

On the 1-nucleolus

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Abstract This paper analyzes the 1-nucleolus and, in particular, its relation to the nucleolus. It is seen that, contrary to the nucleolus, the 1-nucleolus can be computed in polynomial time due to a characterization using a combination of standard bankruptcy rules for associated bankruptcy problems. Sufficient conditions on a compromise stable game are derived such that the 1-nucleolus and the nucleolus coincide.

Keywords 1-nucleolus · Compromise stable games · Aumann–Maschler rule · Nucleolus

1 Introduction

Cooperative transferable utility games (TU-games) have proven effective to analyze problems where the joint profits obtained by a joint collaboration have to be shared among the individuals involved (the grand coalition). In order to decide on a “fair” or “just” distribution of the joint profits (a solution), benchmarks are used: the joint profits that any subgroup of individuals (a coalition) could obtain by cooperation without any help from the other members of the grand coalition that are outside this

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subgroup. This means that the description of a cooperative game in general requires the computation of 2^n values, with n being the number of members of the grand coalition. For this reason, the computation of solutions that are based on all coalitional values is NP-hard.

In a general framework, there are several solutions to TU-games available that, in principle, need as input all coalitional values. Among the most studied are the core (Gillies 1953), the nucleolus (Schmeidler 1969), the Shapley value (Shapley 1953) and the compromise value (Tijs 1981).

This paper analyzes the possibility of reducing the number of input variables with respect to the nucleolus of a TU-game by studying the 1-nucleolus, a member of the class of generalized nucleoli introduced in Maschler et al. (1992) and conceptually related to the notion of the k -core cover as studied in Sánchez-Rodríguez et al. (2015). For TU-games with a nonempty imputation set, the nucleolus is a solution based on the idea that a fair distribution of the total worth should (lexicographically) minimize the sorted vector of the excesses (or complaints) associated with all possible coalitions. Given an imputation x and a coalition S , the excess measures the dissatisfaction of S at x . There are different algorithms to compute the nucleolus, see the Kopelowitz algorithm (Kopelowitz 1967) or the Maschler-Peleg-Shapley algorithm (Maschler et al. 1979). The complexity of these algorithms, however, is exponential in the number of players, and therefore useful only for relatively small games. Still, there are classes of games, such as assignment games (Shapley and Shubik 1972), where the complexity of these algorithms only grows polynomially in the number of players. That fact has allowed to develop special algorithms to obtain the nucleolus when the game has a special underlying structure. Nevertheless, in most applications where many players are involved, the task of computing the nucleolus can be very difficult.

This paper focuses on the 1-nucleolus which is based on computing the excesses only of coalitions of size 1 and $n - 1$. By reducing the number of variables involved in the computation of the 1-nucleolus with respect to the nucleolus, we also decrease its computational difficulty. We characterize the 1-nucleolus using a combination of standard bankruptcy rules of an associated bankruptcy problem, the computations of which are done in polynomial time. Next, we analyze under which conditions the nucleolus and 1-nucleolus of compromise stable games coincide. Finally, without aiming for a full characterization of the 1-nucleolus, we study several properties on the class of TU-games with a nonempty core and use them to compare the 1-nucleolus and the nucleolus. In particular, it is seen that the 1-nucleolus satisfies individual superadditivity gains to the grand coalition and first agent consistency, which are not satisfied by the nucleolus. Unfortunately, just like the nucleolus, the 1-nucleolus does not satisfy aggregate monotonicity either.

The outline of the paper is as follows. Section 2 recalls basic concepts and results that will be used throughout the paper. Section 3 formally defines the 1-nucleoli. In Sect. 4, we characterize the 1-nucleolus by means of a combination of bankruptcy rules. Section 5 analyzes the 1-nucleolus in relation with the nucleolus for compromise stable games and Sect. 6 is dedicated to open questions and to comparing the 1-nucleolus and nucleolus based on some desirable properties.

2 Preliminaries

In this section, we survey some well-known concepts and results that will be used in the subsequent sections.

For $x, y \in \mathbb{R}^n$, we say that x is *lexicographically smaller* than y , $x <_L y$, if there is $m \in \{1, \dots, n\}$ such that $x_l = y_l$ for every $1 \leq l < m$ and $x_m < y_m$. Moreover, $x \leq_L y$ if either $x = y$, or $x <_L y$.

A *transferable utility game* (TU-game) is a pair (N, v) where N is a finite set of players and $v : 2^N \rightarrow \mathbb{R}$ satisfies $v(\emptyset) = 0$, where 2^N denotes the set of subsets or *coalitions* of N . In general, $v(S)$ represents the value of coalition S , that is, the joint payoff that can be obtained by this coalition when its members decide to cooperate. Let G^N be the set of all TU-games with player set N . Given $S \subseteq N$, let $|S|$ be the number of players in S .

The main focus within a cooperative setting is on how to share the total joint payoff obtained when all players decide to cooperate. Given a TU-game $v \in G^N$, the *imputation set* of v , $I(v)$, is the set of efficient allocations that are individually rational. Formally,

$$I(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

Note that the imputation set is nonempty if, and only if,

$$\sum_{i \in N} v(\{i\}) \leq v(N).$$

We denote by I^N the set of all TU-games with player set N and nonempty imputation set.

The *core* of $v \in G^N$, $Core(v)$, was first introduced in Gillies (1953) and is defined as the set of efficient allocations that are stable, in the sense that no coalition has an incentive to deviate. Formally,

$$Core(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \right\}.$$

Bondareva (1963) and Shapley (1967) established that a game $v \in G^N$ has a nonempty core if, and only if, it is balanced. Before introducing balanced games, we need to fix some notation.

