Sharing costs in highways: a game theoretic approach.*

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Abstract

This paper introduces a new class of games, *highway games*, which arise from situations where there is a common resource that agents will jointly use. That resource is an ordered set of several indivisible sections, where each section has an associated fixed cost and each agent requires some consecutive sections. We present an easy formula to calculate the Shapley value, and we present an efficient procedure to calculate the nucleolus for this class of games.

Keywords: game theory, highway games, nucleolus, Shapley value.

1 Introduction

In this paper, we consider situations where several agents use the same public resource, where the resource consists of a finite number of ordered sections, each with its corresponding cost, and where each agent makes use of a consecutive set of these sections. A simple example that illustrates the situation is the case of a linear highway. The sections are delimited by the entry and exit points and its cost depends on the section length and on the number of vehicles that use the section. Each car or truck only needs the highway sections between his entry and exit point. Obviously, this is a simplification of a highway. In a general highway other issues of primary importance, as the congestion problem, can be considered. In our context, congestion does not appear explicitly, but it may be implicit in the cost of each section. The

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congestion problem of a highway (network) has been studied from a non-cooperative game point of view (Rosenthal, 1973) and, more recently, from an evolutionary game theory point of view (Sandholm, 2002). In a classical congestion problem, the agents have several options to get from a point to another in the network and the cost of an agent depends on the route he chooses, the amount of traffic on his route, and the toll. The agents have to choose a probability distribution over their options and one tries to find an equilibrium. A linear highway model does not lend itself for this approach since agents have only one way to get from a point to another. Moreover, congestion is not really the problem for our practical application in Spain and the paying toll is a tool of the highway managers to be paid back part (or all) of its investment. From now on, for clarity of presentation, we consider a linear highway as the public resource.

We address the problem of sharing the cost of such a highway among its users. We are interested in sharing cost methods which are based on the following principles: to exactly allocate the total cost of the highway; and to be axiomatized by axioms which many consider as reasonable and fair. Although these principles are widely accepted, the results on highway cost allocation are very often controversial. Allocating a cost share involves several factors such as different classes of vehicles, traffic characteristics, congestion, environmental factors, and so on. Some of the procedures to assign costs are proportional to the road length traveled, the passenger car equivalence, the single axle load equivalence, etc. Other methods are known as incremental: vehicle classes are introduced sequentially starting from the lightest one, and each time a new class is introduced the corresponding increment is allocated to this class. In turn, the U. S. Federal Highway Administration (1997) allocates construction costs and maintenance costs separately.

Cooperative game theory has proved to be very useful for finding "fair" methods for sharing common costs. The reader is referred to e.g. Young (1985) for applications. In the particular case of highway cost allocation, game theory was successfully applied in Villarreal-Cavazos and García-Díaz (1985), Makrigeorgis (1991), and Castaño-Pardo and García-Díaz (1995), where the cost shares were allocated to the different vehicle classes. In contrast, Dong et al. (2012) allocates these shares between all the potential users, instead of the vehicle classes.

In this paper, we propose a simple transferable utility (TU) game model called *highway game* for this problem. In the model we assume that an agent is completely characterized by the sections it uses, which means that all vehicles in our model are of the same class. We also assume that each section has a fixed cost, which corresponds to either initial construction costs or to periodical maintenance costs. We then analyze the behavior of two well-known solutions concepts for TU games on the class of highway games, and we apply them on realistic data. Our model is similar to that in Dong et al. (2012). The main difference with our work is in its purpose. They deal with axiomatizations of a cost sharing method (*toll pricing method*), while we focus on the computation of the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969).

It turns out that the Shapley value corresponds to a very simple way of sharing costs. We show that it allocates the cost of each section equally among the agents who use it. Another classes of games where Shapley value adopts very simple formulation are airport games (Littlechild and Owen, 1973), tree games (Megiddo, 1978; Granot et al., 2002), convex big boss games (Muto et al., 1988), sequencing games (Curiel et al., 1989), infrastructure games (Fragnelli et al., 2000), peer group games (Brânzei et al., 2002), and weighted majority games (Algaba et al., 2003) among others. Dong et al. (2012) show that the Shapley value coincides with the toll pricing method defined there. Moreover they also axiomatize it in the class of highway toll problems.

The nucleolus is more difficult to analyze. In Luskin et al. (2001) several methods are compared, and these authors recommend the nucleolus, despite the difficulties in computing it. In Villarreal-Cavazos and García-Díaz (1985) the use of the nucleolus (called by these authors the *generalized method*) is also suggested for sharing costs in a practical highway situation. In addition to these practical recommendations, the nucleolus is also supported by several axiomatizations. Perhaps one of the nicest appears in Sobolev (1975) (actually it is an axiomatization for the prenucleolus). Almost all the axioms used in Sobolev's axiomatization are satisfied by the nucleolus in the class of highway games. Nevertheless, the axiom of covariance can not be applied for highway games since, if an additive game is added to a highway game, the result is not necessarily a highway game. One could think in using axioms defined for the highway model setting, as those defined by Dong et al. (2012). Nucleolus satisfies stand alone test and dummy but it does not satisfy cost recovery, routing-proofness, additivity and toll upper bound, as they are stated in Dong et al. (2012). It seems hard to axiomatize the nucleolus in the class of highway games. It deserves a deeper analysis and we leave it for further research. Instead, we focus on to find a computationally efficient way of calculating the nucleolus of a highway game which takes up most of the length of this paper. We prove in this paper that highway games are concave, and although theoretically computationally efficient algorithms for the nucleolus exist for the entire class of concave games (Kuipers, 1996 and Faigle et al., 2001), these algorithms do not perform well in practice. Efficient and practical algorithms have already been developed for calculating the nucleolus of several special classes of games, e.g. for standard tree games (Megiddo, 1978; Granot et al., 1996), assignment games (Solymosi and Raghavan, 1994), flow games (Potters et al., 2006), and peer games (Brânzei et al., 2005). The algorithms described in these papers are all based on the fact that a small family of coalitions determines the nucleolus. We too prove that the nucleolus of highway games is determined by a small family of coalitions, and we exploit it in our algorithm. In our approach, we combine it with the fact that the determining family for the nucleolus must satisfy a special version of the Kohlberg criterium for concave games, a theorem proved by Arin and Iñarra (1998). We then apply the theory of reduced games (Maschler et al., 1972) and prove that the problem reduces to one or two smaller games that are also highway games. This yields a recursive algorithm for the nucleolus, for which we derived a worst-case complexity of $\mathcal{O}(m^5)$, where m is the number of sections.

It is worth mentioning that the class of airport games is a special subclass of highway games, and that our results are a generalization of the results in that paper. On the other hand, the class of savings games derived from highway games is a subclass of the class of *realization games*, introduced by Koster et al. (2003) to study the allocation of public goods. The results in that paper for concavity and the Shapley value generalize our results concerning these issues. Besides, one can consider extending our model to the case of a network. This is not an easy task as it is pointed out in Çiftçi et al. (2010). Based on a preliminary version of our paper, they extend our model to a network. They show that, for keeping the concavity property of the games, one has to restrict the shape of the network to a very particular one: weakly cyclic networks where each cycle is composed by exactly three edges (what is called a *weakly triangular graph*).

The paper is organized as follows. In Section 2, we introduce highway problems and their related games, and we derive the results for concavity and the Shapley value of those games. Section 3 is devoted to the nucleolus and to its algorithm. Finally, we present in Section 4 a practical case where our results can be compared with the official rates.

2 Highway problems and highway games

A *highway problem* is a 4-tuple $\Gamma = (N, M, C, T)$, where N is a finite set of *agents*, M is a finite and completely ordered set of *sections*, $C : M \to \mathbb{R}_{++}$ represents the *cost* of each section, and $T : N \to 2^M$ is a mapping that represents, for each agent $i \in N$, the set of sections $T(i) \subseteq M$ used by that agent. Since each agent uses a consecutive set of sections, we require that each $T(i)$ is of the form $\{t \in M \mid a_i \leq t \leq b_i\}$. Here, a_i is the first section (minimal in the ordering) used by agent i, and b_i is the last section (maximal in the ordering) used by i. We also require that every section is used by at least 1 agent, i.e. we require $\cup_{i \in N} T(i) = M$. In case the defining 4-tuple of a highway problem Γ is not stated, the agent set of Γ will be denoted by N_Γ, the set of sections by M_Γ, etcetera.

Remark 2.1. Notice that a highway problem (N, M, C, T) corresponds to an airport problem (Littlechild and Thompson, 1977) if $\min T(i) = \min M$ for all $i \in N$. $\triangleleft$

Let us illustrate the former definition with an example.

Example 2.1. Figure 1 represents a small highway problem for the highway that connects A Coruña with Vigo in Spain. Four agents that use the highway are depicted as line-pieces below the highway. The agents are numbered 1,2,3, and 4. Assuming that these are the only agents, we define $N = \{1, 2, 3, 4\}$. The black dots in the highway represent the entrances and exits which divide the highway into 4 sections: A Coruña−Santiago (1); Santiago−Padrón (2); Padrón−Pontevedra (3); and Pontevedra−Vigo (4). Then, we define the set of sections as $M = \{1, 2, 3, 4\}$ with the natural ordering. Let us assign costs to those sections: $C(1) = 8$,

$C(2) = 4$, $C(3) = 6$, and $C(4) = 6$. Since agent 1 travels only section 1 between A Coruña and Santiago, we define $T(1) = \{1\}$; agent 2 travels between Padrón and Vigo, so $T(2) = \{3, 4\}$; agent 3 between A Coruña and Pontevedra, so $T(3) = \{1, 2, 3\}$; and agent 4 travels between Santiago and Vigo, so $T(4) = \{2, 3, 4\}$.

