Cooperative Game Theory and Inventory Management*

M.G. Fiestras-Janeiro[1] I. García-Jurado[2] A. Meca[3] M.A. Mosquera[4]

[1] *Department of Statistics and Operations Research, Faculty of Economics, Vigo University, 36271 Vigo, Spain.* fiestras@uvigo.es

[2] *Department of Mathematics, Faculty of Computer Science, Coruña University, 15071 Coruña, Spain.* igjurado@udc.es

[3] *Operations Research Center, Miguel Hernández University, 03202 Elche, Alicante, Spain.* ana.meca@umh.es

[4] *Department of Statistics and Operations Research, Faculty of Business Administration and Tourism, Vigo University, 32004 Ourense, Spain.* mamrguez@uvigo.es

Abstract

Supply chain management is related to the coordination of materials, products and information flows among suppliers, manufacturers, distributors, retailers and customers involved in producing and delivering a final product or service. In this setting the centralization of inventory management and coordination of actions, to further reduce costs and improve customer service level, is a relevant issue. In this paper, we provide a review of the applications of cooperative game theory in the management of centralized inventory systems. Besides, we introduce and study a new model of centralized inventory: a multi-client distribution network.

Keywords: Game theory, cooperative games, inventory models, centralized inventory management.

1 Introduction

Game theory is the mathematical theory of interactive decision situations. In one of those situations some agents make decisions, depending on their decisions an outcome

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results, and each agent has his own preferences on the set of possible outcomes. Since one important class of interactive decision situations are parlour games, game theory uses their terminology to designate the elements of the interactive decision situations: these situations are called games, the agents are called players, the agents' plans to make decisions are called strategies, etc. This was already done by John von Neumann and Oskar Morgenstern in their pioneering book "The Theory of Games and Economic Behavior".

Sometimes we are interested in the strategic analysis of games. In that case, we need to model carefully all the relevant aspects of the problem and then look for the best strategies of each player taking into account that the others will also behave searching for their best. We say then that we are adopting a non-cooperative view and should use an appropriate non-cooperative model to perform our analysis. In other cases we just want to deal with the cooperation issues of the problem at hand and propose how the agents must allocate the benefits of their cooperation. This approach assumes that the agents have mechanisms to enforce their cooperation: it is the cooperative approach.

Operational research models are mathematical instruments to solve decision problems. Most of them deal with one decision maker situations. However, in real world, it is very common that the result of our decisions depend also on other decision makers' choices, i.e. in the real world many decision situations are interactive. Thus, one challenging field within operations research is that of game theoretical models in operations research.

In particular, operations management focused on single-firm analysis in the past. Its goal was to provide managers with suitable tools to improve the performance of their firms. Nowadays, business decisions are dominated by the globalization of markets and should take into account the increasing competition among firms. Further, more and more products reach the customer through supply chains that are composed of independent firms. Following these trends, research in supply chain has shifted its focus from single-firm analysis to multi-firm analysis, in particular to improving the efficiency and performance of supply chains under decentralized control. The main characteristics of such chains are that the firms in the chain are independent actors who try to optimize their individual objectives, and that the decisions taken by a firm do also affect the performance of the other parties in the supply chain. These interactions among firms' decisions ask for alignment and coordination of actions and, therefore, game theory is very well suited to deal with these interactions. This has been recognized by researchers in the field, since there is an ever increasing number of papers that apply tools, methods and models from game theory to supply chain problems. A tutorial on the subject is Cachon and Netessine (2004). The authors discuss both non-cooperative and cooperative game theory in static and dynamic settings. Additionally, Cachon (1998) reviews competitive supply chain inventory management, and Cachon

(2003) reviews and extends the supply chain literature on the management of incentive conflicts with contracts. Papers using cooperative game theory to study supply chain management are scarce, but the use of cooperative games in this context is becoming more popular. Nagarajan and Sošić (2008) reviews and extends the problem of bargaining and negotiations in supply chain relationships. A very recent survey on applications of cooperative game theory to supply chain management, the so called supply chain collaboration, is Meca and Timmer (2008). For theoretical issues and a framework for more general supply chain networks we refer to the book by Slikker and van den Nouweland (2001).

An important aspect of supply chain management is a good management of the inventories by the firms or retailers. The management of inventory, or inventory management, started at the beginning of 20th century when manufacturing industries and engineering grew rapidly. As far as we know, a starting paper on mathematical models of inventory management was Harris (1913). Since then, many books on this subject have been published (i.e. Hadley and Whitin, 1963; Hax and Candea, 1984; Tersine, 1994; Zipkin, 2000). Most often, the objective of inventory management is to minimize the average cost per time unit (in the long run) incurred by the inventory system, while guaranteeing a pre-specified minimal level of service.

In the last years, several papers dealing with the applications of cooperative game theory in inventory management have appeared. Though this is a young field, there are already some relevant contributions. In this paper, we review this literature[1]. Besides, we introduce and study a new model to analyse the cooperation in a multi-client distribution network.

2 Cooperation in deterministic inventory situations

This section tackles the study of cooperation in deterministic inventory situations. There are several papers studying various models of cooperation in this area. To start with, we analyse one of the simplest ones, in which all the agents involved in the inventory situation agree to cooperate and the characteristic function is given by an explicit formula. Later on, we review some models of cooperation where the characteristic function is given by the optimal value of an optimization problem. Finally we deal with situations where the cooperation among the agents is not an assumption and the main issue is to analyse the coalition formation process.