Let $\emptyset \neq S \subseteq N$ and let $e^S \in \mathbb{R}^N$ be the characteristic vector of S , defined as $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. A *family* \mathcal{B} of nonempty subcoalitions of S is called *balanced* on S if there are positive weights $\delta = \{\delta_T\}_{T \in \mathcal{B}}$, $\delta_T > 0$ for all $T \in \mathcal{B}$, such that $\sum_{T \in \mathcal{B}} \delta_T e^T = e^S$ or, equivalently, $\sum_{T \in \mathcal{B}: T \ni i} \delta_T = 1$ for all $i \in S$ and $\sum_{T \in \mathcal{B}: T \ni i} \delta_T = 0$ for all $i \in N \setminus S$. We denote by $\mathcal{F}(S)$ the set of balanced families of S . Given a balanced family \mathcal{B} , we denote by $\Delta(\mathcal{B})$ the set of positive weights satisfying

the balancedness condition. A game $v \in G^N$ is called *balanced* if for all balanced families $\mathcal{B} \in \mathcal{F}(N)$ and all $\{\delta_S\}_{S \in \mathcal{B}} \in \Delta(\mathcal{B})$, $\sum_{S \in \mathcal{B}} \delta_S v(S) \leq v(N)$.

A well-established one-point solution concept is the nucleolus, introduced in [Schmeidler \(1969\)](#). Let $v \in I^N$ and let $x \in I(v)$. We denote the *excess* of coalition $S \in 2^N$ with respect to x by

$$e(S, x) = v(S) - \sum_{j \in S} x_j.$$

Moreover, we denote by $\theta(x) \in \mathbb{R}^{2^{|N|}}$ the vector whose coordinates are the excesses $e(S, x)$ arranged in non-increasing order, that is, $\theta_l(x) \geq \theta_m(x)$ for every $1 \leq l \leq m \leq 2^{|N|}$. The *nucleolus* of $v \in G^N$, $nuc(v)$, is defined as

$$nuc(v) = \{x \in I(v) \mid \theta(x) \leq_L \theta(y) \text{ for all } y \in I(v)\}.$$

[Schmeidler \(1969\)](#) showed that the nucleolus of a game with a nonempty imputation set exists and is unique. The nucleolus is invariant with respect to positive affine transformations, i.e. for $v \in I^N$, $\alpha > 0$, and $a \in \mathbb{R}^N$, it follows $nuc(\alpha v + a) = \alpha nuc(v) + a$ with $(\alpha v + a)(S) = \alpha v(S) + \sum_{j \in S} a_j$ for every $S \in 2^N$.

[Tijs and Lipperts \(1982\)](#) introduced the core cover. Let $v \in I^N$ and $i \in N$. The *utopia* value of player i , $M_i(v)$, is defined as

$$M_i(v) = v(N) - v(N \setminus \{i\}).$$

The *minimal right* of player i , $m_i(v)$, is defined as

$$m_i(v) = \max_{S \subseteq N \setminus \{i\}} \left\{ v(S \cup \{i\}) - \sum_{j \in S} M_j(v) \right\}.$$

Note that, in general, we need $2^{|N|-1}$ values to compute the minimal right of a player. Thus, in general, the computation of minimal rights is NP-hard. The *utopia vector* is given by $M(v) = (M_i(v))_{i \in N}$ and the *minimal right* vector is given by $m(v) = (m_i(v))_{i \in N}$. The *core cover* of $v \in G^N$, $\mathcal{CC}(v)$, is defined as

$$\mathcal{CC}(v) = \{x \in I(v) \mid m(v) \leq x \leq M(v)\}.$$

It can be verified that $Core(v) \subseteq \mathcal{CC}(v) \subseteq I(v)$. A TU game $v \in I^N$ is *compromise admissible* if $\mathcal{CC}(v) \neq \emptyset$. A compromise admissible game is *compromise stable* if $Core(v) = \mathcal{CC}(v)$. [Quant et al. \(2005\)](#) characterized the family of compromise stable games.

Theorem 2.1 ([Quant et al. 2005](#)). *Let $v \in I^N$ be compromise admissible. Then, v is compromise stable if, and only if, $v(S) \leq \max\{\sum_{j \in S} m_j(v), v(N) - \sum_{j \in N \setminus S} M_j(v)\}$ for every $S \subseteq N$.*

An important subclass of balanced games is the class of convex games, as introduced in [Shapley \(1971\)](#). A game $v \in G^N$ is *convex* if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

A *bankruptcy problem* is described by (N, E, c) , with N a finite set of players, $E > 0$, and $c \in \mathbb{R}^N$ such that $c_i \geq 0$ for all $i \in N$ and $\sum_{i \in N} c_i \geq E$. [O’Neill \(1982\)](#) defines the *bankruptcy game* associated to a bankruptcy problem (N, E, c) , as

$$v_{E,c}(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} c_i \right\} \text{ for every } S \in 2^N.$$

In fact, [Quant et al. \(2005\)](#) show that a game is convex and compromise stable game if, and only if, it is S -equivalent¹ to a bankruptcy game. [Aumann and Maschler \(1985\)](#) show that the nucleolus of a bankruptcy game corresponds to the Aumann–Maschler rule of the corresponding bankruptcy problem.

3 1-nucleolus

This section focuses on the 1-nucleolus of a game by considering excesses only of coalitions of size at most 1 and at least $|N| - 1$. In order to formally define the 1-nucleolus, we need to fix some notation. Let N be a finite set, we denote

$$\mathcal{C}^1(N) = \{S \in 2^N \mid |S| \leq 1 \text{ or } |S| \geq |N| - 1\}.$$

If no confusion arises, we write \mathcal{C}^1 instead of $\mathcal{C}^1(N)$. Given $v \in I^N$ and $x \in I(v)$, we write $\theta^1(x) \in \mathbb{R}^{2(|N|+1)}$ the vector whose coordinates are the excesses $e(S, x)$, with $S \in \mathcal{C}^1$, arranged in non-increasing order, that is, $\theta_l^1(x) \geq \theta_m^1(x)$ for every $1 \leq l \leq m \leq 2(|N| + 1)$.

Definition 3.1 Let $v \in I^N$. The *1-nucleolus* is defined by

$$nuc^1(v) = \{x \in I(v) \mid \theta^1(x) \leq_L \theta^1(y) \text{ for all } y \in I(v)\}.$$

Note that for $|N| = 3$, $\mathcal{C}^1(N) = 2^N$. As a consequence, the 1-nucleolus and nucleolus of 3-players games coincide. Moreover, just like the nucleolus, the 1-nucleolus is invariant with respect to positive affine transformations.

