$.

The former properties characterize the solution M on $\mathcal{D}$ which assigns to each decision problem its set of maximin actions:

Theorem 6 *The maximin solution M is the unique solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, IWD, DOS, CONT, r-NEM, CONV, and OSR.*

The proof is given in the next section. First, in Remark 7, we relate the current properties to existing properties in the literature, in particular Milnor (1954) and Maskin (1979).

Remark 7 *ANO corresponds to Milnor's Symmetry axiom and Maskin's Property (11). IIA implies Milnor's Row Adjunction axiom and it combines Maskin's properties (1) and (2). DOS corresponds to Milnor's Column Duplication axiom and Maskin's Property (12). CONT is analogous to Milnor's Continuity axiom and Maskin's Property (10). CONV corresponds to Milnor's Convexity axiom and Maskin's Property (9). IWD is a bit stronger than Maskin's Property (4) and it takes the role of Milnor's Strong Domination axiom and Maskin's Property (5), although IWD is essentially different from these two properties. Finally, OSR, INH-NEM and r-NEM are properties related with the emptiness issue that arises in our general setting; they are not connected to any of the properties in Milnor (1954) or in Maskin (1979). OSR is a rather natural property, while INH-NEM and r-NEM are of a more technical nature. Nevertheless, the following example shows that these properties cannot be dispensed with in our characterization of the maximin solution.*

Example 8 *Define the solution φ on $\mathcal{D}$ as follows:*

$$\varphi(A,\Omega,u) = \begin{cases} M(A,\Omega,u) & \text{if for each } a \in A,\ u(a,\cdot) \text{ is a constant function,} \\ \emptyset & \text{otherwise.} \end{cases}$$

One readily verifies that φ satisfies all properties in Theorem 6, except for INH-NEM and r-NEM.

Remark 9 indicates which properties are violated by other well-known solutions; the reader is referred to the papers cited below for their definitions.

Remark 9 *The minimax solution satisfies all properties in Theorem 6 except IWD and r-NEM. The maximax solution satisfies all except CONV and r-NEM. The minimin solution satisfies all with the exception of IWD, r-NEM, and CONV. In our general setting, technical difficulties arise in defining the leximin, protective (cf. Barberà and Jackson, 1988), and leximax solutions (cf. Naeve, 2000), as well as the Laplacian criterion of insufficient reason (cf. Milnor, 1954). If one were to restrict attention to decision problems with a finite set of states, the leximin solution does not satisfy DOS, CONT, and r-NEM, the protective solution does not satisfy CONT and r-NEM; the leximax solution does not satisfy DOS, CONT, r-NEM, and CONV, and, finally, the Laplacian principle of insufficient reason does not satisfy DOS and r-NEM.*

Remark 10 *Theorem 6 remains valid on other domains of decision problems, as long as these domains are closed under certain transformations mentioned in our characterizing properties (like the duplication of states). For instance, Theorem 6 is still true if we deal with the subsets of $\mathcal{D}$ where A is finite, where Ω is finite, or where both A and Ω are finite. In fact, some of the properties (like r-NEM in case of finite action sets) may be dispensed with.*

3 Proof of the characterization theorem

The purpose of this section is to prove our characterization theorem. The proof is based on a series of lemmas.

The properties ANO and IIA of a solution guarantee that if an action has the same payoff function as an element of the solution set of the problem — up to relabeling of the states — then also the former action belongs to the solution set. We only use a simple version:

Lemma 11 *Let φ be a solution on $\mathcal{D}$ satisfying ANO and IIA, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that, for some $\omega_1, \omega_2 \in \Omega$,*

(i) $u(a', \omega_1) = u(a^*, \omega_2)$ *and* $u(a', \omega_2) = u(a^*, \omega_1)$,

(ii) $u(a', \omega) = u(a^*, \omega)$ *for all* $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$,

then $a' \in \varphi(D)$.