[1] When making the first revision of this paper we found another (unpublished) survey of inventory games by Dror and Hartman (2008). Both surveys are very different and concentrate on different problems and classes of games. Our paper, moreover, includes section 4 which analyses a completely new model.

2.1 Characteristic function given by an explicit formula

When several agents face similar inventory problems they may make some savings if they cooperate. For instance, if there is a fixed cost per order, agents will pay less if they order simultaneously as a group than if they make their orders separately. This raises an allocation problem: how should these savings be divided among the agents? This problem was analysed in Meca et al. (2003), on which part of this subsection is based.

Assume that there are n agents, $N = \{1, \dots, n\}$, each of them facing an Economic Production Quantity (EPQ) problem with shortages.[2] An EPQ model with shortages considers an agent i who places orders of a certain good that he sells. The (deterministic) *demand* that he must fulfill equals to d_i units per time unit ($d_i \geq 0$). The *cost of keeping in stock* one unit of this good per time unit is h_i ($h_i > 0$). The fixed *cost of an order* is a. Agent i considers the possibility of not fulfilling all the demand in time, but allowing a shortage of the good. The *cost of a shortage* of one unit of the good for one time unit is $s_i > 0$. When an order is placed, after a deterministic and constant lead time (which can be assumed to be zero, without loss of generality), agent i receives the order gradually; more precisely, r_i units of the good are received per time unit. It is assumed that $r_i > d_i$ (otherwise the model makes little sense). We call r_i the *replacement rate* of agent i. The agent must choose an order size $\hat{Q}_i$ and a maximum shortage $\hat{M}_i$ minimizing his average inventory cost per time unit given by:

$$C(Q_i, M_i) = a\frac{d_i}{Q_i} + h_i \frac{\left(Q_i\left(1 - \frac{d_i}{r_i}\right) - M_i\right)^2}{2Q_i\left(1 - \frac{d_i}{r_i}\right)} + s_i \frac{M_i^2}{2Q_i\left(1 - \frac{d_i}{r_i}\right)}.$$

By using elementary mathematical techniques it results that:

$$\hat{Q}_i = \sqrt{\frac{2ad_i}{h_i\left(1 - \frac{d_i}{r_i}\right)} \frac{h_i + s_i}{s_i}} \quad \text{and} \quad \hat{M}_i = \sqrt{\frac{2ad_i h_i}{s_i(h_i + s_i)}\left(1 - \frac{d_i}{r_i}\right)}.$$

Moreover, by denoting $\hat{m}_i = \frac{d_i}{\hat{Q}_i}$ as the optimal number of orders that i must place per time unit,

$$C(\hat{Q}_i, \hat{M}_i) = 2a\hat{m}_i.$$

Now, assume that the agents in a coalition $S \subset N$ decide to place their orders jointly to save part of the ordering costs; so they spend a instead of $|S|a$ every time an order is placed. We claim that, in order to minimize the sum of the average inventory costs per time unit, the agents must coordinate their orders, so $\frac{Q_i^*}{d_i} = \frac{Q_j^*}{d_j}$ for all $i, j \in N$, Q_i^*

[2] An exhaustive analysis of the EPQ model with shortages can be found in Tersine (1994).

and Q_j^* denoting the optimal order sizes for i and j if agents in S cooperate. To see this suppose that, optimally, firm 1 has a longer cycle than firm 2. Then, overall costs decrease when firm 1 shortens its cycle length to that of firm 2. Indeed, the overall ordering cost decreases because few orders are placed and holding costs decreases, because the level of the inventory of firm 1 goes down.

Then, the total average cost per time unit can be written as follows,

$$C(Q_1,(M_j)_{j\in S}) = \frac{ad_1}{Q_1} + \frac{1}{2}\sum_{j\in S}\left(h_j\left(\frac{d_j}{d_1}Q_1\left(1-\frac{d_j}{r_j}\right)-2M_j\right)+(h_j+s_j)\frac{d_1M_j^2}{d_jQ_1\left(1-\frac{d_j}{r_j}\right)}\right).$$

Using standard techniques of differential analysis, it can be checked that the optimal values which minimize C are given by:

$$Q_i^* = \sqrt{\frac{2ad_i^2}{\sum_{j\in S} d_jh_j\frac{s_j}{h_j+s_j}\left(1-\frac{d_j}{r_j}\right)}} \quad \text{and} \quad M_i^* = Q_i^*\frac{h_i\left(1-\frac{d_j}{r_j}\right)}{h_i+s_i}$$

for all $i \in S$. Moreover,

$$C(Q_1^*,(M_j^*)_{j\in S}) = 2a\sqrt{\sum_{j\in S}\hat{m}_j^2}.$$

By comparing this minimal cost with the sum of the individual minimum costs if agents do not cooperate, one can observe that the agents do obtain some savings. Then, the next question arises: how to assign these savings among all agents? Game theory provides some tools to reply this question.

Note that the minimal costs only depend on the parameter a and the optimal number of orders $\hat{m}_i$ of the agents. Hence, in order to compute the minimal costs, it suffices that each agent $i \in N$ reveals his optimal value $\hat{m}_i$, and keeps private the parameters d_i, r_i, h_i, and s_i (more details can be found in Meca et al., 2004). Then, the multi-agent inventory cost situation can be characterized by the triplet (N,a,m), where $m = (m_i)_{i\in N}$ and m_i is the optimal number of orders per time unit for agent i if he does not cooperate.