Theorem 3.2 (cf. [Schmeidler 1969](#)) *Let $v \in I^N$. Then, $nuc^1(v)$ exists and is unique.*

The characterization of the nucleolus in [Kohlberg \(1971\)](#) can be translated to the 1-nucleolus as already pointed out in [Maschler et al. \(1992\)](#). For this, we need to obtain a partition of the coalitions in N of size at most 1 or at least $|N| - 1$. Let $v \in I^N$ and let $x \in I(v)$. Let $B_0^1(x, v) = \{\{i\} \subseteq N \mid x_i = v(\{i\})\}$ and define recursively

¹ Two games $v, w \in G^N$ are S -equivalent if there exists $\alpha > 0$ and $a \in \mathbb{R}^N$ such that $v(S) = \alpha w(S) + \sum_{i \in S} a_i$ for every $S \subseteq N$.

$$B_l^1(x, v) = \left\{ S \in \mathcal{C}^1(N) \setminus (\cup_{m=1}^{l-1} B_m^1(x, v)) \mid e(S, x) \geq e(R, x) \right. \\ \left. \text{for every } R \in \mathcal{C}^1(N) \setminus (\cup_{m=1}^{l-1} B_m^1(x, v)) \right\}$$

for $l \in \{1, \dots, p\}$, with p such that $B_p^1(x, v) \neq \emptyset$ and $\langle B_1^1(x, v), \dots, B_p^1(x, v) \rangle$ forms a partition of the set of coalitions of $\mathcal{C}^1(N)$. For $l \in \{1, \dots, p\}$, let $\mathcal{B}^{1,l}(x, v) = \cup_{m=1}^l B_m^1(x, v)$.

Theorem 3.3 (cf. Kohlberg 1971; Maschler et al. 1992). *Let $v \in I^N$. Then, x is the 1-nucleolus of v if, and only if, for every $l \in \{1, \dots, p\}$, there exists $B_0^{1,l}(x, v) \subseteq B_0^1(x, v)$ such that $B_0^{1,l}(x, v) \cup \mathcal{B}^{1,l}(x, v)$ is balanced.*

4 1-nucleolus and bankruptcy

The 1-nucleolus only takes into account the information provided by the value of the singletons (individual coalitions), the value of the $|N| - 1$ player coalitions, and the value of the grand coalition. Thus, the information needed stems from $2|N| + 1$ coalitions.

This section shows that the 1-nucleolus is related to the Aumann–Maschler rule of bankruptcy problems (see Aumann and Maschler 1985), the constrained equal losses rule for bankruptcy problems, and the equal share rule. Moreover, the 1-nucleolus of a balanced game can be described as the Aumann–Maschler rule of an associated bankruptcy problem. As a consequence, it turns out that the 1-nucleolus and the nucleolus of bankruptcy games coincide (see O’Neill 1982; Aumann and Maschler 1985; Quant et al. 2005). We recall some well-known bankruptcy rules in the literature. The *equal share rule* of a bankruptcy problem (N, E, c) , $ES(N, E, c)$, assigns

$$ES_j(N, E, c) = \frac{E}{|N|} \text{ for every } j \in N.$$

The *constrained equal awards rule* of a bankruptcy problem (N, E, c) , $CEA(N, E, c)$, assigns

$$CEA_j(N, E, c) = \min\{\lambda, c_j\} \text{ to every } j \in N,$$

with $\lambda \in \mathbb{R}_+$ chosen such that $\sum_{j \in N} CEA_j(N, E, c) = E$. The *Aumann–Maschler rule* of a bankruptcy problem (N, E, c) , $AM(N, E, c)$, is given by

$$AM(N, E, c) = \begin{cases} CEA(N, E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2} \sum_{j \in N} c_j, \\ c - CEA(N, \sum_{j \in N} c_j - E, \frac{1}{2}c) & \text{otherwise.} \end{cases}$$

Aumann and Maschler (1985) showed that the Aumann–Maschler rule of a bankruptcy problem corresponds to the nucleolus of the associated bankruptcy game.

To conclude, the *constrained equal losses rule* of a bankruptcy problem (N, E, c) , $CEL(N, E, c)$, is defined as

$$CEL_j(N, E, c) = \max\{0, c_j - \lambda\} \text{ for every } j \in N,$$

where λ is chosen such that $\sum_{j \in N} CEL_j(N, E, c) = E$.

For $v \in I^N$, we define the *zero-normalization* of v , $v_0 \in I^N$, as

$$v_0(S) = v(S) - \sum_{j \in S} v(\{j\}) \text{ for every } S \in 2^N.$$

Note that $nuc^1(v_0) = nuc^1(v) - (v(\{j\}))_{j \in N}$. Therefore, when describing the 1-nucleolus, we can assume that $v = v_0$, that is, that v is zero-normalized.

The following result fully describes the 1-nucleolus by means of a combination of standard bankruptcy solutions to associated bankruptcy problems.

Theorem 4.1 *Let $v \in I^N$ with $v = v_0$. Let $E = v(N)$ and let $c \in \mathbb{R}^N$ be defined by $c_j = v(N) - v(N \setminus \{j\})$ for every $j \in N$.*

(i) *If $c_j \geq 0$ for every $j \in N$, then,*

$$nuc^1(v) = \begin{cases} AM(N, E, c) & \text{if } E \leq \sum_{j \in N} c_j, \\ c + ES(N, E - \sum_{j \in N} c_j, c) & \text{if } E > \sum_{j \in N} c_j. \end{cases}$$

(ii) *If $c_j < 0$ for some $j \in N$, let $c^+ \in \mathbb{R}^N$ be defined by $c_j^+ = \max\{0, c_j\}$ for every $j \in N$ and let $c^{\min} \in \mathbb{R}^N$ be defined as $c_j^{\min} = c_j - \min\{c_l | l \in N\}$ for every $j \in N$. Then,*

$$nuc^1(v) = \begin{cases} AM(N, E, c^+) & \text{if } E \leq \sum_{j \in N} c_j^+, \\ CEL(N, E, c^{\min}) & \text{if } \sum_{j \in N} c_j^+ < E \leq \sum_{j \in N} c_j^{\min}, \\ c + ES(N, E - \sum_{j \in N} c_j, c) & \text{if } E > \sum_{j \in N} c_j^{\min}. \end{cases}$$

Proof See ‘‘Appendix’’.