PROOF. Assume that $u(a^*, \omega_1) \neq u(a^*, \omega_2)$ (otherwise ANO concludes the result). The utility functions for actions a^* and a' are represented in the table below, where $\square$ and $\times$ represent two different values:

Actions \ States	$\cdots$	ω_1	$\cdots$	ω_2	$\cdots$
a^*	$\cdots$	$\square$	$\cdots$	$\times$	$\cdots$
	$\Vert$		$\Vert$		$\Vert$
a'	$\cdots$	$\times$	$\cdots$	$\square$	$\cdots$

Consider decision problems

$$D_1 = (\{a^*, a'\}, \Omega, u_{|\{a^*, a'\} \times \Omega}), \ D_2 = (\{a^*, a'\}, \Omega, v),$$

where the utility for a^* and a' is interchanged, i.e.

$$v(a^*, \omega_1) = v(a', \omega_2) := u(a^*, \omega_2),$$

$$v(a^*, \omega_2) = v(a', \omega_1) := u(a^*, \omega_1),$$

and $v(b, \omega) := u(b, \omega)$ for all other $(b, \omega) \in \{a^*, a'\} \times (\Omega \setminus \{\omega_1, \omega_2\})$. By *(i)* and *(ii)*, D_2 is isomorphic to D_1, either via switching the labels of a^* and a' or via switching the labels of ω_1 and ω_2.

Note that D can be obtained from D_1 by adding actions and, moreover, $a^* \in \varphi(D) \cap \{a^*, a'\}$. Therefore, by IIA:

$$\varphi(D_1) = \varphi(D) \cap \{a^*, a'\}, \tag{1}$$

so that $a^* \in \varphi(D_1)$. It is shown that also $a' \in \varphi(D_1)$. Consider the bijection $f : \{a^*, a'\} \to \{a^*, a'\}$ with $f(a^*) = a', f(a') = a^*$ and let $g : \Omega \to \Omega$ be the identity function. Since $u(a, \omega) = v(f(a), g(\omega))$ for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, ANO implies that $\varphi(D_2) = f(\varphi(D_1))$, so $a' = f(a^*) \in \varphi(D_2)$. Next, consider the bijection $\bar{g} : \Omega \to \Omega$ with $\bar{g}(\omega_1) = \omega_2$, $\bar{g}(\omega_2) = \omega_1$, keeping other states unchanged, and let $\bar{f} : \{a^*, a'\} \to \{a^*, a'\}$ be the identity function. Since $v(a, \omega) = u(\bar{f}(a), \bar{g}(\omega))$ for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, ANO implies that $\varphi(D_1) = \bar{f}(\varphi(D_2)) = \varphi(D_2)$. Remember that $a' \in \varphi(D_2)$, so $a' \in \varphi(D_1)$. This shows that $\{a^*, a'\} = \varphi(D_1)$.

Finally, by (1), $a' \in \varphi(D)$. $\square$

With the INH-NEM property and Lemma 11 one can establish the following consequence. If we add an action to a decision problem with the same utility as an action in the solution set of the original problem, except in two states where the utilities are interchanged, then both actions belong to the solution set of the new problem:

Lemma 12 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, and INH-NEM, and let $D = (A, \Omega, u) \in \mathcal{D}$. Take $D' = (A', \Omega, u') \in \mathcal{D}$ satisfying that $A' = A \cup \{a'\}$ for some $a' \notin A$ and $u'_{|A\times\Omega} = u$. Suppose that there exist $a^* \in \varphi(A, \Omega, u)$ and $\omega_1, \omega_2 \in \Omega$ such that*

(i) $u'(a', \omega_1) = u'(a^*, \omega_2)$ *and* $u'(a', \omega_2) = u'(a^*, \omega_1)$,

(ii) $u'(a', \omega) = u'(a^*, \omega)$ *for all* $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$.

Then $\{a^*, a'\} \subseteq \varphi(D')$.