For each inventory cost situation (N,a,m), we can associate a cost game[3], called *inventory cost game* (N,c), where, for each coalition $S \neq \emptyset$, $c(S) = 2am_S$ with

$$m_S = \sqrt{\sum_{i\in S} m_i^2},$$

[3]A cost game is a pair (N,c), where N is the finite set of players who will perform a joint action, and c is a map which associates a real number $c(S)$ with every possible subset of N; $c(\emptyset)$ is zero and, for any other $S \subset N$, $c(S)$ is the cost of the joint action if only the players in S are involved in it. For more information on cooperative game theory González-Díaz et al. (2010) can be consulted.

and $c(\emptyset) = 0$. Note that, for each $S \subset N$, $c^2(S) = \sum_{i \in S} c^2(i)$ and, hence, an inventory game is fully characterized by defining the individual costs. It is easy to see that inventory cost games have a non-empty core[4]. Indeed, we present now a cost allocation rule for this class of games that always selects core allocations: it is the *share the ordering cost* rule, SOC-rule for short. This rule is so called because it proposes that agent i pays:

- a part of the fixed ordering cost proportional to his input parameter m_i^2,
- his own holding and stock-out costs.

Formally, after some calculation, the SOC-rule can be written as the following proportional rule

$$\mathrm{SOC}_i(N, c) = \frac{c^2(i)}{\sum_{j \in N} c^2(j)} c(N) = \frac{c^2(i)}{c(N)}.$$

In Meca et al. (2004) the following result is proved.

Proposition 2.1. *Let (N, a, m) be an inventory cost situation and let (N, c) be the corresponding inventory cost game. The SOC-rule gives an allocation in the core of the game.*

Several characterizations of this rule have appeared in the literature (Meca et al., 2003, 2004; Mosquera et al., 2008). We therefore present two of these characterizations. The first characterization uses the properties of efficiency, symmetry and monotonicity (cf. Meca et al., 2004). A cost allocation rule ψ is said to be *efficient* (EFF) if $\sum_{i \in N} \psi_i(N, c) = c(N)$ for all inventory cost game (N, c). It satisfies *symmetry* (SYM) if $\psi_i(N, c) = \psi_j(N, c)$ for all symmetric agents i and j in (N, c) and for all inventory cost game (N, c) ($i, j \in N$ are symmetric in (N, c) if $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$). Finally, ψ satisfies *monotonicity* (MON) if for all inventory cost games (N, c) and $(N, \bar{c})$ we have that $c(N)\psi_i(N, c) \geq \bar{c}(N)\psi_i(N, \bar{c})$ whenever $c(i) \geq \bar{c}(i)$. The monotonicity property basically says that, if we have two inventory cost situations with the same total costs to share and an agent generates more costs on his own in one situation than in the other, then he should pay more in the former situation than in the latter.

Theorem 2.1. *The SOC-rule is the unique cost allocation rule on the class of inventory cost games satisfying EFF, SYM, and MON.*

An alternative characterization can be found in Mosquera et al. (2008). It is based on the non-manipulability of the SOC-rule. We describe this property below.

[4]Remember that the core of a cost game (N, c) is the set of its stable allocations, which is given by:

$$\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S), \text{ for all } S \subset N\}.$$

Definition 2.1. *Let* (N, c) *and* $(\tilde{N}, \tilde{c})$ *be two inventory cost games and let* $S \subset N$. *We say that* $(\tilde{N}, \tilde{c})$ *is an S-manipulation of* (N, c) *if:*

1. *There is* $i \in S$ *such that* $\tilde{N} = \{i\} \cup (N \setminus S)$*; we denote agent* i *by* i_S.
2. *For each* $T \subset \tilde{N} \setminus \{i_S\}$, $\tilde{c}(T) = c(T)$.
3. *For each* $T \subset \tilde{N}$ *with* $i_S \in T$, $\tilde{c}(T) = c(S \cup T)$.

A cost allocation rule ψ satisfies *immunity to coalitional manipulation* (ICM) if, for each inventory game (N, c) and for each $(\tilde{N}, \tilde{c})$, an S-manipulation of (N, c), then it holds that

$$\psi_{i_S}(\tilde{N}, \tilde{c}) = \sum_{j \in S} \psi_j(N, c).$$

This property describes how the rule behaves if a player splits in several players or several players merge into a unique player. In both situations, the splitting or merging do not benefit players. Hence, if avoiding strategic merging and splitting is an issue, ICM is a desirable property.

Theorem 2.2. *The SOC-rule is the unique cost allocation rule on the class of inventory cost games satisfying EFF and ICM.*

Until now, we have considered cooperation by making joint orders, but another cooperative behaviour can be considered: coordination by storing all goods in the cheapest warehouse of a coalition. This new kind of cooperation gives raise to the so-called *holding cost games* introduced by Meca et al. (2004). They show that these games have non-empty cores, and introduce a cost allocation rule which always provides core allocations. Later, Meca (2007) studies this class of games in depth by presenting a more general class of games called *generalized holding cost games*. She shows that these games are totally balanced[5], she defines a family of core allocations for these games, and she studies a proportional cost allocation rule which proposes allocations in the aforementioned family.