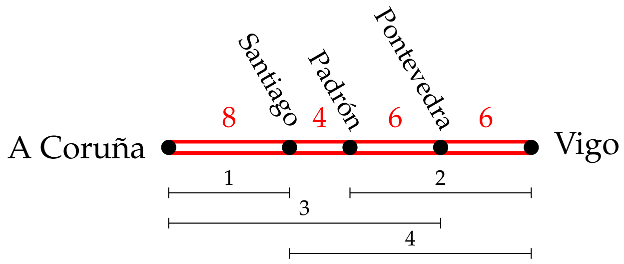

Figure 1: Linear highway of Example 2.1.

Notice that the sets $T(i)$ are all consecutive and that each section is in least 1 of the sets $T(i)$. Hence, the 4-tuple $\Gamma = (N, M, C, T)$ is indeed a highway problem, and Figure 1 is its informal representation. $\diamond$

Let us refer to $\sum_{t \in M} C(t)$ as the *cost* of a highway problem (N, M, C, T). It is of course possible to allocate the cost of a highway problem without any game-theoretic model. A natural way of sharing costs in a highway problem is to share the cost of each resource section equally among the agents who use it. We define the *allocation rule* ζ by

$$\zeta_i(\Gamma) = \sum_{t \in T(i)} \frac{C(t)}{|\{j \in N \mid t \in T(j)\}|} \tag{1}$$

for every highway problem $\Gamma = (N, M, C, T)$, and every agent $i \in N$. Notice that ζ is well defined and allocates the total cost due to the fact $\cup_{i \in N} T(i) = M$. For Example 2.1, one can verify that $\zeta = (4, 5, 8, 7)$.

Let us now recall some basic game theory definitions. A *cooperative cost game with transferable utility*, or *game*, is a pair (N, c), where N is a finite set of *players*, and $c : 2^N \longrightarrow \mathbb{R}$ is the *characteristic function*, which is a mapping c that assigns to each subset $S \subseteq N$ of players a cost $c(S)$, with $c(\emptyset) = 0$. The nonempty subsets of N are called *coalitions*.

Given a highway problem $\Gamma = (N, M, C, T)$, we define an associated game (N, c) by

$$c(S) = C(T(S)) \quad \text{for all } S \subseteq N.$$

Here, for any $S \subseteq N$, the notation $T(S)$ denotes $\cup_{i \in S} T(i)$, and for any $M' \subseteq M$, the notation $C(M')$ denotes $\sum_{t \in M'} C(t)$. That is, the cost incurred by coalition S is defined as the total cost of the sections used by members of S. We say that (N, c) is the *highway game* associated with Γ.

Example 2.2. The highway game associated with the highway problem defined in Example 2.1

is

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$
$c(S)$	0	8	12	18	16	20	18	24	24	16

S	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$c(S)$	24	24	24	24	24	24

Remark 2.2. Koster et al. (2003) introduced the class of *realization problems* to study allocation of public goods. One of their approaches was to associate a *cooperative realization game* with each problem. Let $\Gamma = (N, M, C, T)$ be a highway problem. If we define $w_i = c(\{i\})$, then the 5-tuple $\Omega = (N, M, T, w, C)$ is a realization problem as defined in the aforementioned work. Moreover, the realization game (N, v) associated with Ω and the highway game associated with Γ have the relationship $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$. Thus, the *savings games* derived from highway games form a subclass of realization games. The results we present in this introductory section are, after a transformation from cost to savings game, specializations of more general results in Koster et al. (2003). $\triangleleft$

In cooperative game theory, a vector $x \in \mathbb{R}^N$ can be interpreted as the share of the cost allocated to player $i \in N$ in any game (N, c) and is called an *allocation* for (N, c). An allocation $x \in \mathbb{R}^N$ is *efficient* for (N, c) if $\sum_{i \in N} x_i = c(N)$, i.e. if the shares of the players sum up to the total cost. A *value* for a class of games is a mapping ψ that associates with each game (N, c) in the class an efficient allocation $\psi(N, c) \in \mathbb{R}^N$. A value for a class of games defines an allocation rule for a highway problem in an obvious way, provided that the class contains all highway games.

A well-known value is the *Shapley value* (Shapley, 1953), which is defined for the class of all games. The Shapley value associates with each game (N, c) the allocation $\Phi(N, c)$ defined by

$$\Phi_i(N, c) = \sum_{S \subseteq N, S \ni i} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \left(c(S) - c(S \setminus \{i\})\right) \tag{2}$$

for each player $i \in N$. Our first result relates the Shapley value with the highway allocation rule ζ defined at the beginning of this section.

Proposition 2.1. *Let Γ be a highway problem and let (N, c) be its associated highway game. Then*

$$\Phi(N, c) = \zeta(\Gamma).$$

Proof. Let $\Gamma = (N, M, C, T)$. For every $t \in M$, define the game (N, c_t) by

$$c_t(S) = \begin{cases} C(t) & \text{if } t \in T(S), \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \subseteq N$. Notice that $c = \sum_{t \in M} c_t$.

It is well-known that the Shapley value is efficient (Shapley, 1953). Further, it is clear from (2) that the Shapley value satisfies the property of symmetry: $\Phi_i(N,c) = \Phi_j(N,c)$ for $i,j \in N$ whenever $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$. It is also clear from (2) that the Shapley value satisfies the null player property: $\Phi_i(N,c) = 0$ whenever $c(S \cup \{i\}) = c(S)$ for all $S \subseteq N \setminus \{i\}$. Then,

$$\Phi_i(N,c_t) = \begin{cases} \dfrac{C(t)}{|\{j \in N \mid t \in T(j)\}|} & \text{if } t \in T(i) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the Shapley value is additive (Shapley, 1953): $\Phi(N,c) + \Phi(N,d) = \Phi(N,c+d)$ for all games $(N,c),(N,d)$. We therefore obtain for each $i \in N$

$$\Phi_i(N,c) = \sum_{t \in M} \Phi_i(N,c_t) = \sum_{t \in T(i)} \Phi_i(N,c_t) = \zeta(\Gamma).$$

□

We conclude this section with two properties of highway games that are relevant for this paper. A game (N,c) is said to be *monotone* if $c(S) \leq c(T)$ for all S,T with $S \subseteq T \subseteq N$. The game is said to be *concave* if $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for each $S,T \subseteq N$.

Proposition 2.2. *Let (N,M,C,T) be a highway problem. Then the associated game (N,c) is monotone and concave.*

Proof. Let $S \subseteq R \subseteq N$. Since $T(S) \subseteq T(R)$ and $C(t) \geq 0$ for each $t \in M$, we have that $c(S) = \sum_{t \in T(S)} C(t) \leq \sum_{t \in T(R)} C(t) = c(R)$. Therefore, (N,c) is monotone.

Let $S,R \subseteq N$. Then,

$$\begin{aligned} c(S) + c(R) &= C(T(S)) + C(T(R)) \\ &= C(T(S) \cup T(R)) + C(T(S) \cap T(R)) \\ &\geq C(T(S \cup R)) + C(T(S \cap R)) \\ &= c(S \cup R) + c(S \cap R), \end{aligned}$$

where the inequality follows since $C(t) \geq 0$ for each $t \in M$, $T(S \cup R) = T(S) \cup T(R)$ and $T(S \cap T) \subseteq T(S) \cap T(R)$. Therefore, (N,c) is concave. □

3 The nucleolus of highway games

The nucleolus is another well-known value for games. It is defined for games (N,c) that satisfy the condition $c(N) \leq \sum_{i \in N} c(\{i\})$. This condition equates to the existence of an *imputation*, i.e. an efficient allocation x satisfying $x_i \leq c(\{i\})$ for all $i \in N$. Notice that highway games satisfy the condition. If we measure the 'happiness'of a coalition S with respect to an allocation x by the number $c(S) - x(S)$, then the nucleolus is the unique imputation such that no coalition can increase its happiness without decreasing the happiness of a coalition with less happiness already.

Formally, for a game (N, c), an allocation $x \in \mathbb{R}^N$, and a coalition $S \subseteq N$, the *excess of S at x* is defined by $e(S, x) := c(S) - x(S)$. The vector in $\mathbb{R}^{2^{|N|}}$ obtained by arranging the excesses of all coalitions at x in non-decreasing order is denoted by $\theta(x)$. Further, for $\phi, \psi \in \mathbb{R}^{2^{|N|}}$, it is said that ϕ is *lexicographically greater* than ψ if $s \in \{1, \ldots, 2^{|N|}\}$ exists such that $\phi_k = \psi_k$ for each $k \in \{1, \ldots, s-1\}$ and $\phi_s > \psi_s$. If ϕ is lexicographically greater than ψ or if $\phi = \psi$, we write $\phi \geq_L \psi$. Now, the *nucleolus* $\nu(N, c)$ of (N, c) is defined (Schmeidler, 1969) as the set of imputations that lexicographically maximizes $\theta(x)$ over the set of all imputations, i.e.

$$\nu(N, c) = \{x \in \mathcal{I}(N, c) \mid \theta(x) \geq_L \theta(y) \text{ for all } y \in \mathcal{I}(N, c)\},$$

where $\mathcal{I}(N, c)$ denotes the set of imputations for (N, c). It was proved by Schmeidler (1969) that $\nu(N, c)$ consists of a unique imputation if $\mathcal{I}(N, c) \neq \emptyset$ (otherwise $\nu(N, c) = \emptyset$). If no confusion arises, we write ν for the nucleolus of a game (N, c).