PROOF. Note that D' is well-defined. Suppose that $a' \notin \varphi(D')$. Since φ satisfies INH-NEM, $A' \setminus A = \{a'\}$ is a finite set, and $\varphi(D) \neq \emptyset$: $\varphi(D') \neq \emptyset$. So $\varphi(D') \cap A \neq \emptyset$ and IIA implies that $\varphi(D') \cap A = \varphi(D)$. Therefore $a^* \in \varphi(D')$. By Lemma 11, also $a' \in \varphi(D')$, a contradiction. Hence, $a' \in \varphi(D')$ and using Lemma 11 again it follows that $a^* \in \varphi(D')$. So $\{a^*, a'\} \subset \varphi(D')$. □

Consider the following modification of weak dominance. In a decision problem $(A, \Omega, u) \in \mathcal{D}$, action $a' \in A$ *quasi-dominates* action $a \in A$ if there exist $\omega_1, \omega_2 \in \Omega$ such that:

(i) $u(a', \omega) \geq u(a, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$, and

(ii) $u(a', \omega_2) \geq u(a, \omega_1) > u(a', \omega_1) \geq u(a, \omega_2)$.

Intuitively, a' quasi-dominates a if it is at least as good as a in all states except some ω_1, and the loss from choosing a' in state ω_1 is compensated for by a utility gain in another state ω_2.

The next Lemma shows that a solution satisfying ANO, IIA, INH-NEM, and IWD, satisfies the following property: if an action quasi-dominates an action in the solution set, then the former action also belongs to the solution set.

Lemma 13 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, and IWD, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ quasi-dominates a^*, then $a' \in \varphi(D)$.*

PROOF. Let $\omega_1, \omega_2 \in \Omega$ be as in the definition of quasi-dominance. Define the decision problem $D' = (A \cup \{\alpha\}, \Omega, u') \in \mathcal{D}$ with $\alpha \notin A$, $u'_{|A\times\Omega} = u$, $u'(\alpha, \omega) = u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$, $u'(\alpha, \omega_1) = u(a^*, \omega_2)$, and $u'(\alpha, \omega_2) = u(a^*, \omega_1)$. By Lemma 12: $\{a^*, \alpha\} \subset \varphi(D')$. Now a' weakly dominates α unless $u'(a', \omega) = u'(\alpha, \omega)$ for all $\omega \in \Omega$ (in which case $a' \in \varphi(D')$ by ANO). So, by IWD, $a' \in \varphi(D')$.

Hence, $\{a^*, \alpha, a'\} \subset \varphi(D')$. Now $\varphi(D) = \varphi(D') \cap A$ by IIA, so $a' \in \varphi(D)$. □

If a solution satisfies ANO, IIA, INH-NEM, IWD, DOS, and CONT, then whether or not an action belongs to the solution set of a decision problem depends exclusively on the infimum and supremum of its payoffs.

Lemma 14 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, IWD, DOS, and CONT, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that*

$$\inf_{\omega\in\Omega} u(a', \omega) = \inf_{\omega\in\Omega} u(a^*, \omega) = m \quad \text{and} \quad \sup_{\omega\in\Omega} u(a', \omega) = \sup_{\omega\in\Omega} u(a^*, \omega) = M,$$

then $a' \in \varphi(D)$.

PROOF. If $m = M$, then a^* and a' yield the same, constant payoff, regardless of ω, so ANO and $a^* \in \varphi(D)$ imply that $a' \in \varphi(D)$. So henceforth assume that $m < M$. This means that Ω has at least two elements. Let $\omega_1 \in \Omega$. Define for each $(\varepsilon, \delta) \in \mathbb{R}^2_+$ the decision problem $D_{\varepsilon,\delta} = (A \cup \{\alpha, \beta\}, \Omega, u_{\varepsilon,\delta})$ with $\alpha, \beta \notin A$ as follows. For all $(\tilde{a}, \omega) \in (A \cup \{\alpha, \beta\}) \times \Omega$,

$$u_{\varepsilon,\delta}(\tilde{a}, \omega) = \begin{cases} u(a', \omega) + \delta & \text{if } \tilde{a} = a', \\ m + \varepsilon & \text{if } (\tilde{a}, \omega) = (\alpha, \omega_1), \\ m & \text{if } \tilde{a} = \beta \text{ and } \omega \neq \omega_1, \\ M & \text{if } (\tilde{a}, \omega) = (\beta, \omega_1) \text{ or } (\tilde{a} = \alpha \text{ and } \omega \neq \omega_1), \\ u(\tilde{a}, \omega) & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of $D_{\varepsilon,\delta}$.