Since all rules used in Theorem 4.1 are computed in polynomial time and only the values of coalitions of size 1, $|N| - 1$, and $|N|$ are used, the 1-nucleolus is also computed in polynomial time. The next result provides an explicit connection of the 1-nucleolus for balanced games to the Aumann–Maschler rule.

Theorem 4.2 *Let $v \in I^N$ be a balanced game with $v = v_0$. Then,*

$$nuc^1(v) = AM(N, E, c)$$

with $E = v(N)$ and $c \in \mathbb{R}^N$ defined as $c_j = v(N) - v(N \setminus \{j\})$ for every $j \in N$.

Proof By Theorem 4.1 (i), it suffices to show that $c_j \geq 0$ for every $j \in N$ and that $E \leq \sum_{j \in N} c_j$.

Let $j \in N$. Since v is balanced, we have that $v(N) \geq v(\{j\}) + v(N \setminus \{j\}) = v(N \setminus \{j\})$. Therefore, $c_j = v(N) - v(N \setminus \{j\}) \geq 0$.

Moreover, since v is balanced, we have that $\sum_{j \in N} \frac{1}{|N|-1} v(N \setminus \{j\}) \leq v(N)$. Therefore,

$$\begin{aligned} E &= v(N) = v(N) + (|N| - 1) \sum_{j \in N} \frac{1}{|N| - 1} v(N \setminus \{j\}) - \sum_{j \in N} v(N \setminus \{j\}) \\ &\leq v(N) + (|N| - 1)v(N) - \sum_{j \in N} v(N \setminus \{j\}) = \sum_{j \in N} (v(N) - v(N \setminus \{j\})) = \sum_{j \in N} c_j. \end{aligned}$$

As a consequence, we have

Theorem 4.3 *Let (N, E, c) be a bankruptcy problem and let $(N, v_{E,c})$ be the corresponding bankruptcy game. Then, $nuc(v_{E,c}) = nuc^1(v_{E,c})$.*

Proof Let $w \in G^N$ be the zero-normalization of $v_{E,c}$, that is $w(S) = v_{E,c}(S) - \sum_{j \in S} v_{E,c}(\{j\})$ for all $S \in 2^N$. Then,

$$\begin{aligned} nuc^1(v_{E,c}) &= (v_{E,c}(\{j\}))_{j \in N} + nuc^1(w) \\ &= (v_{E,c}(\{j\}))_{j \in N} + AM(N, w(N), (w(N) - w(N \setminus \{j\}))_{j \in N}) \\ &= (v_{E,c}(\{j\}))_{j \in N} \\ &\quad + AM\left(N, v_{E,c}(N) - \sum_{j \in N} v_{E,c}(\{j\}), M(v_{E,c}) - (v_{E,c}(\{j\}))_{j \in N}\right) \\ &= m(v_{E,c}) + AM(N, v_{E,c}(N) - \sum_{j \in N} m_j(v_{E,c}), M(v_{E,c}) - m(v_{E,c})) \\ &= AM(N, v_{E,c}(N), M(v_{E,c})) \\ &= nuc(v_{E,c}) \end{aligned}$$

where the third equality follows from $M_i(w) = w(N) - w(N \setminus \{i\}) = M_i(v_{E,c}) - v_{E,c}(\{i\})$ for every $i \in N$, the fourth equality is a direct consequence of $v_{E,c}(\{j\}) = m_j(v_{E,c})$ for every $j \in N$, and the fifth equality follows from the fact that the Aumann–Maschler rule satisfies the property of minimal rights first (see Thomson 2003).

As a consequence of Theorem 4.3, the nucleolus and 1-nucleolus of convex and compromise stable games coincide since every convex and compromise stable game is S -equivalent to a bankruptcy game (cf. Quant et al. 2005).

5 1-nucleolus and nucleolus

Quant et al. (2005) characterize the nucleolus of compromise stable games.

Theorem 5.1 (Quant et al. 2005). *Let $v \in I^N$ be a compromise stable game. Then,*

$$nuc(v) = m(v) + AM(N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)).$$

By Theorem 5.1, the nucleolus of a compromise stable game depends on the minimal right vector of the game and its computation is, therefore, still NP-hard. The following example shows that the 1-nucleolus might not belong to the core cover of a compromise stable game. Furthermore, it illustrates that the 1-nucleolus and the nucleolus of such a game need not coincide.

Example 5.2 Consider $v \in I^N$ with $N = \{1, 2, 3, 4\}$,

$$\begin{aligned} v(\{1\}) &= 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0, \quad v(\{4\}) = 0, \\ v(\{1, 2\}) &= 0, \quad v(\{1, 3\}) = 0, \quad v(\{1, 4\}) = 1, \quad v(\{2, 3\}) = 3, \\ v(\{2, 4\}) &= 0, \quad v(\{3, 4\}) = 4, \\ v(\{1, 2, 3\}) &= 0, \quad v(\{1, 2, 4\}) = 1, \quad v(\{1, 3, 4\}) = 5, \quad v(\{2, 3, 4\}) = 5, \quad v(N) = 5. \end{aligned}$$

Here, $m(v) = (0, 0, 3, 1)$ and $M(v) = (0, 0, 4, 5)$. One readily verifies (using Theorem 2.1) that v is compromise stable. Using Theorem 5.1, we have

$$nuc(v) = (0, 0, 3, 1) + AM(N, 1, (0, 0, 1, 4)) = (0, 0, 3.5, 1.5) \in \mathcal{CC}(v).$$

However, using Theorem 4.1, we have

$$nuc^1(v) = AM(N, 5, (0, 0, 4, 5)) = (0, 0, 2, 3) \notin \mathcal{CC}(v).$$

Notice that in the example above, both the nucleolus and the 1-nucleolus are obtained through the Aumann–Maschler rule, but they provide different allocations. This difference arises from the fact that we first allocate the minimal rights in the nucleolus and then we apply the Aumann–Maschler rule, while in the case of the 1-nucleolus we first allocate the vector $(v(\{i\}))_{i \in N}$ and then we apply the Aumann–Maschler rule. Thus, some of the coordinates of the 1-nucleolus may be smaller than the corresponding coordinates of the minimal rights vector. Precisely that difference makes the 1-nucleolus of a compromise stable game easier to compute than the nucleolus, since one does not need the minimal rights vector. Next, we provide some conditions for the nucleolus and 1-nucleolus of a compromise stable game to coincide.