3.1 The nucleolus of a concave game

For a game (N, c), let $\mathcal{D}(N, c)$ denote the family of proper coalitions of N with minimal excess at ν, i.e.

$$\mathcal{D}(N, c) = \{S \subset N \mid e(S, \nu) \leq e(T, \nu), \text{ for all } T \subset N,\ S, T \neq \emptyset\}.$$

We write $\mathcal{D}$ if no confusion can arise. A result for the nucleolus of concave games due to Arin and Iñarra (1998)[1] is crucial in our method for determining the nucleolus of highway games. In the terminology of Arin and Iñarra, a family $\mathcal{A}$ of coalitions is an *antipartition* of N if $\{N \setminus S \mid S \in \mathcal{A}\}$ is a partition of N. Their result is then stated as

Proposition 3.1 (Arin and Iñarra, 1998). *For any concave game (N, c) the family $\mathcal{D}$ contains a partition or an antipartition of N.*

Proposition 3.1 can be used to determine the nucleolus of a concave game as follows. Let (N, c) be a game, let $x \in \mathbb{R}^n$ be an eficient allocation, and let $\mathcal{A}$ be a nonempty family of coalitions of N. Define the *average excess of $\mathcal{A}$ at x* by

$$e(\mathcal{A}, x) = \frac{\sum_{S \in \mathcal{A}} e(S, x)}{|\mathcal{A}|}$$

If $\mathcal{A}$ is a partition of N then

$$e(\mathcal{A}, x) = \frac{\sum_{S \in \mathcal{A}} c(S) - c(N)}{|\mathcal{A}|},$$

[1] They stated Proposition 3.1 in terms of convex games.

and if $\mathcal{A}$ is an antipartition of N then

$$e(\mathcal{A}, x) = \frac{\sum_{S \in \mathcal{A}} c(S) - (|\mathcal{A}| - 1)c(N)}{|\mathcal{A}|}.$$

Notice that in both cases, the average excess does not depend on x. It follows by Proposition 3.1 that

$$e(\mathcal{D}, \nu) = \min\{e(\mathcal{A}, \nu) \mid \mathcal{A} \text{ is a partition or an antipartition of } N\}.$$

Since the numbers $e(\mathcal{A}, \nu)$ in this minimization do not depend on ν, it is possible to determine $e(\mathcal{D}, \nu)$ as well as a partition or antipartition $\mathcal{A}$ contained in it, simply by enumerating all partitions and antipartitions of N. In general, this is impractical because of the huge number of partitions and antipartitions. However, if the partitions and antipartitions must come from only a small family of *relevant* coalitions, it can be an efficient method. Such is the case for highway games.

In the following we will give a refinement of Proposition 3.1 for a subclass of concave games. A coalition $S \subseteq N$ is said to be *essential* if there exists no nontrivial partition $\mathcal{P}$ of S such that $c(S) \geq \sum_{R \in \mathcal{P}} c(R)$. A coalition $S \subset N$ is said to be *saturated* if there exists no coalition R such that $S \subset R$ and $c(S) \geq c(R)$. A coalition $S \neq \emptyset, N$ is said to be *relevant* if it is both essential and saturated. We denote by $\mathcal{RC}(N, c)$ the set of relevant coalitions for a game (N, c), and we define $\overline{\mathcal{RC}}(N, c) = \mathcal{RC}(N, c) \cup \{N \setminus \{i\} \mid i \in N\}$. If no confusion arises, we also write $\mathcal{RC}$ and $\overline{\mathcal{RC}}$.

Lemma 3.1. *Let (N, c) be a concave game such that N is essential, and such that $c(N \setminus \{i\}) \leq c(N)$ for all $i \in N$. Then*

- *(A) $e(S, \nu) > 0$ for all $S \neq \emptyset, N$,*
- *(B) $\nu_i > 0$ for all $i \in N$,*
- *(C) $\mathcal{D} \subseteq \overline{\mathcal{RC}}$.*

Proof. In order to prove (A), note that for every $S \subseteq N$ a core element x^S exists such that $x^S(N \setminus S) = c(N \setminus S)$. Namely, since (N, c) is concave, it suffices to take any marginal vector corresponding to a permutation where the players of $N \setminus S$ precede the players of S. Then, for each coalition S, we have $x^S(S) = c(N) - c(N \setminus S) < c(S)$, where the strict inequality follows because N is essential and $S \neq \emptyset, N$. Now, define x as the average over all vectors x^S with $S \neq \emptyset, N$. Then $e(S, x) > 0$ for all $S \neq \emptyset, N$ since $e(S, x)$ is the average of nonnegative numbers with one of them, $e(S, x^S)$, strictly positive. Since the minimum excess is maximized at the nucleolus, it follows that

$$\min\{e(S, \nu) \mid S \neq \emptyset, N\} \geq \min\{e(S, x) \mid S \neq \emptyset, N\} > 0.$$

Hence, $e(S, \nu) > 0$ for all $S \neq \emptyset, N$.

Part (B) of the lemma follows because

$$v_i = c(N) - v(N \setminus \{i\}) \geq c(N \setminus \{i\}) - v(N \setminus \{i\}) = e(N \setminus \{i\}, v) > 0.$$

Finally, we prove (C). Let $S \in \mathcal{D}$, and suppose $|S| < |N| - 1$. We prove that $S \in \mathcal{RC}$ by contradiction. So assume that S is not essential or not saturated. If S is not essential, then a nontrivial partition $\mathcal{P}$ of S exists such that $c(S) \geq \sum_{P \in \mathcal{P}} c(P)$. It follows that $e(S, v) \geq \sum_{P \in \mathcal{P}} e(P, v)$. Together with the fact $e(S, v) > 0$ for all $S \neq \emptyset, N$, this implies $e(P, v) < e(S, v)$ for all $P \in \mathcal{P}$, which contradicts $S \in \mathcal{D}$. If S is not saturated, then $R \supset S$ exists such that $c(S) \geq c(R)$. We claim that we may choose $R \subset N$. Indeed, if $R = N$, then we may also choose $R = N \setminus \{i\}$ for any $i \notin S$. This follows, since $c(S) \geq c(R) = c(N) \geq c(N \setminus \{i\})$ for all $i \notin S$, and since $S \subset N \setminus \{i\}$ for all $i \notin S$. Now, $v > 0$ implies that $e(R, v) < e(S, v)$, which contradicts $S \in \mathcal{D}$. □

The following corollary is a direct consequence of Proposition 3.1 and Lemma 3.1(C).

Corollary 3.1. *Let (N, c) be a concave game such that N is essential, and such that $c(N \setminus \{i\}) \leq c(N)$ for all $i \in N$. Then $\mathcal{D} \cap \overline{\mathcal{RC}}$ contains a partition or an antipartition of N.*

3.2 The minimal excess at the nucleolus of a highway problem

In the class of highway games the relevant coalitions can be easily identified. Let (N, c) be a highway game associated with a highway problem $\Gamma = (N, M, C, T)$. Then coalition $S \subset N$ is saturated if $S = \{i \in N \mid \min T(S) \leq \min T(i) \leq \max T(i) \leq \max T(S)\}$, and it is essential if no partition $\{R, R'\}$ of S exists such that $T(R) \cap T(R') = \emptyset$. Notice that there can be at most $\frac{1}{2}|M|(|M| + 1) - 1$ relevant coalitions for a highway game, since $\frac{1}{2}|M|(|M| + 1)$ is the number of possible combinations for $\min T(S)$ and $\max T(S)$, and since by definition N cannot be relevant.

Remark 3.1. In a highway problem we say that two agents are of the same type, if they use exactly the same sections. In the highway game agents of the same type correspond to symmetric players, and the nucleolus allocation is the same for symmetric players. As a consequence, the complexity of the algorithm presented in this paper does not depend on the number of agents, but only on the number of types of agents. ◁

By Corollary 3.1, in order to compute the minimal excess at the nucleolus of (N, c), one only has to calculate the average excess of all the partitions and antipartitions of N contained in $\overline{\mathcal{RC}}$, provided that N is essential. Let us say that the highway problem Γ is *decomposable* if a nontrivial partition $\{L, R\}$ of N exists, such that $\{T(L), T(R)\}$ forms a partition of M. Observe that N is not essential in (N, c) if Γ is decomposable, since then $c(N) = C(T(N)) = C(T(L)) + C(T(R)) = c(L) + c(R)$. Conversely, if N is not essential in (N, c), then a partition $\{L, R\}$ of N exists such that $c(N) = c(L) + c(R)$, which is only possible if L and R are such that

$\{T(L), T(R)\}$ is a partition of M. Observe also that the coalitions L and R in the partition of N of a decomposable highway problem must be relevant coalitions. Thus, we have the following corollary.