Actions \ States	ω_1	$\omega \in \Omega \setminus \{\omega_1\}$
a^*	$u(a^*, \omega_1)$	$u(a^*, \omega)$
a'	$u(a', \omega_1) + \delta$	$u(a', \omega) + \delta$
α	$m + \varepsilon$	M
β	M	m
all other a	$u(a, \omega_1)$	$u(a, \omega)$

Let $D' = (A \setminus \{a'\}, \Omega, u_{|(A \setminus \{a'\}) \times \Omega}) \in \mathcal{D}$. Since $a^* \in \varphi(D) \cap (A \setminus \{a'\})$, IIA implies that $\varphi(D') = \varphi(D) \cap (A \setminus \{a'\}) \neq \emptyset$. For all $(\varepsilon, \delta) \in \mathbb{R}^2_+$, $D_{\varepsilon,\delta}$ is obtained from D' by adding finitely many actions, so INH-NEM implies that

$\varphi(D_{\varepsilon,\delta}) \neq \emptyset$.

Step 1: Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with $\lim_{k\to\infty} \varepsilon_k = 0$. We show that $\alpha \in \varphi(D_{\varepsilon_k,0})$ for all $k \in \mathbb{N}$. By CONT, we then have $\alpha \in \varphi(D_{0,0})$.

Let $k \in \mathbb{N}$ and suppose, to the contrary, that $\alpha \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$, we have two cases:

- $\beta \in \varphi(D_{\varepsilon_k,0})$. This is not possible, because α quasi-dominates β and applying Lemma 13 one obtains that $\alpha \in \varphi(D_{\varepsilon_k,0})$.
- $\beta \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$ and $\alpha, \beta \notin \varphi(D_{\varepsilon_k,0})$ there is an $a \in \varphi(D_{\varepsilon_k,0}) \cap A$. By IIA: $\varphi(D_{\varepsilon_k,0}) \cap A = \varphi(D)$, so $a^* \in \varphi(D_{\varepsilon_k,0})$.
  - If $u(a^*, \omega_1) \leq m + \varepsilon_k$, then α weakly dominates a^*: $u(a^*, \omega) \leq u(\alpha, \omega)$ for all $\omega \in \Omega$, and there is an $\omega_0 \in \Omega$ such that $u(a^*, \omega_0) < u(\alpha, \omega_0)$, because otherwise $u(a^*, \omega) = u(\alpha, \omega)$ for all $\omega \in \Omega$, so that $m = \inf_{\omega\in\Omega} u(a^*, \omega) = \inf_{\omega\in\Omega} u(\alpha, \omega) = \min\{m + \varepsilon_k, M\} > m$, a contradiction. Using IWD, it follows that $\alpha \in \varphi(D_{\varepsilon_k,0})$.
  - If $u(a^*, \omega_1) > m+\varepsilon_k$, then α quasi-dominates a^*: $u(\alpha, \omega) \geq u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$ and by definition of $m = \inf_{\omega\in\Omega} u(a^*, \omega)$, there is an $\omega_2 \in \Omega$, different from ω_1 (since $u(a^*, \omega_1) > m + \varepsilon_k$) with $u(a^*, \omega_2) \leq m + \varepsilon_k$. This implies that $M = u(\alpha, \omega_2) \geq u(a^*, \omega_1) > m + \varepsilon_k \geq u(a^*, \omega_2)$. By Lemma 13, $\alpha \in \varphi(D_{\varepsilon_k,0})$.

  In both subcases, we established that $\alpha \in \varphi(D_{\varepsilon_k,0})$, in contradiction with our assumption. Conclude that $\alpha \in \varphi(D_{\varepsilon_k,0})$.

Step 2: We show that $\beta \in \varphi(D_{0,0})$.