Theorem 5.3 *Let $v \in I^N$ be compromise stable. Let $E = v(N) - \sum_{j \in N} v(\{j\})$ and $c_j = M_j(v) - v(\{j\})$ for every $j \in N$.*

- (i) *If $m_j(v_{E,c}) = m_j(v) - v(\{j\})$ for every $j \in N$, then, $nuc^1(v) = nuc(v)$.*
- (ii) *If $m_j(v) = \max\{v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\}$ for every $j \in N$, then, $nuc^1(v) = nuc(v)$.*
- (iii) *If either $m(v) < M(v)$, or $m(v) = M(v)$, then, $nuc^1(v) = nuc(v)$.*

Proof We assume, without loss of generality, that $v = v_0$. Note that $E = v(N) - \sum_{j \in N} v(\{j\}) = v(N)$ and $c_j = M_j(v) - v(\{j\}) = M_j(v)$ for every $j \in N$. Since v is compromise stable, v is balanced and, therefore, $nuc^1(v) = AM(N, E, c)$ by Theorem 4.2.

(i) Let $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$. Then,

$$\begin{aligned} nuc^1(v) &= AM(N, E, c) \\ &= m(v_{E,c}) + AM(N, E - \sum_{j \in N} m_j(v_{E,c}), c - m(v_{E,c})) \\ &= m(v) + AM(N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)) \\ &= nuc(v) \end{aligned}$$

where the second equality follows from the fact that the Aumann–Maschler rule satisfies minimal rights first (see Thomson 2003), the third one is a direct consequence of $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$, and the last one follows from Theorem 5.1.

(ii) Let $m_j(v) = \max \{v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\}$ for every $j \in N$. We show that $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$.

Since $(N, v_{E,c})$ is convex, we have $m_j(v_{E,c}) = v_{E,c}(\{j\})$ for every $j \in N$. Then, for $j \in N$,

$$\begin{aligned} m_j(v_{E,c}) &= v_{E,c}(\{j\}) \\ &= \max \left\{ 0, E - \sum_{k \in N \setminus \{j\}} c_k \right\} \\ &= \max \left\{ 0, v(N) - \sum_{k \in N} v(\{k\}) - \sum_{k \in N \setminus \{j\}} (M_k(v) - v(\{k\})) \right\} \\ &= \max \left\{ 0, v(N) - \sum_{k \in N \setminus \{j\}} M_k(v) \right\} \\ &= \max \left\{ v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v) \right\} \\ &= m_j(v) - v(\{j\}) \end{aligned}$$

Then, by (i), we have that $nuc^1(v) = nuc(v)$.

(iii) First, let $m(v) < M(v)$. We show that $m_j(v) = \max\{v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\}$ for every $j \in N$. By (ii), we then have that $nuc^1(v) = nuc(v)$.

On the contrary, suppose that there exists $i \in N$ and $S \in 2^N \setminus \{\emptyset, N\}$ with $S \ni i$ such that $m_i(v) = v(S) - \sum_{k \in S \setminus \{i\}} M_k(v) > \max\{v(\{i\}), v(N) - \sum_{k \in N \setminus \{i\}} M_k(v)\}$. Then, we arrive at a contradiction since

$$\begin{aligned}
 m_i(v) &= v(S) - \sum_{k \in S \setminus \{i\}} M_k(v) \\
 &\leq \max \left\{ \sum_{k \in S} m_k(v), v(N) - \sum_{k \in N \setminus S} M_k(v) \right\} - \sum_{k \in S \setminus \{i\}} M_k(v) \\
 &= \max \left\{ m_i(v) + \sum_{k \in S \setminus \{i\}} (m_k(v) - M_k(v)), v(N) - \sum_{k \in N \setminus \{i\}} M_k(v) \right\} \\
 &< m_i(v)
 \end{aligned}$$

where the first inequality follows from Theorem 2.1 and the second one is a direct consequence of $M(v) > m(v)$ and our supposition.

Second, let $m(v) = M(v)$. Since $v \in I^N$ is a compromise stable game and $m(v) = M(v)$, it follows that $\sum_{i \in N} m_i(v) = v(N) = \sum_{i \in N} M_i(v)$ and $nuc(v) = M(v) = AM(N, E, c) = nuc^1(v)$.

Remark 5.1 As a consequence of Theorem 5.3, we can identify several well-known classes of compromise stable games for which the nucleolus and the 1-nucleolus coincide: big boss games (see Muto et al. 1988), clan games (see Potters et al. 1989), 1-convex games (see Driessen 1983) and 2-convex games (see Driessen 1983).

6 Concluding remarks

Similarly to the 1-nucleolus, one can consider k -nucleoli, with $k \in \{1, \dots, |N|\}$, where only coalitions of size at most k and at least $|N| - k$ are taken into account. Note that for² $k \geq \lfloor \frac{|N|}{2} \rfloor$, $nuc^k(v) = nuc(v)$ since all coalitions are considered. From a computational perspective, it could be interesting to further study the 2-nucleolus and to analyze explicit relationships between the 2-nucleolus and the nucleolus for special classes of games.

Another solution that selects an allocation on the basis of a restricted number of coalitional values is the Rawls rule (Tissdell and Harrison 1992), which considers only individual coalitions. The Rawls rule is also known as the *centre of the imputation set* (CIS-value, cf. Driessen and Funaki 1991) in the context of TU games. The 1-nucleolus, in general, is different from the CIS-value. For instance, consider $N = \{1, 2, 3\}$, $v(\{i\}) = 0$, for every $i \in N$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 0$, $v(\{2, 3\}) = 3$, and $v(N) = 6$. It turns out that $nuc^1(v) = (1.5, 3.5, 1)$, but $CIS(v) = (2, 2, 2)$.

An important line of research in TU-games is the axiomatic characterization of solutions based on desirable properties. As future research, it may be interesting to provide such a characterization of the 1-nucleolus for (special classes of) TU-games. In Table 1, we provide a comparative analysis of some properties regarding the 1-nucleolus and the nucleolus for the class of balanced games. It is readily seen that both the 1-nucleolus and the nucleolus satisfy basic properties as invariance with respect to strategic equivalence, the dummy player property, and symmetry. It turns out that that the 1-nucleolus, contrary to the nucleolus, satisfies individual superadditivity

² For each $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer smaller than or equal to r .