Corollary 3.2. *Let $\Gamma = (N, M, C, T)$ be a highway problem and let (N, c) be its associated highway game. Then $\mathcal{D} \cap \overline{\mathcal{RC}}$ contains a partition or an antipartition of N.*

Proof. If Γ is not decomposable, then N is essential in (N, c). Further, (N, c) is monotone according to Lemma 2.2, hence it satisfies $c(N \setminus \{i\}) \leq c(N)$ for all $i \in N$. In this case, the result follows from Corollary 3.1. If Γ is decomposable, then a partition $\{L, R\}$ of N exists with $L, R \in \mathcal{RC}$ such that $\{T(L), T(R)\}$ forms a partition of M, hence $c(N) = c(L) + c(R)$. Further, (N, c) is concave according to Lemma 2.2, hence the core is nonempty, which implies that $e(S, v) \geq 0$ for all $S \subseteq N$. Since $c(N) = c(L) + c(R)$, we must have $e(L, v) = e(R, v) = 0$. Hence $\{L, R\} \subseteq \mathcal{D} \cap \overline{\mathcal{RC}}$, and the result also follows. □

In fact, as the next results show, for highway games we only need to consider antipartitions in $\overline{\mathcal{RC}}$.

Lemma 3.2. *Let $\Gamma = (N, M, C, T)$ be a highway problem and let (N, c) be its associated highway game. Then $\mathcal{D} \cap \overline{\mathcal{RC}}$ contains an antipartition of N.*

Proof. If Γ is decomposable, then, as argued in the proof of Corollary 3.2, a partition $\{L, R\}$ of N exists with $\{L, R\} \subseteq \mathcal{D} \cap \overline{\mathcal{RC}}$. Since $\{L, R\}$ is also an antipartition, this proves the lemma in this case. Assume further that Γ is not decomposable.

Let $\mathcal{P}$ be a partition of N and define its associated antipartition by $\mathcal{A}_{\mathcal{P}} = \{S^c \mid S \in \mathcal{P}\}$. We will prove that $e(\mathcal{A}_{\mathcal{P}}, v) \leq e(\mathcal{P}, v)$. We know that

$$\begin{aligned} e(\mathcal{A}_{\mathcal{P}}, v) &= \frac{\sum_{S \in \mathcal{A}_{\mathcal{P}}} c(S) - \big(|\mathcal{A}_{\mathcal{P}}| - 1\big) c(N)}{|\mathcal{A}_{\mathcal{P}}|} = \frac{\sum_{S \in \mathcal{P}} c(S^c) - \big(|\mathcal{P}| - 1\big) c(N)}{|\mathcal{P}|}, \\ e(\mathcal{P}, v) &= \frac{\sum_{S \in \mathcal{P}} c(S) - c(N)}{|\mathcal{P}|}. \end{aligned}$$

So, we have to prove that

$$\sum_{S \in \mathcal{P}} c(S^c) - \sum_{S \in \mathcal{P}} c(S) \leq \big(|\mathcal{P}| - 2\big) c(N). \tag{3}$$

For $t \in M$, denote by η_t the number of coalitions $S \in \mathcal{P}$ such that $t \in T(S^c)$, and by ζ_t the number of coalitions $S \in \mathcal{P}$ such that $t \in T(S)$. Then

$$\sum_{S \in \mathcal{P}} c(S^c) - \sum_{S \in \mathcal{P}} c(S) = \sum_{t \in M} \eta_t C(t) - \sum_{t \in M} \zeta_t C(t) = \sum_{t \in M} (\eta_t - \zeta_t) C(t).$$

Thus, it sufficies to prove that $\eta_t - \zeta_t \leq |\mathcal{P}| - 2$ for all $t \in M$. So let $t \in M$. Clearly, $\eta_t \leq |\mathcal{P}|$, so we have nothing to prove if $\zeta_t \geq 2$. Assume therefore that $\zeta_t < 2$. Then $\zeta_t = 1$, and there is precisely one $\hat{S} \in \mathcal{P}$ with $t \in T(\hat{S})$. Since $\hat{S}^c = \cup_{S \in \mathcal{P} \setminus \{\hat{S}\}} S$, it follows that $t \notin T(\hat{S}^c)$, and we deduce that $\eta_t \leq |\mathcal{P}| - 1$. Hence, $\eta_t - \zeta_t = \eta_t - 1 \leq |\mathcal{P}| - 2$. So indeed, $e(\mathcal{A}_\mathcal{P}, v) \leq e(\mathcal{P}, v)$

By Proposition 3.1 we know that $\mathcal{D}$ contains a partition or an antipartition. If $\mathcal{D}$ contains a partition, say $\mathcal{P}$, then $\mathcal{D}$ also contains the antipartition $\mathcal{A}_P$, since $e(\mathcal{A}_\mathcal{P}, v) \leq e(\mathcal{P}, v)$. We can therefore conclude that $\mathcal{D}$ contains an antipartition. Since we assume that Γ is not decomposable, Lemma 3.1(C) applies, hence the antipartition is also contained in $\overline{\mathcal{RC}}$, which proves the lemma. □

Proposition 3.2. *Let $\Gamma = (N, M, C, T)$ be a highway problem and let (N, c) be its associated highway game. At least one of the following statements is true.*

(A) There exist $L, R \in \mathcal{RC}$ with $L \cup R = N$, such that $\{L, R\} \cup \{N \setminus \{i\} \mid i \in L \cap R\} \subseteq \mathcal{D}$.

(B) There exists $S \in \mathcal{RC}$ such that $\{S\} \cup \{N \setminus \{i\} \mid i \in S\} \subseteq \mathcal{D}$.

(C) $\{N \setminus \{i\} \mid i \in N\} \subseteq \mathcal{D}$.

Proof. By Lemma 3.2, $\mathcal{D} \cap \overline{\mathcal{RC}}$ contains an antipartition of N. Let $\mathcal{A} = \{A_1, \ldots, A_k\} \subseteq \mathcal{D} \cap \overline{\mathcal{RC}}$ be such an antipartition. Then $A_\ell \in \mathcal{RC}$ or $|A_\ell| = n - 1$ for each $\ell \in \{1, \ldots, k\}$. Let $r = |\{S \in \mathcal{A} \mid S \in \mathcal{RC}\}|$. The cases (A), (B) and (C) correspond with $r = 2$, $r = 1$ and $r = 0$, respectively. Then, we have to prove that $r \leq 2$.

It will be proven by contradiction. Suppose that $r \geq 3$. Then let $A_1, A_2, A_3 \in \mathcal{RC}$ and assume w.l.o.g. that $\min M \in T(A_1)$ and $\max M \in T(A_2)$. Since $A_1, A_2 \in \mathcal{RC}$, it follows that $\max T(A_1) < \max M$ and $\min T(A_2) > \min M$, hence $\min M \notin T(A_1) \cap T(A_2)$ and $\max M \notin T(A_1) \cap T(A_2)$.

By definition of an antipartition $A_3^c \subseteq A_1 \cap A_2$. Then, $T(A_3^c) \subseteq T(A_1) \cap T(A_2)$. It follows that $M = T(A_3) \cup T(A_3^c) \subseteq T(A_3) \cup (T(A_1) \cap T(A_2))$. Then, $\min M \in T(A_3)$ and $\max M \in T(A_3)$, which contradicts that $A_3 \in \mathcal{RC}$. □

Notice that two coalitions $L, R \in \mathcal{RC}$ can only have the property $L \cup R = N$ if either L or R contains an agent that uses section $\min M$ and if the other coalition contains an agent that uses section $\max M$. We can assume without loss of generality that $\min T(L) = \min M$ and $\max T(R) = \max M$. Then, only $\mathcal{O}(|M|^2)$ combinations are possible in Proposition 3.2 (A). The number of possibilities in Proposition 3.2 (B) is of the same order, hence the minimal excess at the nucleolus of a highway game can be determined by comparing $\mathcal{O}(|M|^2)$ different values.

Let us define,

$$\begin{aligned}
\beta(L, R) &= \frac{c(L) + c(R) + \sum_{i \in L \cap R} c(N \setminus \{i\}) - (|L \cap R| + 1) c(N)}{|L \cap R| + 2} && \text{for } L, R \in \mathcal{RC}, \\
\gamma(S) &= \frac{c(S) + \sum_{i \in S} c(N \setminus \{i\}) - (|S|) c(N)}{|S| + 1} && \text{for } S \in \mathcal{RC}, \\
\delta &= \frac{\sum_{i \in N} c(N \setminus \{i\}) - (|N| - 1) c(N)}{|N|}.
\end{aligned}$$

Define further

$$\begin{array}{rcl} \beta & = & \min\{\beta(L,R) \mid L,R \in \mathcal{RC} \text{ with } L \cup R = N\}, \\ \gamma & = & \min\{\gamma(S) \mid S \in \mathcal{RC}\}, \\ \lambda & = & \min\{\beta,\gamma,\delta\}. \end{array}$$

We then have

Corollary 3.3. *Let Γ be a highway problem and let (N,c) be its associated game. Then $e(\mathcal{D},\nu) = \lambda$. Moreover:*

(A) If $\lambda = \beta = \beta(L,R)$ for $L,R \in \mathcal{RC}$ with $L \cup R = N$, then $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for all $i \in L \cap R$, $\nu(L) = c(L) - \lambda$, and $\nu(R) = c(R) - \lambda$.

(B) If $\lambda = \gamma = \gamma(S)$ for $S \in \mathcal{RC}$, then $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for all $i \in S$ and $\nu(S) = c(S) - \lambda$.

(C) If $\lambda = \delta$, then $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for all $i \in N$.