Let $\omega_2 \in \Omega$, $\omega_2 \neq \omega_1$, and consider the decision problems

$$D_1 = \left(\{\alpha, \beta\}, \{\omega_1, \omega_2\}, u_{0,0|\{\alpha,\beta\}\times\{\omega_1,\omega_2\}}\right) \text{ and } D_2 = \left(\{\alpha, \beta\}, \Omega, u_{0,0|\{\alpha,\beta\}\times\Omega}\right).$$

D_2 can be obtained from $D_{0,0}$ by deleting actions. By step 1, $\varphi(D_{0,0}) \cap \{\alpha, \beta\} \neq \emptyset$. So IIA implies that

$$\varphi(D_{0,0}) \cap \{\alpha, \beta\} = \varphi(D_2). \tag{2}$$

Therefore, $\alpha \in \varphi(D_2)$. By DOS, $\varphi(D_1) = \varphi(D_2)$, so $\alpha \in \varphi(D_1)$. Now Lemma 11 implies that $\beta \in \varphi(D_1)$. Since $\varphi(D_1) = \varphi(D_2)$, equation (2) gives that $\beta \in \varphi(D_{0,0})$.

Step 3: Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with $\lim_{k\to\infty} \delta_k = 0$. We show that $a' \in \varphi(D_{0,\delta_k})$ for all $k \in \mathbb{N}$. By CONT, we then have $a' \in \varphi(D_{0,0})$.

Consider the decision problem

$$D_3 = \left(A_3, \Omega, u_{0,0|A_3\times\Omega}\right)$$

where $A_3 = (A \cup \{\alpha, \beta\}) \setminus \{a'\}$ for some $\alpha, \beta \notin A$. By steps 1 and 2, $\varphi(D_{0,0}) \cap A_3 \neq \emptyset$, so IIA implies that $\varphi(D_{0,0}) \cap A_3 = \varphi(D_3)$. Hence, from step 2, $\beta \in \varphi(D_3)$.

Let $\delta_k > 0$ and suppose that $a' \notin \varphi(D_{0,\delta_k})$. Since $\varphi(D_{0,\delta_k}) \neq \emptyset$ one obtains that $\varphi(D_{0,\delta_k}) \cap A_3 \neq \emptyset$ and then IIA implies that $\beta \in \varphi(D_{0,\delta_k})$. So, reasoning as in step 1: if $u(a', \omega_1) + \delta_k \geq M$, then a' weakly dominates β and, by IWD, $a' \in \varphi(D_{\delta_k,0})$; otherwise, a' quasi-dominates β and by Lemma 13: $a' \in \varphi(D_{\delta_k,0})$. In both cases we reach a contradiction. Conclude that $a' \in \varphi(D_{\delta_k,0})$.

Step 4: Finally, we show that $a' \in \varphi(D)$.

By step 3 $a' \in \varphi(D_{0,0}) \cap A$. Hence, IIA implies $\varphi(D_{0,0}) \cap A = \varphi(D)$, and so $a' \in \varphi(D)$. $\square$

These results will help us prove Theorem 6:

Proof of Thm. 6 It is easy to verify that the solution M satisfies all the properties.

Let φ be a solution on $\mathcal{D}$ satisfying all the properties and let $D = (A, \Omega, u) \in \mathcal{D}$. If $\varphi(D) = \emptyset$, then by r-NEM: $M(D) = \emptyset$. So, assume that $\varphi(D) \neq \emptyset$.

Under the assumption that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs, it is true that $\varphi(D) = M(D)$. Namely, consider the decision problem $\widehat{D} = (A, \widehat{\Omega}, \widehat{u})$ where $|\widehat{\Omega}| = 1$ and $\widehat{u}(a, \widehat{\omega}) = \inf_{\omega \in \Omega} u(a, \omega)$ for all $(a, \widehat{\omega}) \in A \times \widehat{\Omega}$. We show that

$$\varphi(D) = \varphi(\widehat{D}) \tag{3}$$

Consider the decision problem $\widetilde{D} = (\widetilde{A}, \Omega, \widetilde{u}) \in \mathcal{D}$ obtained from D by adding to the action space a replica $r(a)$ of every action $a \in A$, i.e., $\widetilde{A} = \{a, r(a)\}_{a \in A}$ and with payoffs $\widetilde{u}_{|A \times \Omega} = u$ and $\widetilde{u}(r(a), \omega) = \inf_{\omega \in \Omega} u(a, \omega)$ for all $a \in A$ and $\omega \in \Omega$.