Several modifications of the EPQ model with shortages have been studied from a cooperative point of view. Meca et al. (2007) study a special EPQ model with shortages where temporary discounts are offered to the agents when they place an order. The cooperative behaviour consists here of sharing the order costs and the warehouse facilities. This cooperative situation leads to the class of *p-additive games*[6]. Meca et al. (2007) show that p-additive games are totally balanced. Furthermore, they propose and characterize a modified SOC-rule which provides core elements in this context.

[5]Remember that a cost game (N, c) is totally balanced if its core is non-empty and, moreover, so it is the core of every cost game (S, c_S) with $S \subset N$ and with c_S being the restriction of c to S.

[6]Given $p \in \mathbb{R} \setminus \{0\}$, a p-additive game is a pair (N, w) where $w(S) \geq 0$ for all $S \subset N$, $w(S) = 0$ if $w(i) = 0$ for all $i \in S$, and $w(S)^p = \sum_{i \in S_+} w(i)^p$ where $S_+ = \{i \in S \mid c(i) > 0\}$.

Another cost allocation problem associated to inventory management is the so-called *first order interaction joint replenishment model* (cf. Federgruen and Zheng, 1992; Anily and Haviv, 2007). In this model, when a subset of agents simultaneously place an order, the total ordering cost is given by the addition of the major setup cost (a fixed part of the ordering cost) and the corresponding minor setup cost (a part of the ordering cost that is agent-dependent). Anily and Haviv (2007) define the characteristic function assuming that the warehouse does not hold any inventory and agents follow the power-of-two (POT) policy, according to which each agent orders in equidistant time intervals whose lengths equal an integer (positive or negative) power-of-two times a fixed base time unit. (Federgruen and Zheng, 1992; Federgruen et al., 1992). They show that the resulting cooperative game is concave[7] and that, therefore, its core is non-empty (see Shapley, 1971). Then, they describe a particular core allocation, extensible to a set of core allocations with similar properties. Dror and Hartman (2007) also consider this special first order interaction cost structure, but assume, as in Meca et al. (2004), that agents always place orders together by means of an EOQ policy. They provide conditions for the non-emptiness of the core and analyse the sensitivity of the core with respect to the cost parameters. Later on, Zhang (2009) generalizes the model by Anily and Haviv (2007). He shows that under POT policies, the joint replenishment game with a submodular joint setup cost function has a non-empty core, even when the warehouse is allowed to hold inventory.

In Fiestras-Janeiro et al. (2009), it is assumed that when a subset of agents simultaneously place an order, the ordering cost is the addition of the fixed part of the ordering cost and the maximum of the individual transportation costs (part of the ordering cost that is agent-dependent). This is the key difference with the game by Dror and Hartman (2007). This structure of the ordering costs can appear, for instance, when the agents are located on the same distribution route. Unlike Anily and Haviv (2007) and Zhang (2009), who consider a POT policy, Fiestras-Janeiro et al. (2009) focus on the unsplit fixed partition policy (Anily and Bramel, 2004). They prove that if cooperation is profitable (i.e., if the inventory game is subadditive), then the core of the game is non-empty. They further define three context-specific cost allocation rules and study their properties. The first one, which turns out to provide core allocations, is a sharing rule *à la* Shapley. The other two, simpler but not always in the core, are proportional cost allocation rules.

Finally, Bernstein et al. (2009) analyse a production system similar to that of the Japanese firm Toyota. Based on this model, they investigate the benefits and challenges associated with establishing a Knowledge Sharing Network. The main objective of such a network is to share information and knowledge among their suppliers.

[7] Remember that a cost game (N, c) is concave if, for every $i \in N$ and every $S, T \subset N \setminus \{i\}$ with $S \subset T$, $c(S \cup i) - c(S) \geq c(T \cup i) - c(T)$.

This knowledge transfer between suppliers is modeled through a two stage model. At stage one, they analyse how to invest in order to reduce the setup costs in their own EOQ model. For this first type of knowledge sharing, we could think of the assembler (i.e., Toyota) sharing knowledge that allows all suppliers to reduce their fixed setup costs. At stage two, a group of suppliers is interested on learning from the supplier that incurs the lowest fixed setup cost. This is the case of knowledge transfer between suppliers which is modeled throughout a cost game focusing on fixed setup cost reductions. In this framework, Bernstein et al. (2009) explore the feasibility of sharing knowledge by means of the existence of cost allocations which enable the cooperation among suppliers in the system (the core of the corresponding game is non-empty). Then, they propose some appealing core allocations which are very useful in the investment process at stage one. Since the above cooperation (sharing knowledge) lead to more reduced fixed setup costs, the wholesale price (the one paid by the assembler) is also reduced and so, the price of the finished product is lower and then its demand increases. Hence, all the assembler and suppliers get some benefits from this cooperative situation.

2.2 Characteristic function given by the optimal value of an optimization problem

As mentioned before, one of the main objectives of the agents is to reduce their costs. In order to achieve this goal, groups of agents tend to form coalitions to decrease operation costs by making dynamic decisions throughout a finite planning horizon. In tactical planning of enterprises that produce indivisible goods, operation costs mainly consist of production, inventory-holding, and backlogging costs. The coalitions should induce individual and collective cost reductions; in this way, stability is achieved in the process of enterprise cooperation. In this framework a coalition allows each of its members to have access to the technologies owned by the other members of the coalition. Thus, members of a coalition can use the lowest-cost technology of the agents in the coalition. Planning is done throughout a finite time horizon; at the beginning of each period, the costs to the members of a coalition, which depend on the best technology at that point, may change.