Proof. It is a matter of verification to see that

$$\begin{array}{rcll} \beta(L,R) & = & e(\{L,R\} \cup \{N \setminus \{i\} \mid i \in L \cap R\},\nu) & \text{if } L,R \in \mathcal{RC} \text{ with } L \cup R = N, \\ \gamma(S) & = & e(\{S\} \cup \{N \setminus \{i\} \mid i \in S\},\nu) & \text{if } S \in \mathcal{RC}, \\ \delta & = & e(\{N \setminus \{i\} \mid i \in N\},\nu). & \end{array}$$

Then, by Proposition 3.2, the minimum of these numbers corresponds to an antipartition in $\mathcal{D}$. If the minimum is found in (A), then coalitions L and R are in $\mathcal{D}$, hence $\nu(S) = c(S) - \lambda$. Also, $N \setminus \{i\} \in \mathcal{D}$, hence $\nu_i = c(N) - \nu(N \setminus \{i\}) = c(N) - c(N \setminus \{i\}) + \lambda$ for all $i \in L \cap R$. The cases (B) and (C) are proved similarly. □

If $\lambda = \delta$ we immediately obtain the nucleolus for all agents. If $\lambda = \gamma < \delta$, we obtain the value of the nucleolus for agents in a proper coalition $S \in \mathcal{RC}$. To obtain the nucleolus for the remaining agents in S^c, it is possible to formulate a reduced highway problem with agent set S^c, and repeat to the procedure on this smaller problem. If $\lambda = \beta < \min(\gamma,\delta)$, we obtain the value of the nucleolus for agents in a coalition of the form $L \cap R$ with $L,R \in \mathcal{RC}$ and $L \cup R = N$. To obtain the nucleolus for the remaining agents we formulate two reduced highway problems, one for the agents in L^c, and another one for the agents in R^c.

3.3 The reduced highway problem

Let $\Gamma = (N,M,C,T)$ be a highway problem. In the following, we will formulate, for $Z \in \mathcal{RC}$ and $\pi \in [0, \frac{1}{2}c(Z)]$, a reduced highway problem $\Gamma_{\pi,Z} = (N_{\pi,Z}, M_{\pi,Z}, C_{\pi,Z}, T_{\pi,Z})$ with agent set $N_{\pi,Z} = Z^c$. Sections outside $T(Z)$ remain in the reduced problem, and sections in $T(Z)$ are replaced by a set of m new sections $NS = \{t_1, t_2, \ldots, t_m\}$, disjoint from M. We thus have $M_{\pi,Z} = M \setminus T(Z) \cup NS$. The new sections of NS form a consecutive set, ordered $t_1 < t_2 <$

$\ldots < t_m$, and are placed where the consecutive set of sections of $T(Z)$ were removed. The number m and the cost of the new sections is determined by the set of agents who, in the original highway problem, use sections of $T(Z)$ with positive cost, but with a total cost less than π. We denote this set of agents by N^*.

For every $i \in N_{\pi,Z}$, we define $g_i = \min\{\pi, C(T(i) \cap T(Z))\}$. Then $N^* = \{i \in N_{\pi,Z} \mid 0 < g_i < \pi\}$. We distinguish between agents in N^* whose first section belongs to $T(Z)$ and agents in N^* whose last section belongs to $T(Z)$. We define

$$\begin{aligned} N^*_\ell &= \{i \in N^* \mid \max T(i) \in T(Z)\} \\ N^*_r &= \{i \in N^* \mid \min T(i) \in T(Z)\}. \end{aligned}$$

As the next lemma shows, the sets N^*_ℓ and N^*_r together contain all agents in N^*, and the sets do not overlap.

Lemma 3.3. *The sets N^*_ℓ and N^*_r form a partition of N^*.*

Proof. We first prove by contradiction that $N^*_\ell \cap N^*_r = \emptyset$. Choose $i \in N^*$ and suppose $i \in N^*_\ell \cap N^*_r$. Then $\max T(i) \in T(Z)$ and $\min T(i) \in T(Z)$, implying $T(i) \subseteq T(Z)$. Since Z is saturated, it then follows that $i \in Z$, contradicting the choice $i \in N^* \subseteq Z^c$. We next prove that $N^*_\ell \cup N^*_r = N^*$. Choose $i \in N^*$ and suppose that $i \notin N^*_\ell$ and $i \notin N^*_r$. Then $\min T(i) \notin T(Z)$ and $\max T(i) \notin T(Z)$, implying $T(i) \cap T(Z) = \emptyset$ or $T(i) \supseteq T(Z)$. The case $T(i) \cap T(Z) = \emptyset$ contradicts that $g_i > 0$, and the case $T(i) \supseteq T(Z)$ contradicts that $g_i < \pi$. □

Define, for all $i \in N^*$,

$$\varsigma_i = \begin{cases} g_i & \text{if } i \in N^*_\ell, \\ \pi - g_i & \text{if } i \in N^*_r. \end{cases}$$

Now, the number m, i.e. the number of new sections, is the number of different values ς_i plus 1. Let $\xi \in \mathbb{R}^m_{++}$ be the vector of the $m-1$ different values ς_i and the number π, arranged in strictly increasing order. Notice $\xi_m = \pi$, since $\varsigma_i < \pi$ for all $i \in N^*$. We define the cost function $C_{\pi,Z}$ by

$$C_{\pi,Z}(t) = \begin{cases} C(t) & \text{if } t \in M \setminus T(Z), \\ \xi_1 & \text{if } t = t_1, \\ \xi_k - \xi_{k-1} & \text{if } t = t_k \text{ with } 1 < k \leq m. \end{cases}$$

For $i \in N^*$, we denote by $k(i)$ the unique index $k \in \{1, \ldots, m-1\}$ such that $\varsigma_i = \xi_k$. We then define the mapping $T_{\pi,Z} : N_{\pi,Z} \to 2^{M_{\pi,Z}}$ by

$$T_{\pi,Z}(i) = \begin{cases} T(i) & \text{if } g_i = 0, \\ \big(T(i) \setminus T(Z)\big) \cup NS & \text{if } g_i = \pi \\ \big(T(i) \setminus T(Z)\big) \cup \{t_1, \ldots, t_{k(i)}\} & \text{if } i \in N_\ell^*, \\ \big(T(i) \setminus T(Z)\big) \cup \{t_{k(i)+1}, \ldots, t_m\} & \text{if } i \in N_r^*, \end{cases}$$

for all $i \in N_{\pi,Z}$. Notice that the definition ensures that, for each $i \in N_{\pi,Z}$, $T_{\pi,Z}(i)$ contains a consecutive set of sections of $M_{\pi,Z}$. Therefore, $\Gamma_{\pi,Z} = (N_{\pi,Z}, M_{\pi,Z}, C_{\pi,Z}, T_{\pi,Z})$ is a valid highway problem.

For the following lemma, the choice $\pi \leq \frac{1}{2}c(Z)$ is needed.

Lemma 3.4. *$T(N_\ell^*) \cap T(N_r^*) = \emptyset$ and $T(N_\ell^*) \cup T(N_r^*)$ does not contain every section of $T(Z)$.*

Proof. We first prove that $T(i) \cap T(j) = \emptyset$ for all $i \in N_\ell^*$ and $j \in N_r^*$. Suppose to the contrary that $i \in N_\ell^*$ and $j \in N_r^*$ exist with $T(i) \cap T(j) \neq \emptyset$. Then $T(Z) \subseteq T(i) \cup T(j)$, and we obtain

$$g_i + g_j = C(T(i) \cap T(Z)) + C(T(j) \cap T(Z)) \geq C(T(Z)) = c(Z) \geq 2\pi,$$

which contradicts that $g_i < \pi$ and $g_j < \pi$.

So we have indeed $T(i) \cap T(j) = \emptyset$ for all $i \in N_\ell^*$ and $j \in N_r^*$. The claim $T(N_\ell^*) \cap T(N_r^*) = \emptyset$ immediately follows. It also follows that

$$C(T(Z) \cap T(N^*)) = \max\{g_i \mid i \in N_\ell^*\} + \max\{g_i \mid i \in N_r^*\} < 2\pi \leq C(T(Z)).$$

The strict inequality implies $T(Z) \not\subseteq T(N_\ell^*) \cup T(N_r^*)$. □

Corollary 3.4. *The number of sections of the reduced highway problem $\Gamma_{\pi,Z}$ does not exceed the number of sections of Γ.*

Proof. Note that the number of new sections in the reduced highway problem is bounded by $1 + |T(N^*) \cap T(Z)|$. By Lemma 3.4, $T(N^*) \cap T(Z)$ is a proper subset of $T(Z)$, hence $1 + |T(N^*) \cap T(Z)| \leq |T(Z))|$. Thus, the number of new sections is at most the number of replaced sections, and the claim of the corollary follows. □

Proposition 3.3. *Let $\Gamma = (N, M, C, T)$ be a highway problem with associated highway game (N, c). Let further $Z \in \mathcal{RC}$ and $\pi \in [0, \frac{1}{2}c(Z)]$. Then, the characteristic function $c_{\pi,Z}$ of the highway game $(N_{\pi,Z}, c_{\pi,Z})$ associated with the reduced highway problem $\Gamma_{\pi,Z}$ is given by*

$$c_{\pi,Z}(S) = \min\{c(S), \pi + C(T(S) \setminus T(Z))\}$$

for all $S \subseteq N$.

Proof. Let $S \subseteq N_{\pi,Z}$.