By the assumption: $a \in \varphi(D)$ if and only if $\{a, r(a)\} \subseteq \varphi(\widetilde{D})$. Since $\varphi(D) \neq \emptyset$, deletion of all non-replica actions and IIA imply that

$$a \in \varphi(D) \Leftrightarrow r(a) \in \varphi((\{r(a)\}_{a \in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a \in A} \times \Omega})). \tag{4}$$

ANO and DOS imply that

$$r(a) \in \varphi((\{r(a)\}_{a \in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a \in A} \times \Omega})) \Leftrightarrow a \in \varphi(\widehat{D}). \tag{5}$$

The equality (3) now follows from (4) and (5).

Write $\widehat{\Omega} = \{\widehat{\omega}\}$. By OSR we know that $\varphi(\widehat{D}) = M(\widehat{D}) = \arg\max_{a \in A} \widehat{u}(a, \widehat{\omega})$. Finally, since M satisfies all the properties we also have that $M(\widehat{D}) = M(D)$. Therefore $\varphi(D) = M(D)$.

Now it remains to prove that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

Let $a^* \in \varphi(D)$ and let $m = \inf_{\omega \in \Omega} u(a^*, \omega)$ and $M = \sup_{\omega \in \Omega} u(a^*, \omega)$. If $m = M$, then $u(a^*, \omega) = m$ for all $\omega \in \Omega$. Let $a \in A$ be such that $\inf_{\omega \in \Omega} u(a, \omega) = m$. If $\sup_{\omega \in \Omega} u(a, \omega) = m$, then $u(a, \omega) = u(a^*, \omega)$ for all $\omega \in \Omega$ and, by ANO, $a \in \varphi(D)$; otherwise, a weakly dominates a^*, so, by IWD, $a \in \varphi(D)$. Therefore, if $m = M$, then whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

So henceforth assume that $m < M$. This implies in particular that Ω contains at least two elements. Choose $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$.

Take $D' = (A, \Omega', u') \in \mathcal{D}$ where $\Omega' = \{\omega_1, \omega_2, \omega_3\}$ with $\omega_3 \notin \Omega$ and, for all $a \in A$:

$$u'(a, \omega') = \begin{cases} \sup\limits_{\omega \in \Omega} u(a, \omega) & \text{if } \omega' = \omega_1 \\ \\ \inf\limits_{\omega \in \Omega} u(a, \omega) & \text{otherwise} \end{cases}$$

The table below summarizes the definition of D'.

States / Actions	ω_1	ω_2	ω_3
$\vdots$		$\vdots$	
a^*	M	m	m
a	$\sup_{\omega\in\Omega} u(a,\omega)$	$\inf_{\omega\in\Omega} u(a,\omega)$	$\inf_{\omega\in\Omega} u(a,\omega)$
$\vdots$		$\vdots$	

Similar to the proof of (3), using Lemma 14 instead of the assumption, it follows that $\varphi(D) = \varphi(D')$.

Define the sequence of decision problems $\{D_k\}_{k\in\mathbb{N}} = \{(A \cup \{\alpha, \beta, \gamma\}, \Omega', u_k)\}_{k\in\mathbb{N}}$ where $\alpha, \beta, \gamma \notin A$, $u_{k|A\times\Omega'} = u'$ and, for all $(a,\omega) \in \{\alpha, \beta, \gamma\} \times \Omega'$,

$$u_k(a,\omega) = \begin{cases} m + \frac{1}{2^{k-1}}(M - m) & \text{if } (a,\omega) \in \{(\alpha,\omega_1), (\beta,\omega_2)\} \\ m + \frac{1}{2^k}(M - m) & \text{if } (a,\omega) \in \{(\gamma,\omega_1), (\gamma,\omega_2)\} \\ m & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of D_k.

States / Actions	ω_1	ω_2	ω_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^*	M	m	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$
α	$m + \frac{1}{2^{k-1}}(M-m)$	m	m
β	m	$m + \frac{1}{2^{k-1}}(M-m)$	m
γ	$m + \frac{1}{2^k}(M-m)$	$m + \frac{1}{2^k}(M-m)$	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For all $k \in \mathbb{N}$, D_k can be obtained from D' by adding three actions. So, $\varphi(D') \neq \emptyset$ and INH-NEM imply that $\varphi(D_k) \neq \emptyset$. We show by induction that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Step 1: $\gamma \in \varphi(D_1)$.