The model that represents such a situation is the dynamic, discrete, finite planning horizon production-inventory problem with backlogging. Here, the objective of any group of agents is to satisfy the demand for indivisible goods in each period at a minimum cost. This is a well-known combinatorial optimization problem for which the algorithm by Wagner and Whitin provides optimal solutions by dynamic programming techniques. The optimal solutions of this problem lead to the best production-inventory policy for the group of agents. These policies generate an optimal operation cost for the entire group. The question is what portion of this cost is to be supported by each agent. Once more, cooperative game theory provides the natural tools for answering this question.

In Guardiola et al. (2009) a class of production-inventory games is introduced. The authors consider a group of agents, each one facing a production-inventory problem, that decide to cooperate in order to reduce costs. For each such a production-inventory situation, a corresponding production-inventory game is defined (henceforth, PI-game). The main results of the paper include the total balancedness of PI-games and an explicit form for their characteristic functions. The study of PI-games is completed by showing that the Owen set of a PI-situation (the set of allocations that are achievable through dual solutions, see Owen, 1975) shrinks to a singleton: the Owen point. Guardiola et al. (2009) propose the Owen point as a core allocation for a PI-game which is easy to calculate and satisfies good properties. In addition, a necessary and sufficient condition for the core of a PI-game to be a singleton is presented. Finally, the authors point out the relationship of the Owen point with some well-known worth allocation rules in cooperative game theory. In addition, Guardiola et al. (2008) present three different axiomatic characterizations of the Owen point in this context.

There are other papers that focus on cooperation in periodic review inventory situations by means of cooperative game theory. One of those papers is van den Heuvel et al. (2007), which studies coordination in economic lot sizing situations (henceforth, ELS-situations). In that finite horizon model, agents should satisfy the demand in each period by producing in that period or carrying inventory from previous periods; backlogging is not allowed. The main difference between that model and the one given by Guardiola et al. (2009) is that the former considers setup costs but assumes that costs are the same for all players in every period. Therefore, ELS- and PI-situations are distinct, in general. The main result in van den Heuvel et al. (2007) is that ELS-games (games induced by ELS-situations) have a non-empty core. In another paper, Chen and Zhang (2008) propose an integer programming formulation for the concave minimization problem that results from an ELS-situation. They consider the ELS-game with general concave ordering cost. When both the inventory holding cost and backlogging cost are linear functions, it can be shown that the core of the resulting game is non-empty. Their approach is based on duality in linear programming.

2.3 Coalition Formation

In cooperative games it is usually assumed that the grand coalition is formed whenever it leads to some profit. Then, one of the goals of cooperative game theory is to find allocations of the total profit in such a way that no subset of players has incentives to leave the grand coalition and form its own coalition, i.e., allocations that are stable with regard to one-step deviations of coalitions.

Nevertheless, other types of stability can be considered. For instance, think of cooperation among competitors. In this case, although the cooperation may produce benefits for a particular player, it also may profit some of his possible competitors. Then, players might have some doubts about joining or not to the grand coalition because of the consequence this can have in the long run. The core of a game is a concept which belongs to the first type of stability: it only sees one-step deviations of a coalition and does not consider further deviations of the other coalitions or even further deviations of a player to join other players outside his coalition. These issues are captured by the so-called *far-sighted coalitional stability*: every coalition considers that, once it reacts, another coalition can react, and then yet another, and so on. There exist in the literature some solution concepts which incorporate this far-sighted coalitional stability, as the *largest consistent set* (LCS) defined by Chwe (1994), the *largest cautious consistent set* (LCCS) defined by Mauleón and Vannetelbosch (2004) and the *equilibrium process of coalition formation* (EPCF) defined by Konishi and Ray (2003).

Coalition formation in inventory situations and supply chain management has become more popular in recent years. Nevertheless, one of the first papers dealing with coalition formation in supply chain management is Chacko (1961), where the effects of coalition formation in joint profits are analysed. After that paper, there is a gap in this topic until 2005 when Granot and Sošić (2005) use the far-sighted coalitional stability in supply chain management. They study a simple model of three retailers which have almost substitutable products and who form coalitions. They study the LCS in several situations depending on the degree of substitutability of the products and on the portion of the profit that each retailer receives.

Nagarajan and Bassok (2008) study an assembly problem in which a single assembler buys complementary components from n suppliers and assembles the final product in anticipation of demand. Basically, the coalition formation process is analysed here using a two-stage approach. In stage 1, suppliers form coalitions for sending a kit of components to the assembler. In stage 2, there is an interaction between the assembler and the coalitions of suppliers formed in stage 1. In the coalition formation process they assume that the players are far-sighted and they study the structure of the supply chain using the concept of LCS. More papers in this line are Granot and Yin (2008) and Nagarajan and Sošić (2009). Granot and Yin (2008) introduce to the assembly problem two schemes of negotiation among the assembler and the suppliers: push and pull. They show that, using the concept of LCCS, the grand coalition is stable under the push scheme and any coalition structure can be stable under the pull scheme. Nagarajan and Sošić (2009) study the assembly problem in three different scenarios depending on whether the retailers or the assembler are the Stackelberg leaders or whether both make decisions simultaneously. They find the coalition structures which are coalitional far-sighted stable using the concepts of LCS and EPCF.