Assume first that (i) $g_i < \pi$ for all $i \in S$ and that (ii) $g_i + g_j < \pi$ for all $i \in S \cap N_\ell^*$ and $j \in S \cap N_r^*$. We then claim that at least one of the sections in NS is not used by any of the agents of S in the reduced highway problem. This is obvious if $S \cap N_\ell^* = \emptyset$, since then t_1 is not used, and it is also obvious if $S \cap N_r^* = \emptyset$, since then t_m is not used. If $S \cap N_\ell^* \neq \emptyset$ and $S \cap N_r^* \neq \emptyset$, then $\xi_{k(i)} = g_i < \pi - g_j = \xi_{k(j)}$, hence $k(i) < k(j)$, for all $i \in S \cap N_\ell^*$, $j \in S \cap N_r^*$. It follows that section t_k, where $k = \min\{k(j) \mid j \in S \cap N_r^*\}$, is not used by any of the agents in S. Then, the agent set S can be partitioned into $\{S_\ell, S_r\}$, where S_ℓ consists of the agents that only use sections smaller than the unused section, and where S_r consists of the agents that only use sections larger than the unused section. Note then that also $\{T_{\pi,Z}(S_\ell), T_{\pi,Z}(S_r)\}$ forms a partition of $T_{\pi,Z}(S)$. Moreover, $T(S_\ell) \cap T(S_r) = \emptyset$ by Lemma 3.4, implying that $\{T(S_\ell), T(S_r)\}$ forms a partition of $T(S)$. Therefore, conditions (i) and (ii) imply that

$$\begin{aligned} &c_{\pi,Z}(S) = C_{\pi,Z}(T_{\pi,Z}(S)) = C_{\pi,Z}(T_{\pi,Z}(S_\ell)) + C_{\pi,Z}(T_{\pi,Z}(S_r)) = \\ &C(T(S_\ell)) + C(T(S_r)) = C(T(S)) = c(S). \end{aligned}$$

Conditions (i) and (ii) also imply that

$$C(T(S) \cap T(Z)) = \max\{g_i \mid i \in S_\ell\} + \max\{g_i \mid i \in S_r\} < \pi,$$

hence $c(S) = C(T(S) \cap T(Z)) + C(T(S) \setminus T(Z)) < \pi + C(T(S) \setminus T(Z))$. This proves that the claim of the proposition holds if conditions (i) and (ii) hold.

Now assume that condition (i) does not hold, i.e. assume that $i \in S$ exists with $g_i = \pi$. Then, in the reduced highway problem, agent i uses all new sections, hence $NS \subseteq T_{\pi,Z}(S)$. It follows that $T_{\pi,Z}(S) = NS \cup T(S) \setminus T(Z)$ and that $c_{\pi,Z}(S) = \pi + C(T(S) \setminus T(Z))$. Moreover, $c(S) = C(T(S) \cap T(Z)) + C(T(S) \setminus T(Z)) \geq \pi + C(T(S) \setminus T(Z))$, which proves that the claim of the proposition holds in this case as well.

Finally assume that condition (ii) does not hold, i.e. assume that $i \in S \cap N_\ell^*$ and $j \in S \cap N_r^*$ exist with $g_i + g_j \geq \pi$. Then $\xi_{k(i)} = g_i \geq \pi - g_j = \xi_{k(j)}$, hence $k(i) \geq k(j)$. Since, in the reduced problem, i uses the sections $t_1, \ldots, t_{k(i)}$ and j uses $t_{k(j)+1}, \ldots, t_m$, it follows that $NS \subseteq T_{\pi,Z}(i) \cup T_{\pi,Z}(j)$. Then $NS \subseteq T_{\pi,Z}(S)$, and $T_{\pi,Z}(S) = NS \cup T(S) \setminus T(Z)$. It follows that $c_{\pi,Z}(S) = \pi + C(T(S) \setminus T(Z))$. Moreover, $c(S) = C(T(S) \cap T(Z)) + C(T(S) \setminus T(Z)) \geq g_i + g_j + C(T(S) \setminus T(Z)) \geq \pi + C(T(S) \setminus T(Z))$. This proves the claim of the proposition for this final case. □

The next proposition provides a formula for the Davis-Maschler reduced game. In the following we denote by $(Q^c, c^{x,Q})$ the Davis-Maschler reduced game of a game (N, c), where $Q \subseteq N$ is the player set that left the game, and where x is an allocation for (N, c).

Proposition 3.4. *Let (N, M, C, T) be a highway problem, let x be a core element of the associated highway game (N, c), and let $Z \in \mathcal{D}(x) := \{S \subset N \mid e(S, x) \leq e(T, x)$ for all $T \subset N$, $S, T \neq \emptyset\}$.*

Then, the characteristic function $c^{x,Z}$ of the Davis-Maschler reduced game is given by

$$c^{x,Z}(S) = \min\{c(S), \pi + C(T(S) \setminus T(Z))\} \quad \textit{for all } S \subseteq N,$$

where $\pi = c(Z) - x(Z)$.

Proof. Since (N, c) is a concave game, x is a core element of the game, and $Z \in \mathcal{D}(x)$, Lemma 5.7 in Maschler et al. (1972), restated for concave games, says that,

$$c^{x,Z}(S) = \min\{c(S), c(S \cup Z) - x(Z)\}.$$

Moreover

$$c(S \cup Z) - x(Z) = c(Z) - x(Z) + c(S \cup Z) - c(Z) = c(Z) - x(Z) + C(T(S) \setminus T(Z)).$$

The proposition follows. □

The formulas for the reduced highway game and the Davis-Maschler reduced game look the same, but the conditions under which they hold are different. The next corollary shows that both conditions are satisfied if the reduction is with respect to the nucleolus ν and a coalition in $\mathcal{D} \cap \mathcal{RC}$.

Corollary 3.5. *Let (N, M, C, T) be a highway problem with associated game (N, c), and let $Z \in \mathcal{D} \cap \mathcal{RC}$. Then $c^{\nu,Z} = c_{\pi,Z}$, where $\pi = c(Z) - \nu(Z)$.*

Proof. Since ν is a core element and $Z \in \mathcal{D}$, Proposition 3.4 applies. For Proposition 3.3, the condition $Z \in \mathcal{RC}$ is satisfied, but it remains to verify that $\pi \in [0, \frac{1}{2}c(Z)]$.

Since $Z \in \mathcal{D}$, the results from the Section 3.2 show that we have one of the following three possibilities: $\pi = \gamma(Z)$, $\pi = \beta(Z, R)$, or $\pi = \beta(L, Z)$.
If $\pi = \gamma(Z)$, then

$$\pi = \frac{c(Z) + \sum_{i \in Z} c(N \setminus \{i\}) - |Z| c(N)}{1 + |Z|} \leq \frac{c(Z)}{2}.$$

If $\pi = \beta(Z, R)$, then

$$\pi = \frac{c(Z) + c(R) + \sum_{i \in Z \cap R} c(N \setminus \{i\}) - (1 + |Z \cap R|) c(N)}{2 + |Z \cap R|} \leq \frac{c(Z) + c(R) - c(N)}{2} \leq \frac{1}{2} c(Z).$$

The case $\pi = \beta(L, Z)$ is similar to the case $\pi = \beta(R, Z)$. Since both Propositions 3.4 and 3.3 apply, we have indeed $c^{\nu,Z} = c_{\pi,Z}$. □

3.4 Algorithm for computing the nucleolus of a highway game

The results from sections 3.2 and 3.3 suggest a recursive approach for calculating the nucleolus of a highway game. We present the algorithm in the form of a function $\nu = \texttt{nucleolus}(\Gamma)$, where the function `nucleolus` takes its input from an arbitrary highway problem Γ and delivers its output in ν, the nucleolus of the associated highway game. In the algorithm, we use the convention $\min \emptyset = \infty$.

```
function ν = nucleolus(Γ);
    β = min{β(L,R) | L,R ∈ RC and L ∪ R = N};
    γ := min{γ(S) | S ∈ RC};
    λ := min(β,γ,δ);
    if λ = ∞ then
        ν := c(N);
    else if λ = δ then
        ν_i := c(N) − c(N \ {i}) + λ for all i ∈ N;
    else if λ = γ then
        choose S ∈ RC such that λ = γ(S);
        ν_i := c(N) − c(N \ {i}) + λ for all i ∈ S;
        ν_{N\S} := nucleolus(Γ_{λ,S});
    else if λ = β then
        choose L,R ∈ RC such that λ = β(L,R) and L ∪ R = N;
        ν_i := c(N) − c(N \ {i}) + λ for all i ∈ L ∩ R;
        ν_{N\L} := nucleolus(Γ_{λ,L});
        ν_{N\R} := nucleolus(Γ_{λ,R});
    end;
end;
```

The correctness of the algorithm relies on results established in the Corollaries 3.3 and 3.5.

Corollary 3.6. *The function* `nucleolus` *calculates the nucleolus of any highway problem in finite time.*

Proof. The proof is by induction on the cardinality of the agent set of the highway problem.

If there is only one agent, then $\beta = \gamma = \delta = \infty$. Hence $\lambda = \infty$, and the function immediately returns the trivial correct answer. Now let $\Gamma = (N, M, C, T)$ be a highway problem with $|N| = n > 1$ and assume that the function `nucleolus` returns the nucleolus in finite time, whenever it is given an input highway problem with less than n agents. Then $\lambda \leq \delta < \infty$, so we need to distinguish three cases.