D_1 can be obtained from D' by adding actions α, β, and γ in two steps:

First, add α and β to obtain the decision problem $D_1' = (A \cup \{\alpha, \beta\}, \Omega', u_1')$ with $u_1' = u_{1|(A\cup\{\alpha,\beta\})\times\Omega'}$. Lemma 14 implies that $\alpha \in \varphi(D_1')$ if and only if $\beta \in \varphi(D_1')$. Suppose that $\alpha, \beta \notin \varphi(D_1')$. INH-NEM and $\varphi(D') \neq \emptyset$ imply that $\varphi(D_1') \neq \emptyset$, so there is an $a \in \varphi(D_1') \cap A$. Then, by IIA, $\varphi(D_1') \cap A = \varphi(D')$. Hence, $a^* \in \varphi(D_1')$. Lemma 14 then implies that $\alpha, \beta \in \varphi(D_1')$, which is a contradiction. Thus $\alpha, \beta \in \varphi(D_1')$.

Second, add action γ, whose utility is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the utility of the actions α and β, and by CONV: $\gamma \in \varphi(D_1)$.

Step 2: Let $k \in \mathbb{N}$ and assume that $\gamma \in \varphi(D_n)$ for all $n \in \mathbb{N}, n \leq k$. We show that $\gamma \in \varphi(D_{k+1})$.

The decision problem D_{k+1} can be obtained from D_k in two steps:

First, delete actions α and β from D_k to obtain a new decision problem. By IIA and the assumption that $\gamma \in \varphi(D_k)$, its solution set contains γ. Next, introduce actions α and β again, but now with their utility functions equal to those in the problem D_{k+1}. Since α and β have the same infimum and supremum, α belongs to the solution set if and only if β belongs to the solution set of this new problem. Suppose that α and β do not belong to the solution set. By INH-NEM and IIA, γ belongs to the solution set. But then Lemma 14 implies that α and β should belong to the solution set, which is a contradiction. Thus α and β belong to the solution set.

Second, delete γ from this new problem to obtain the decision problem $D'_{k+1} = (A \cup \{\alpha, \beta\}, \Omega', u'_{k+1})$ with $u'_{k+1} = u_{k+1|(A\cup\{\alpha,\beta\})\times\Omega'}$. By IIA $\alpha, \beta \in \varphi(D'_{k+1})$. Next, introduce action γ again, but now with utility function equal to the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the payoffs of actions α and β in D'_{k+1}, so the decision problem D_{k+1} is obtained. By CONV it follows that $\gamma \in \varphi(D_{k+1})$.

Conclude, by induction, that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Let $D_\infty = (A \cup \{\alpha, \beta, \gamma\}, \Omega', u_\infty)$ be the limiting decision problem of the sequence $\{D_k\}_{k\in\mathbb{N}}$. Notice that $u_{\infty|A\times\Omega'} = u'$ and $u_\infty(\alpha, \omega) = u_\infty(\beta, \omega) = u_\infty(\gamma, \omega) = m$ for all $\omega \in \Omega'$. Since $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$, CONT implies that $\gamma \in \varphi(D_\infty)$.

Take $a \in A$ such that $\inf_{\omega\in\Omega'} u'(a, \omega) = m$. If $\sup_{\omega\in\Omega'} u'(a, \omega) = m$, then $u_\infty(a, \omega) = u'(a, \omega) = m = u_\infty(\gamma, \omega)$ for all $\omega \in \Omega$, so that $a \in \varphi(D_\infty)$ by

ANO. Otherwise, a weakly dominates γ and, by IWD, $a \in \varphi(D_\infty)$. Hence $a \in \varphi(D_\infty) \cap A$, and using IIA it follows that $\varphi(D_\infty) \cap A = \varphi(D') = \varphi(D)$.

Hence, $a \in \varphi(D)$ for all $a \in A$ with $\inf_{\omega \in \Omega} u(a, \omega) = \inf_{\omega \in \Omega} u(a^*, \omega) = m$. □

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