Following this approach of negotiation in two stages, Nagarajan and Sošić (2007) study dynamic coalition formation among agents in competitive markets. In their framework, agents selling substitutable goods compete by setting prices and inventory levels. They use the concepts of LCS and EPCF for modelling the coalitional far-sighted stability. They show that the grand coalition is stable even when agents benefit myopically by defecting and that there exist conditions under which the situation of a large coalition against a few lone agents is stable.

Sošić (2006) studies a decentralized distribution system where the process of coalition formation is also analysed in two stages. Nevertheless, these stages are different from the ones considered in the papers above. In this paper, at stage 1, the agents must order their initial inventory before its stochastic demand occurs. Then, at stage 2, after demand realization, retailers decide how much of their unsold inventory or unsatisfied demand want to share with other agents. They show that, when the profits from the cooperation are distributed according to the Shapley value, the grand coalition is coalitional far-sighted stable. For that purpose they use the stability concepts of LCS and EPCF.

Finally, Nagarajan et al. (2009) study the stability of Group Purchasing Organizations (GPOs). GPOs benefit its members through quantity discounts and negotiation power when dealing with suppliers. They propose three possible allocation rules for the cooperative benefits: the Shapley value, the equal allocation principle, and the quantity-based proportional allocation rule. They study the coalition formation process using the concept of LCS. They show that, when buyers are homogeneous, the grand coalition is stable for these three allocation rules. However, when the buyers are heterogeneous, only the Shapley value produces an allocation that ensures the stability of the grand coalition. For the other two allocation rules, the buyers with large orders prefer to form their own GPO, leaving the other buyers outside the consortium.

3 Cooperation in stochastic inventory situation

This section deals with non-deterministic inventory situations. There are a number of papers that examine combined problems of optimizing and allocating the savings in non-deterministic centralized inventory systems. The focus of most of those papers is on news-vendor type problems.

In a news-vendor setting, the agents might benefit from cooperation through coordinated ordering and inventory centralization. Here, we concentrate on the news-vendor inventory centralization problem with identical news-vendors as introduced by Hartman et al. (2000). Consider a finite set of stores (news-vendors) that respond to a periodic random demand (of newspapers) by ordering a certain quantity at the start of every period. Since the demand is random, a store will face in each period one of the two next cases: (1) the ordered quantity is less than the realized demand, resulting in lost profit for the store; (2) the ordered quantity exceeds the realized demand, resulting in a disposal cost for the store, since the items (the newspapers) are perishable.

Formally, we consider a set $N = \{1, ..., n\}$ of stores. Each store $i \in N$ faces a non-negative random demand x_i, with distribution function F_i and mean μ_i. The disposal cost per unit is $h > 0$, and the penalty cost per unsold unit is $p > 0$. These costs are identical for all stores. The product is ordered once at the start of each period (not reordered), and items on hand at the beginning of the period cannot be returned. There is no order cost and no quantity discounts. Both the demand distributions and the costs are common knowledge. This situation is stationary and infinitely repeated period after period. The cost resulting from an order quantity of q is:

$$\Psi(x, q) = \begin{cases} h(q - x) & \text{if } q \geq x \\ p(x - q) & \text{if } q < x. \end{cases}$$

Consider a coalition $S \subset N$ of stores facing the joint demand $x_S = \sum_{i \in S} x_i$ with distribution function F_S and expected value μ_S. Assume that, for all coalitions $S \subset N$, $E\left[\Psi(x_S, q)\right] < \infty$ for all $q \in \mathbb{R}$. This is the classic news-vendor problem and for all $S \subset N$, we can find a value q_S that minimizes $E\left[\Psi(x_S, q)\right]$; *i.e.*, q_S is the optimal order size for S.

The *news-vendor expected game* (N, c_E) (henceforth E-game) is the cost game with characteristic function $c_E(S) := E\left[\Psi(x_S, q_S)\right]$ for all $\emptyset \neq S \subset N$. For all $S \subset N, c_E(S) \geq 0$ represents the optimal expected cost of holding or shortage.

Hartman et al. (2000) proves that the cores of the E-games are non-empty for demands with symmetric distribution and for joint multivariate normal demand distribution. Müller et al. (2002) generalizes the result above for all possible joint distributions of the random demands. This non-emptiness result is still valid even when there are infinitely many stores, as proved by Montrucchio and Scarsini (2007). Slikker et al. (2005) enrich the finite model by allowing the stores to use transshipment (at a positive cost) after demand realization is known. They also show that news-vendor games with transshipments have a non-empty core even if the stores have different retail and wholesale prices. Moreover, news-vendor games are not concave in general. Özen et al. (2005) study the concavity of news-vendor games under special assumptions about the demand distributions. Their analysis focuses on the class of news-vendor games with independent symmetric unimodal demand distributions. Several interesting subclasses, which only contain concave games, are identified.

Suppose now that in a given period, coalition $S \subset N$ decides on an optimal order size q_S. Then, at the end of the period, each store $i \in S$ observes its demand realization, say $\hat{q}_i$. The total demand realization for S is $\hat{q}(S) = \sum_{i \in S} \hat{q}_i$. Just as for a single store there are two possibilities: (1) $\hat{q}(S) \leq q_S$, and the cost for this centralized system is $h(q_S - \hat{q}(S))$; (2) $\hat{q}(S) \geq q_S$ and the cost is equal to $p(\hat{q}(S) - q_S)$.