Table 1 Comparative analysis of the 1-nucleolus and the nucleolus by means of properties for the class of balanced games

Property	1-nucleolus	Nucleolus
Aggregate monotonicity	No	No
Individual superadditivity gains to the grand coalition	Yes	No
First agent consistency	Yes	No

gains to the grand coalition and first agent consistency (which is based on the idea of first-player consistency in [Potters and Sudhölter \(1999\)](#) and this, in turn, is based on [Sobolev \(1975\)](#)). Unfortunately, just like the nucleolus, the 1-nucleolus does not satisfy aggregate monotonicity either.

Let f be a one point solution for TU-games. We say that f satisfies

- (i) *aggregate monotonicity* if for every balanced TU-games (N, v) and (N, w) satisfying $v(S) = w(S)$ for every $S \subseteq N, S \neq N$, and $w(N) > v(N)$, $f_i(N, w) \geq f_i(N, v)$ for every $i \in N$;
- (ii) *individual superadditivity gains to the grand coalition* if for every balanced TU-game (N, v) and every players $i, j \in N$ satisfying $v(N) - v(N \setminus \{i\}) - v(\{i\}) \leq v(N) - v(N \setminus \{j\}) - v(\{j\})$, $f_i(N, v) \leq f_j(N, v)$;
- (iii) *first agent consistency* if for every balanced TU-game (N, v) with $N = \{1, \dots, n\}$ such that $v(N \setminus \{1\}) + v(\{1\}) \geq v(N \setminus \{2\}) + v(\{2\}) \geq \dots \geq v(N \setminus \{n\}) + v(\{n\})$ and such that $(N \setminus \{1\}, v_{1, f_1(N, v)})$ is balanced, $f_i(N, v) = f_i(N \setminus \{1\}, v_{1, f_1(N, v)})$ for every $i \in N \setminus \{1\}$; here, for $i \in N$ and $x \in \mathbb{R}$, the game $(N \setminus \{i\}, v_{i, x})$ is defined by

$$v_{i,x}(S) = \begin{cases} v(S) & \text{if } S \subseteq N \setminus \{i\}, |S| \leq |N| - 3, \\ v(S \cup \{i\}) - x & \text{if } S \subseteq N \setminus \{i\}, |N| - 2 \leq |S| \leq |N| - 1. \end{cases}$$

for every $S \subseteq N \setminus \{i\}$.

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Appendix: Proof of Theorem 4.1

We assume, without loss of generality, that $N = \{1, \dots, n\}$ and $c_1 \leq c_2 \leq \dots \leq c_n$. Note that if $x \in I(v)$, then,

$$e(\{j\}, x) = v(\{j\}) - x_j = -x_j$$

and

$$e(N \setminus \{j\}, x) = v(N \setminus \{j\}) - \sum_{k \in N \setminus \{j\}} x_k = v(N \setminus \{j\}) - (v(N) - x_j) = x_j - c_j.$$

(i) We have $c_j \geq 0$ for every $j \in N$.

Case (i.a) $E \leq \sum_{j \in N} c_j$.

We show that $nuc^1(v) = AM(N, E, c)$. Note that (N, E, c) is a bankruptcy problem. We distinguish between two situations: $E \leq \frac{1}{2} \sum_{j \in N} c_j$ and $E > \frac{1}{2} \sum_{j \in N} c_j$.

Case (i.a.1) $E \leq \frac{1}{2} \sum_{j \in N} c_j$.

By definition of the Aumann–Maschler rule, $AM(N, E, c) = CEA(N, E, \frac{1}{2}c)$ where $CEA_j(N, E, \frac{1}{2}c) = \min\{\lambda, \frac{1}{2}c_j\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{j \in N} CEA_j(N, E, \frac{1}{2}c_j) = E$. Let $c_0 = 0$ and let $i \in N$ satisfy

$$\frac{c_{i-1}}{2} \leq \lambda < \frac{c_i}{2}.$$

Then, $\lambda = \frac{1}{|N|-i+1} (E - \sum_{k=0}^{i-1} \frac{c_k}{2})$ and

$$AM_j(N, E, c) = \begin{cases} \frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = AM(N, E, c)$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} -\frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ -\lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = \begin{cases} -\frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ \lambda - c_j & \text{if } i \leq j \leq n. \end{cases}$$

Therefore,

$$\begin{aligned} e(\{1\}, x) &= e(N \setminus \{1\}, x) \geq \dots \geq e(\{i - 1\}, x) \\ &= e(N \setminus \{i - 1\}, x) = -\frac{c_{i-1}}{2} \geq -\lambda \\ &= e(\{i\}, x) = \dots = e(\{n\}, x) = -\lambda > -\lambda + 2\lambda - c_i = \lambda - c_i \\ &= e(N \setminus \{i\}, x) \geq \dots \geq e(N \setminus \{n\}, x). \end{aligned}$$

Let $i_1, i_2, \dots, i_p \in \{1, \dots, i - 1\}$ be such that

$$c_1 = \dots = c_{i_1} < c_{i_1+1} = \dots = c_{i_2} < \dots < c_{i_p+1} = \dots = c_{i-1} < c_i$$

and let $j_1, j_2, \dots, j_q \in \{i, \dots, n\}$ be such that

$$c_i = \dots = c_{j_1} < c_{j_1+1} = \dots = c_{j_2} < \dots < c_{j_q+1} = \dots = c_n.$$

Let $i_0 = 0, i_{p+1} = i - 1, j_0 = i - 1$ and $j_{q+1} = n$. Using the same notation as in Theorem 3.3, we have $B_0^1(x, v) = \emptyset$ and

$$B_m^1(x, v) = \begin{cases} \{\{i_{m-1} + 1\}, N \setminus \{i_{m-1} + 1\}, \dots, \{i_m\}, N \setminus \{i_m\}\} & \text{if } 1 \leq m \leq p + 1, \\ \{\{i\}, \dots, \{n\}\} & \text{if } m = p + 2, \\ \{N \setminus \{j_{m-p-3} + 1\}, \dots, N \setminus \{j_{m-p-2}\}\} & \text{if } p + 3 \leq m \leq p + q + 3. \end{cases}$$

Then, $\mathcal{B}^{1,l}(x, v) = \cup_{m=1}^l B_m^1(x, v)$ is balanced for every $l \in \{1, 2, \dots, p + q + 3\}$ and, by Theorem 3.3, we have that $x = nuc^1(v)$.