Case 1: $\lambda = \delta$. By Corollary 3.3(C), we have $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for all $i \in N$, which shows that the function indeed returns the nucleolus.

Case 2: $\lambda = \gamma$. By Corollary 3.3(B), we have $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for $i \in S$, which shows that the function returns the nucleolus for the players in S. For the players in $N \setminus S$, the function calls itself with the reduced highway problem $\Gamma_{\lambda,S}$ as input. By induction, this returns

$\nu(\Gamma_{\lambda,S})$, the nucleolus of the reduced highway problem, in finite time. By Corollary 3.5, this is indeed the nucleolus for the players in $N \setminus S$.

Case 3: $\lambda = \beta$. By Corollary 3.3(A), we have $\nu_i = c(N) - c(N \setminus \{i\}) + \lambda$ for $i \in L \cap R$, so the function returns the nucleolus for the players in $L \cap R$. For the players in $N \setminus L$ and $N \setminus R$, there are a recursive calls of the function with the reduced highway problems $\Gamma_{\lambda,L}$ and $\Gamma_{\lambda,R}$ as input respectively. The argument that this returns the nucleolus for these players is similar to Case 2. □

For the following proposition we assume that the function `nucleolus` calculates the nucleolus payoff for each *type* of agent, not for each individual agent. This will make the complexity independent of the number of agents.

Proposition 3.5. *The function* `nucleolus` *runs in* $\mathcal{O}(m + m^3k + k^2)$ *time, where m is the number of sections of the input highway problem, and k is the number of relevant coalitions.*

Proof. The computation of one number $\beta(L, R)$ can be done in $\mathcal{O}(m)$ time: it requires one division and at most one addition or subtraction per section. Since there are $\mathcal{O}(m^2)$ possible combinations for L and R, the computation of all numbers $\beta(L, R)$ can certainly be done in $\mathcal{O}(m^3)$ time. Simililarly, the computation of each number $\gamma(S)$ takes $\mathcal{O}(m)$ time, and since there are k possibilities for S, the computation of all numbers $\gamma(S)$ can be done in $\mathcal{O}(mk)$ time. Other calculations in the body of the function, the explicit ones like the computation of δ, and also the implicit ones like calculating the data of a reduced game, are also easily seen to require at most $\mathcal{O}(m^3 + mk)$ work. Thus, the function runs in $\mathcal{O}(m^3 + mk)$ time plus the time needed for the recursive calls. This means we can choose K such that the number of numerical operations, not in the recursive calls, is bounded by $K(m^3 + mk)$. We now prove by induction on k that the number of numerical operations in the function `nucleolus`, including operations inside the recursive calls, is bounded by $K(m + m^3k + mk^2)$.

If $k = 0$, then $\beta = \gamma = \infty$. Then no recursive calls are made, no numbers $\beta(L, R)$ and $\gamma(S)$ are computed, and the number of numerical operations is linear in m, say it is bounded by Cm. We may assume without loss of generality that $K \geq C$, and the claim that the number of operations is bounded by $K(m + m^3k + mk^2)$ follows. Now, let $k > 0$ and assume the claim is true for all integers smaller than k. If $\lambda = \delta$, no recursive call is made, and the claim is trivially true. So assume $\lambda = \beta$ or $\lambda = \gamma$.

If $\lambda = \gamma = \gamma(S)$, there will be one recursive call of the function. Let m_S, k_S denote the number of sections and relevant coalitions respectively of the reduced highway problem $\Gamma_{\lambda,S}$. Since a relevant coalition in Γ can correspond to at most one relevant coalition in $\Gamma_{\lambda,S}$ and since S itself does not correspond to a relevant coalition in the reduced problem, we have $k_S \leq k - 1$. Also $m_S \leq m$ by Corollary 3.4. By induction, the amount of numerical operations in the recursive call is bounded by $K(m_S + m_S^3k_S + m_Sk_S^2)$. Then, for the total number of operations, we have an upper bound of

$$\begin{array}{ll} K(m_S + m_S^3 k_S + m_S k_S^2) + K(m^3 + mk) & \leq \\ Km + Km^3(k-1) + Km(k-1)^2 + K(m^3 + mk) & = \\ K(m + m^3 k + mk^2) - Km(k-1) & \leq \\ K(m + m^3 k + mk^2). & \end{array}$$

If $\lambda = \beta = \beta(L, R)$, there will be two recursive calls of the function. Notice that this can only be the case if $k > 1$. Let m_L, m_R denote the number of sections and let k_L, k_R denote the number of relevant coalitions of the two reduced highway problems $\Gamma_{\lambda,L}$ and $\Gamma_{\lambda,R}$ respectively. Since $(N \setminus L) \cap (N \setminus R) = \emptyset$, a relevant coalition in Γ can correspond to at most one relevant coalition in either $\Gamma_{\lambda,L}$ or $\Gamma_{\lambda,R}$. This shows that $k_L + k_R \leq k$, but we wish to prove $k_L + k_R \leq k - 1$. If $L \cap R = \emptyset$, then coalitions L and R are not relevant in either of the reduced problems, so we even have $k_L + k_R \leq k - 2$. If $L \cap R \neq \emptyset$, then $L \cap R$ is relevant in Γ, but not in either of the reduced problems, so also in this case we have $k_L + k_R \leq k - 1$. We further have $m_L, m_R \leq m$ by Corollary 3.4. By induction, the amount of numerical operations in the two recursive calls is bounded by $K(m_L + m_L^3 k_L + m_L k_L^2)$ and $K(m_R + m_R^3 k_R + m_R k_R^2)$ respectively. Then, for the total number of operations, we have an upper bound of

$$\begin{array}{ll} K(m_L + m_L^3 k_L + m_L k_L^2) + K(m_R + m_R^3 k_R + m_R k_R^2) + K(m^3 + mk) & \leq \\ K(m + m^3 k_L + mk_L^2) + K(m + m^3 k_R + mk_R^2) + K(m^3 + mk) & = \\ Km^3(k_L + k_R) + Km(k_L^2 + k_R^2) + K(m^3 + mk + 2m) & \leq \\ Km^3(k-1) + Km(k-1)^2 + K(m^3 + mk + 2m) & = \\ K(m + m^3 k + mk^2) - Km(k-2) & \leq \\ K(m + m^3 k + mk^2), & \end{array}$$

where the last inequality follows, since $k > 1$. □

Since the number of relevant coalitions is bounded by $\frac{1}{2}m(m+1)$ for a highway problem with m sections, it is also true that the complexity is at most $\mathcal{O}(m^5)$. If Proposition 3.5 is applied to the subclass of airport games, the derived complexity bound becomes $\mathcal{O}(m^4)$ by substituting $k = \mathcal{O}(m)$ relevant coalitions. However, it is easy to derive a sharper complexity bound of $\mathcal{O}(m^3)$ for the function `nucleolus` in the case of airport games, by observing that the numbers $\beta(L, R)$ need not be computed. This is the same complexity bound as the procedure for airport games, described by Littlechild (1974).

4 A practical application: AP68 highway in Spain

To conclude, we applied the results of this paper to the AP68 highway located in the north of Spain. It connects Bilbao to Zaragoza and it has 23 entry/exit points along its route. The total length of the highway is 295 km. Figure 2 shows its route.

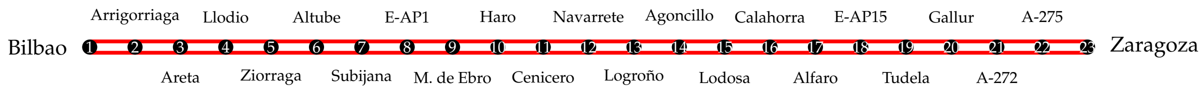

Figure 2: Route of the AP68 highway in Spain.

AP68 is a toll highway. The concessionaire firm is *Autopistas Vasco-Aragonesas*. There exist three kinds of rates according to the different classes of vehicles: *light vehicles* (cars, motorbikes and delivery cars), *light trucks* and *heavy trucks*. For simplicity, we restricted to the light vehicles class. Table 2 provides the official rates applied in 2007.[2] The total number of journeys using each part of the highway during 2007 are given in Table 3. Take into account that the direction is not important in our model and that the numbers in Table 3 represent the number of vehicles using a particular part of the highway in either direction. The bold numbers in the first row and first column of Table 2, Table 3, Table 5 and Table 6 represent the entrance/exit points. Their correspondences with cities are given in Table 1. There are no exits at points 5, 15 and 22, and there is no entrance at point 3. At point 2 there is an entrance, but the machine that handles the vehicles from point 1 is located at point 2, so it does not make any difference whether a vehicle enters at 1 or at 2. To estimate the costs, we assumed that the cost of each section is the total amount that the highway concessionaire firm collected in that section (we have considered other alternatives as well, but we have included only this case for illustrating purposes). These costs are given in Table 4. These data are the ingredients for defining the highway problem and it associated highway game.

1	Bilbao	6	Altube	11	Cenicero	16	Calahorra	21	A-272
2	Arrigorriaga	7	Subijana	12	Navarrete	17	Alfaro	22	A-275
3	Areta	8	E-AP1	13	Logroño	18	E-AP15	23	Zaragoza
4	Llodio	9	Miranda de Ebro	14	Agoncillo	19	Tudela		
5	Ziorraga	10	Haro	15	Lodosa	20	Gallur		

Table 1: Legend for Table 2, Table 3, Table 5, and Table 6.