Then, the *news-vendor realization game* (henceforth R-game), (N, c_R), introduced by Hartman and Dror (2005), is defined by $c_R(S) := \max\{p(\hat{q}(S) - q_S), h(q_S - \hat{q}(S))\}$ for all $\emptyset \neq S \subset N$, where q_S is the demand of S in the E-game. This non-negative game measures, from a pessimistic point of view, the actual cost of the demand realization for every $S \subset N$. Hartman and Dror (2005) show that R-games may have empty cores, by providing a realization example for a joint multivariate normal demand distribution.

The reader may note that E-games and R-games are related by means of a long-term expectation property: $E[c_R(S)] = E[\Psi(x, \hat{q}(S))] = c_E(S)$ for all $\emptyset \neq S \subset N$. The above property means that the long-term average cost of coalition S, for repeated realizations of the actual cost game c_R, is the same as its cost in the expected cost game c_E, provided that the underlying individual demand distributions do not change.

There are other papers which examine the existence of stable profit allocations among cooperative agents by means of the so-called stochastic cooperative decision situations (see Özen, 2007). Özen et al. (2006) analyse the stability of cooperation among several outlets who coordinate in order to benefit from inventory centralization. The authors focus on news-vendor situations with delivery restrictions. In these situations, the outlets place some restrictions on the number of items that should be delivered to them if they join a coalition to benefit from joint ordering. They show that the associated cooperative game has a non-empty core. Afterwards, they concentrate on a dynamic situation where the outlets change their delivery restrictions. They then investigate how the profit allocation might be affected by these changes.

Another example of news-vendor situations is considered in Özen et al. (2008). They study news-vendor situations with multiple warehouses, where the stores can cooperate to benefit from inventory pooling. The warehouses offer alternative ways of supplying the goods to the stores, which might become more useful when the stores form coalitions. The authors study the corresponding cooperative games and prove that the cores of these games are non-empty.

A very recent paper by Chen and Zhang (2009) presents a unified approach to analyse news-vendor games using the duality theory of stochastic programming (Rockafellar and Wets, 1976). The optimizations problems corresponding to the news-vendor games are formulated as stochastic programs. The authors observe that the strong duality of stochastic linear programming not only implies the non-emptiness of the cores of those games, but also suggests a way to find core allocations.

The news-vendor inventory centralization problem examined in the literature is geared mainly to the expected value cost analysis. The analysis of a dynamic and repeated cost allocation system is the main topic of Dror et al. (2008). They examine a related inventory centralization game based on demand realizations and propose a repeated cost allocation scheme for dynamic realization games based on allocation processes introduced by Lehrer (2002). It is proved that the cost sub-sequences of the dynamic realization game process, based on Lehrer's rules, converge almost surely to either a least square value or to a point in the core of the expected game. To complete this study, they extend the above results to more general dynamic cost games and relax the independence hypothesis of the sequence of players' demands at different stages.

4 Cooperation in a multi-client distribution network

There are many problems to explore within the general topic of cooperation in centralized inventory models. We consider that this is a very wide field which will produce many results in the next years. We conclude this paper introducing and exploring a new model in this context.

As we have already mentioned, Meca et al. (2004) study the class of holding cost games. In those games, a group of agents facing EOQ problems decide to coordinate by making joint orders and by storing all goods in the cheapest warehouse of the group. However, in many real cases, storing in the cheapest warehouse will not be possible because of several reasons, like geographical issues, market restrictions, differences among the agents, etc. Anyway, this holding coordination may still be possible inside smaller groups of agents. These situations appear, for instance, in multi-client distributions networks, in which the common provider serves the products from a central warehouse to intermediate warehouses, from which firms receive their goods as they need them. We study now this new pattern of centralized inventory which, to our knowledge, has not been treated in the literature.

Let $N = \{1, \dots, n\}$ be a set of agents facing EOQ problems. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a partition of N. For each $k \in M = \{1, \dots, m\}$, P_k represents a group of agents who can coordinate by storing their goods in the cheapest warehouse of that group. We will distinguish two steps in the coordination process. First, the agents in each group P_k agree to coordinate their orders and storing all their goods in their cheapest warehouse. We denote $h_{P_k} = \min_{j \in P_k} h_j$. And second, assuming that the agents in a group P_k act as a block, the groups agree to coordinate their orders. Notice that, we mean by acting as a block that, when agents $i \in P_k$ and $j \in P_\ell$ agree to cooperate, then all the agents in $P_k \cup P_\ell$ also cooperate.

Like in inventory games, at the optimum, the agents have cycles of the same length, i.e. $\frac{Q_i^*}{d_i} = \frac{Q_j^*}{d_j}$ for all pair $i, j \in N$ where Q_i^* denotes the optimal order size for $i \in N$.

Then, the total average cost per time unit can be written as follows,

$$C(Q_1) = \frac{ad_1}{Q_1} + \frac{Q_1}{2d_1} \sum_{k \in M} \left(h_{P_k} \sum_{j \in P_k} d_j \right).$$

Using standard techniques of differential analysis, it can be checked that the optimal values which minimize C are given by:

$$Q_i^* = \sqrt{\frac{2ad_i^2}{\sum_{k \in M} \left(h_{P_k} \sum_{j \in P_k} d_j \right)}}$$

for all $i \in N$. Moreover,

$$C(Q_1^*) = 2a \sqrt{\sum_{k \in M} \hat{m}_{P_k}^2}$$

where $\hat{m}_{P_k} = \sqrt{\frac{\sum_{j \in P_k} d_j h_{P_k}}{2a}}$ denotes the optimal number of orders for the coalition P_k in the holding cost situation $(P_k, a, \{h_i, d_i\}_{i \in P_k})$.