Case (i.a.2) $E > \frac{1}{2} \sum_{j \in N} c_j$.

By definition of the Aumann–Maschler rule, $AM(N, E, c) = c - CEA(N, \sum_{k \in N} c_k - E, \frac{1}{2}c)$ where $CEA_j(N, \sum_{k \in N} c_k - E, \frac{1}{2}c) = \min\{\lambda, \frac{1}{2}c_j\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{j \in N} CEA_j(N, \sum_{k \in N} c_k - E, \frac{1}{2}c) = \sum_{k \in N} c_k - E$.

Let $c_0 = 0$ and let $i \in N$ satisfy

$$\frac{c_{i-1}}{2} \leq \lambda < \frac{c_i}{2}.$$

Then, $\lambda = \frac{1}{|N|-i+1} \left(\sum_{k=1}^{|N|} c_k - E - \sum_{k=0}^{i-1} \frac{c_k}{2} \right)$ and

$$AM_j(N, E, c) = \begin{cases} \frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ c_j - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = AM(N, E, c)$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} -\frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ -c_j + \lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = \begin{cases} -\frac{c_j}{2} & \text{if } 1 \leq j \leq i - 1, \\ -\lambda & \text{if } i \leq j \leq n. \end{cases}$$

Therefore,

$$\begin{aligned} e(\{1\}, x) &= e(N \setminus \{1\}, x) \geq \dots \geq e(\{i - 1\}, x) = e(N \setminus \{i - 1\}, x) \\ &= -\frac{c_{i-1}}{2} \geq -\lambda \\ &= e(N \setminus \{i\}, x) = \dots = e(N \setminus \{n\}, x) = -\lambda > -\lambda + 2\lambda - c_i = \lambda - c_i \\ &= e(\{i\}, x) \geq \dots \geq e(\{n\}, x). \end{aligned}$$

Then, similarly as in Case (i.a.1), we have that $x = nuc^1(v)$ by Theorem 3.3.

Case (i.b) $E > \sum_{j \in N} c_j$.

We show that $nuc^1(v) = c + ES(N, E - \sum_{j \in N} c_j, c)$. Let $x = c + ES(N, E - \sum_{j \in N} c_j, c)$. Then, $x_j = c_j + \frac{E - \sum_{k \in N} c_k}{|N|}$,

$$e(\{j\}, x) = -x_j = -c_j - \frac{E - \sum_{k \in N} c_k}{|N|}, \quad \text{and}$$

$$e(N \setminus \{j\}, x) = x_j - c_j = \frac{E - \sum_{k \in N} c_k}{|N|}$$

for every $j \in N$. Therefore,

$$e(N \setminus \{1\}, x) = \dots = e(N \setminus \{n\}, x) > e(\{1\}, x) \geq \dots \geq e(\{n\}, x)$$

where the strict inequality is a direct consequence of the fact that $\frac{E - \sum_{k \in N} c_k}{|N|} > 0 > -c_1 - \frac{E - \sum_{k \in N} c_k}{|N|}$. Then, similarly as in Case (i.a.1), we have that $x = nuc^1(v)$ by Theorem 3.3.

- (ii) We have $c_j < 0$ for some $j \in N$. Assume, without loss of generality, that $c_1 \leq \dots \leq c_{\bar{k}} < 0 \leq c_{\bar{k}+1} \leq \dots \leq c_n$ with $\bar{k} \in \{1, \dots, n\}$.

Case (ii.a) $E \leq \sum_{j \in N} c_j^+$.

We show that $nuc^1(v) = AM(N, E, c^+)$. Note that $c_1^+ = \dots = c_{\bar{k}}^+ = 0$ and $c_j^+ = c_j$ for every $j \in \{\bar{k} + 1, \dots, n\}$. Moreover, (N, E, c^+) is a bankruptcy problem. By definition of the Aumann–Maschler rule, $AM_j(N, E, c^+) = 0$ for every $j \in \{1, \dots, \bar{k}\}$. Let $x = AM(N, E, c^+)$. Then,

$$e(\{j\}, x) = -x_j = 0 \text{ and } e(N \setminus \{j\}, x) = x_j - c_j = -c_j > 0$$

for every $j \in \{1, \dots, \bar{k}\}$.

Since $AM_j(N, E, c^+) > 0$ for every $j \in N$ with $c_j^+ > 0$, it follows that $B_0^1(x, v) = \{1, \dots, \bar{k}\}$. Moreover, $B_1^1(x, v) = \{N \setminus \{1\}, \dots, N \setminus \{\bar{k}\}\}$ and $B_0^1(x, v) \cup B_1^1(x, v)$ is balanced. Further, following the same lines as in Case (i.a.1) of this proof, one can see that $x = nuc^1(v)$.

Case (ii.b) $\sum_{j \in N} c_j^+ < E \leq \sum_{j \in N} c_j^{\min}$.

We show that $nuc^1(v) = CEL(N, E, c^{\min})$, with $CEL_j(N, E, c^{\min}) = \max\{0, c_j^{\min} - \lambda\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ chosen such that $\sum_{j \in N} CEL_j(N, E, c^{\min}) = E$. Note that $c_j^{\min} = c_j - \min\{c_k | k \in N\} = c_j - c_1$ for every $j \in N$ and $0 = c_1^{\min} \leq c_2^{\min} \leq \dots \leq c_n^{\min}$. Moreover, it follows that (N, E, c^{\min}) is a bankruptcy problem.