It can be observed that the rates according to the Shapley value are close to the official rates. This is actually no surprise, as the cost of a section was estimated by the collected toll fee at that section, and the Shapley value distributes this cost among the users of that section. So although the methods of aggregating costs and distributing costs are different, they are similar. The nucleolus gives almost a flat rate. Only users of the highway near Bilbao have reduced rates, but other users all pay the same toll. The reduced rates near Bilbao can be explained by the fact that 57% of the journeys occur in the aforementioned part.

[2]The authors would like to thank the AP68 managers, particularly to Domingo Sobrón, for having provided all data that the authors asked for.

Recall that, in our game model, the cost of a section is interpreted as a cost independent of traffic. Then it may make sense to charge for the use of the entire highway, independent of traveled sections, as traveling does not incur costs. The nucleolus behaves somewhat in this way, but makes exceptions for sections with exceptionally high or low cost and high or low usage. The Shapley value however takes into account the number of traveled sections, charging as if damage was caused to each traveled section. The logic of the nucleolus appears more in line with the interpretation of the game model.

The official rates are based on maintenance costs that are not independent of traffic. Therefore, these should not be compared to the nucleolus.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
1	0.00	0.65	1.55	1.55	-	4.15	6.05	7.55	8.15	9.10	10.70	11.25	11.95	12.95	-	15.15	17.05	17.45	18.35	20.75	22.80	-	24.40
2		0.00	1.55	1.55	-	4.15	6.05	7.55	8.15	9.10	10.70	11.25	11.95	12.95	-	15.15	17.05	17.45	18.35	20.75	22.80	-	24.40
3			0.00	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4				0.00	-	2.75	4.60	6.10	6.75	7.70	9.30	9.85	10.55	11.50	-	13.75	15.65	16.05	16.90	19.35	21.35	-	23.00
5					0.00	1.00	2.85	4.35	5.00	5.95	7.50	8.10	8.80	9.75	-	11.95	13.85	14.30	15.15	17.60	19.60	-	21.25
6						0.00	2.15	3.65	4.30	5.20	6.80	7.35	8.05	9.05	-	11.25	13.15	13.60	14.45	16.90	18.90	-	20.50
7							0.00	1.60	2.25	3.20	4.75	5.35	6.05	7.00	-	9.20	11.10	11.55	12.40	14.85	16.85	-	18.50
8								0.00	0.55	1.50	3.05	3.65	4.35	5.30	-	7.50	9.45	9.85	10.70	13.15	15.15	-	16.80
9									0.00	1.20	2.75	3.35	4.05	5.00	-	7.20	9.10	9.55	10.40	12.85	14.85	-	16.50
10										0.00	1.70	2.25	2.95	3.95	-	6.15	8.05	8.50	9.35	11.80	13.80	-	15.40
11											0.00	0.70	1.40	2.40	-	4.60	6.50	6.90	7.75	10.20	12.20	-	13.85
12												0.00	0.80	1.80	-	4.00	5.90	6.35	7.20	9.65	11.65	-	13.25
13													0.00	1.25	-	3.45	5.35	5.80	6.65	9.10	11.10	-	12.75
14														0.00	-	2.30	4.20	4.65	5.50	7.95	9.95	-	11.55
15															0.00	0.85	2.75	3.15	4.05	6.45	8.45	-	10.10
16																0.00	2.00	2.45	3.30	5.75	7.75	-	9.35
17																	0.00	0.60	1.50	3.90	5.90	-	7.55
18																		0.00	1.05	3.50	5.50	-	7.15
19																			0.00	2.60	4.60	-	6.25
20																				0.00	2.15	-	3.80
21																					0.00	-	2.05
22																						0.00	1.75
23																							0.00

Table 2: Official rates in euros for light vehicles applied in the AP68 highway during 2007.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	1792	3376	4851	0	13061	181	6855	561	1114	222	278	1072	128	0	241	101	6	144	130	103	0	1394
2		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4				0	0	929	11	139	28	27	5	5	24	4	0	4	4	0	4	3	2	0	24
5					0	316	11	87	18	20	4	6	15	2	0	5	2	0	2	4	2	0	21
6						0	19	63	18	15	4	4	13	2	0	4	2	0	2	2	1	0	20
7							0	70	30	20	4	4	10	2	0	3	1	0	1	1	1	0	13
8								0	107	297	97	115	697	66	0	120	47	5	66	53	75	0	1222
9									0	442	56	145	390	37	0	53	21	1	29	39	19	0	223
10										0	76	207	862	44	0	94	33	2	37	21	25	0	347
11											0	24	151	35	0	35	14	0	12	8	7	0	84
12												0	110	187	0	101	28	2	25	18	17	0	191
13													0	969	0	1073	227	5	143	74	67	0	737
14														0	0	758	147	1	89	40	38	0	339
15															0	279	64	1	70	20	13	0	105
16																0	86	16	181	55	57	0	449
17																	0	29	44	29	41	0	333
18																		0	937	392	322	0	2869
19																			0	77	168	0	1095
20																				0	306	0	1869
21																					0	0	39
22																						0	1297
23																							0

Table 3: Number of vehicles using AP68 highway in the year 2007.

1 − 2	20385,79	**6 − 7**	24325,86	**11 − 12**	6618,932	**16 − 17**	15951,27	**21 − 22**	2742,027
2 − 3	23065,18	**7 − 8**	20136,34	**12 − 13**	6512,166	**17 − 18**	4009,55	**22 − 23**	20549,93
3 − 4	6736,976	**8 − 9**	3120,606	**13 − 14**	10597,76	**18 − 19**	9667,746		
4 − 5	49815,77	**9 − 10**	12023,41	**14 − 15**	14476,77	**19 − 20**	27460,07		
5 − 6	18021,02	**10 − 11**	16455,22	**15 − 16**	8045,568	**20 − 21**	23431,98		

Table 4: Cost of each section of AP68 in the year 2007.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0.00	4.85	4.85	4.85	-	4.85	5.80	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
2		0.00	4.85	4.85	-	4.85	5.80	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
3			0.00	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4				0.00	-	4.85	5.80	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
5					0.00	4.85	5.80	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
6						0.00	5.80	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
7							0.00	5.80	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
8								0.00	5.80	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
9									0.00	6.21	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
10										0.00	6.31	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
11											0.00	6.31	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
12												0.00	6.31	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
13													0.00	6.31	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
14														0.00	-	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
15															0.00	6.31	6.31	6.31	6.31	6.31	6.31	-	6.31
16																0.00	6.31	6.31	6.31	6.31	6.31	-	6.31
17																	0.00	6.31	6.31	6.31	6.31	-	6.31
18																		0.00	6.31	6.31	6.31	-	6.31
19																			0.00	6.31	6.31	-	6.31
20																				0.00	6.31	-	6.31
21																					0.00	-	6.31
22																						0.00	6.31
23																							0.00

Table 5: Rates in euros for light vehicles using the nucleolus.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0.00	0.57	1.25	1.48	-	3.99	5.84	7.37	7.73	8.98	10.74	11.45	12.16	13.32	-	15.76	17.83	18.38	19.19	21.60	23.46	-	25.33
2		0.00	0.68	0.90	-	3.42	5.27	6.80	7.15	8.41	10.16	10.88	11.59	12.75	-	15.18	17.26	17.80	18.62	21.03	22.89	-	24.75
3			0.00	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4				0.00	-	2.52	4.36	5.90	6.25	7.51	9.26	9.97	10.68	11.85	-	14.28	16.36	16.90	17.72	20.13	21.99	-	23.85
5					0.00	0.66	2.51	4.04	4.39	5.65	7.40	8.12	8.83	9.99	-	12.42	14.50	15.04	15.86	18.27	20.13	-	21.99
6						0.00	1.85	3.38	3.73	4.99	6.74	7.46	8.17	9.33	-	11.76	13.84	14.38	15.20	17.61	19.47	-	21.33
7							0.00	1.53	1.89	3.14	4.90	5.61	6.32	7.48	-	9.92	11.99	12.54	13.35	15.76	17.62	-	19.49
8								0.00	0.35	1.61	3.36	4.08	4.79	5.95	-	8.38	10.46	11.00	11.82	14.23	16.09	-	17.95
9									0.00	1.26	3.01	3.72	4.43	5.60	-	8.03	10.11	10.65	11.47	13.88	15.74	-	17.60
10										0.00	1.75	2.47	3.18	4.34	-	6.77	8.85	9.39	10.21	12.62	14.48	-	16.34
11											0.00	0.71	1.42	2.58	-	5.02	7.10	7.64	8.46	10.87	12.73	-	14.59
12												0.00	0.71	1.87	-	4.31	6.38	6.93	7.74	10.15	12.01	-	13.88
13													0.00	1.16	-	3.60	5.67	6.22	7.03	9.44	11.30	-	13.17
14														0.00	-	2.44	4.51	5.05	5.87	8.28	10.14	-	12.00
15															0.00	0.84	2.91	3.46	4.27	6.68	8.54	-	10.41
16																0.00	2.08	2.62	3.44	5.85	7.71	-	9.57
17																	0.00	0.54	1.36	3.77	5.63	-	7.49
18																		0.00	0.82	3.23	5.09	-	6.95
19																			0.00	2.41	4.27	-	6.13
20																				0.00	1.86	-	3.72
21																					0.00	-	1.86
22																						0.00	1.62
23																							0.00

Table 6: Rates in euros for light vehicles using the Shapley value.

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