The optimal average cost only depends on the parameters a and $\hat{m}_{P_k}$. Then, each group of agents only has to reveal its optimal number of orders $\hat{m}_{P_k}$. Nevertheless, for computing each $\hat{m}_{P_k}$, it is needed a complete disclosure of information inside each P_k. We denote a *partial holding cost situation* by the tuple $(N, \mathcal{P}, a, \{m_{P_k}\}_{k \in M})$. Let us note that, if $\mathcal{P} = \{\{1\}, \ldots, \{n\}\}$, i.e. the cooperation is only by making joint orders, then the agents face an inventory situation as in Meca et al. (2004). Nevertheless, if $\mathcal{P} = \{N\}$, i.e. all the agents can cooperate by making joint orders and by storing in the cheapest warehouse, then the agents again face a holding cost situation as in Meca et al. (2004). Therefore, a partial holding cost situation generalizes both inventory and holding cost situations.

Our main objective is to allocate the optimal total average cost per time unit $C(Q_1^*)$ among the agents in N. Since the agents in each group P_k act as a block, the only coalitions that can break the grand coalition are those formed by complete groups of agents, i.e. those $S \subset N$ such that $S = \bigcup_{k \in R} P_k$ for some $R \subset M$. Now we can define a cost game played only by the groups of agents. Taking into account our coordination procedure, they play an inventory game associated with the inventory situation $(M, a, (m_{P_k})_{k \in M})$, more precisely, the cost game $(M, c_{\mathcal{P}})$ where $c_{\mathcal{P}}(\emptyset) = 0$ and, for each $\emptyset \neq R \subset M$,

$$c_{\mathcal{P}}(R) = 2a \sqrt{\sum_{k \in R} m_{P_k}^2}.$$

In order to share the total costs among the agents, and taking into account the game played by the groups, we define the set of *reasonable allocations* as the allocations for the players such that yield core elements for the groups, i.e.,

$$\mathcal{R}(N,\mathcal{P},a,(m_{P_k})_{k\in M}) = \left\{ x \in \mathbb{R}^n \;\middle|\; x(N) = c_{\mathcal{P}}(M) \text{ and } x\big(\bigcup_{k\in T} P_k\big) \leq c_{\mathcal{P}}(T)\ \forall\, T \subset M \right\}$$

where $x(S) = \sum_{i\in S} x_i$. It is straightforward to prove that $R(N,\mathcal{P},a,(m_{P_k})_{k\in M}) \neq \emptyset$.

A cost allocation rule on the class of partial holding cost situations is a function ψ which assigns to each situation $(N,\mathcal{P},a,(m_{P_k})_{k\in M})$ a unique vector $\psi(N,\mathcal{P},a,(m_{P_k})_{k\in M}) \in \mathbb{R}^N$. Following the steps of the coordination process of our model, we can define the following cost allocation rule:

- Share $c_{\mathcal{P}}(M)$ among the groups according to the SOC-rule of $(M,c_{\mathcal{P}})$.
- Share the amount that each group P_k has to pay among the agents in P_k proportionally to their demands (d_i for all $i \in P_k$).

We call this rule the *Share the Cost among the Groups* rule (briefly, SCG-rule). After some algebra, the SCG-rule can be written as,

$$\text{SCG}_i(N,\mathcal{P},a,(m_{P_k})_{k\in M}) = \frac{2ad_ih_{P_\ell}}{\sum_{k\in M} c_{\mathcal{P}}(k)^2}\, c_{\mathcal{P}}(M)$$

where $\ell \in M$ is such that $i \in P_\ell$.

The next result[8] shows that the SCG-rule is a reasonable allocation and that it generalizes the SOC-rule and the demand proportional rule for holding cost games (see Meca et al., 2004).

Theorem 4.1. *Let $(N,\mathcal{P},a,(m_{P_k})_{k\in M})$ be a partial holding cost situation. Then,*

1. $\text{SCG}(N,\mathcal{P},a,(m_{P_k})_{k\in M}) \in \mathcal{R}(N,\mathcal{P},a,(m_{P_k})_{k\in M})$.
2. *If $\mathcal{P} = \{\{1\},\ldots,\{n\}\}$, then the* SCG*-rule equals the* SOC*-rule for inventory games.*
3. *If $\mathcal{P} = \{N\}$, then the* SCG*-rule equals the demand proportional rule for holding cost games.*

To finish this subsection we show that the SCG-rule is the unique cost allocation rule on the class of partial holding cost situations which satisfies a bundle of nice properties. Let ψ be a cost allocation rule on the class of partial holding cost situations. ψ is said to be *immune to coalitional manipulation in the quotient* (CMQ) if the groups cannot benefit by splitting themselves in several subgroups or by merging in a larger group. ψ is said to be *immune to coalitional manipulation inside the groups* (CMG) if the agents inside

[8]The proofs of the results in this section follow the lines of similar results in inventory games and are omitted. They can be however obtained from the authors upon request.

a group cannot benefit by splitting themselves in several agents or by merging[9] in a larger agent.

Theorem 4.2. *The unique cost allocation rule on the class of partial holding cost situations satisfying EFF, CMQ, and CMG is the* SCG*-rule. Moreover, these three properties are independent.*

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[9]Merging agents of different groups is not allowed.

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