Let $i \in N$ satisfy

$$c_{i-1}^{\min} \leq \lambda < c_i^{\min}.$$

Then, $\lambda = \frac{1}{|N|-i+1} \left(\sum_{k=i}^{|N|} c_k^{\min} - E \right)$ and

$$\text{CEL}_j(N, E, c^{\min}) = \begin{cases} 0 & \text{if } 1 \leq j \leq i - 1, \\ c_j^{\min} - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = \text{CEL}(N, E, c^{\min})$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} 0 & \text{if } 1 \leq j \leq i - 1, \\ -c_j^{\min} + \lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = x_j - c_j^{\min} - c_1 = \begin{cases} -c_j & \text{if } 1 \leq j \leq i - 1, \\ -c_1 - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Before we write the excesses in non-increasing order, we show that

$$i - 1 \leq \bar{k}. \tag{1}$$

First, note that

$$-c_1 > \lambda$$

since

$$\begin{aligned} \lambda &= \frac{1}{|N| - i + 1} \left(\sum_{k=i}^{|N|} c_k^{\min} - E \right) = \frac{1}{|N| - i + 1} \left(\sum_{k=i}^{|N|} (c_k - c_1) - E \right) \\ &= \frac{1}{|N| - i + 1} \left(\sum_{k=i}^{|N|} c_k - E \right) - c_1 \leq \frac{1}{|N| - i + 1} \left(\sum_{k \in N} c_k^+ - E \right) - c_1 < -c_1, \end{aligned}$$

where the weak inequality is a direct consequence of the definition of c^+ and the strict inequality follows by the assumption $\sum_{k \in N} c_k^+ < E$.

Next, we show that $i - 1 \leq \bar{k}$ by contradiction. Suppose, on the contrary, that $i - 1 > \bar{k}$. Then, $c_{i-1} > 0$ by definition of \bar{k} and $c_{i-1}^{\min} = c_{i-1} - c_1 > -c_1 > \lambda$. This establishes a contradiction with the definition of i . Therefore, we have $i - 1 \leq \bar{k}$. Then,

$$B_0^1(x, v) = \{1, \dots, i - 1\}$$

and

$$\begin{aligned} e(N \setminus \{1\}, x) &\geq \dots \geq e(N \setminus \{i - 1\}, x) \geq e(N \setminus \{i\}, x) = \dots = e(N \setminus \{n\}, x) \\ &> e(\{1\}, x) = \dots = e(\{i - 1\}, x) > e(\{i\}, x) \geq \dots \geq e(\{n\}, x) \end{aligned}$$

where $e(N \setminus \{i - 1\}, x) \geq e(N \setminus \{i\}, x)$ because $-c_{i-1} = -c_1 - c_{i-1}^{\min} \geq -c_1 - \lambda$ by definition of i ; $e(N \setminus \{n\}, x) > e(\{1\}, x)$ since $-c_1 > \lambda$ and, then, $-c_1 - \lambda > 0$;

$e(\{i - 1\}, x) > e(\{i\}, x)$ since $c_i^{\min} > \lambda$ by definition of i . Then, similarly as in Case (i.a.1), we have that $x = nuc^1(v)$ by Theorem 3.3.

Case (ii.c) $E > \sum_{j \in N} c_j^{\min}$.

We show that $nuc^1(v) = c + ES(N, E - \sum_{j \in N} c_j, c)$. Let $x = c + ES(N, E - \sum_{j \in N} c_j, c)$. Then, $x_j = c_j + \frac{E - \sum_{k \in N} c_k}{|N|}$,

$$e(\{j\}, x) = -x_j = -c_j - \frac{E - \sum_{k \in N} c_k}{|N|}, \quad \text{and}$$

$$e(N \setminus \{j\}, x) = x_j - c_j = \frac{E - \sum_{k \in N} c_k}{|N|}$$

for every $j \in N$. Therefore,

$$e(N \setminus \{1\}, x) = \dots = e(N \setminus \{n\}, x) > e(\{1\}, x) \geq \dots \geq e(\{n\}, x)$$

where the strict inequality is a direct consequence of the fact that $\frac{E - \sum_{k \in N} c_k}{|N|} > 0 > -c_1 - \frac{E - \sum_{k \in N} c_k}{|N|}$. Then, similarly as in Case (i.a.1), we have that $x = nuc^1(v)$ by Theorem 3.3. □

7 Proofs or counterexamples to the properties on Table 1

To see that the nucleolus and the 1-nucleolus do not satisfy aggregate monotonicity, we refer to the example in Tauman and Zapechelnyuk (2010), where both the nucleolus and 1-nucleolus coincide.

Since the 1-nucleolus of a balanced game (N, v) is given by $nuc^1(v) = v(\{i\}) + AM_i(N, E, c)$ for every $i \in N$, with (N, E, c) the bankruptcy problem given by $E = v(N) - \sum_{j \in N} v(\{j\})$ and $c_j = v(N) - v(N \setminus \{j\}) - v(\{j\})$ for every $j \in N$, it immediately follows that the 1-nucleolus satisfies individual superadditivity gains to the grand coalition since the Aumann–Maschler rule satisfies order preservation (cf. Thomson 2003). Example 5.2 shows that the nucleolus does not need to satisfy individual superadditivity gains to the grand coalition.

With respect to first agent consistency, Example 5.2 provides a balanced game for which the nucleolus does not satisfy first agent consistency. Next, we show that the 1-nucleolus satisfies first agent consistency. Let (N, v) be a balanced TU-game with $N = \{1, \dots, n\}$ such that $v(N \setminus \{1\}) + v(\{1\}) \geq v(N \setminus \{2\}) + v(\{2\}) \geq \dots \geq v(N \setminus \{n\}) + v(\{n\})$ and such that $(N \setminus \{1\}, v_{1, f_1(N, v)})$ is balanced. We show that $f_i(N, v) = f_i(N \setminus \{1\}, v_{1, f_1(N, v)})$ for every $i \in N \setminus \{1\}$. We can assume, without loss of generality, that $v = v_0$. Then, by Theorem 4.2, $nuc^1(v) = AM(N, E, c)$ with $E = v(N)$ and $c \in \mathbb{R}^N$ defined as $c_j = v(N) - v(N \setminus \{j\})$ for every $j \in N$. Note that $c_1 \leq c_2 \leq \dots \leq c_n$. We distinguish between two cases: $E \leq \frac{1}{2} \sum_{j \in N} c_j$ and $E > \frac{1}{2} \sum_{j \in N} c_j$.

Case 1 $E \leq \frac{1}{2} \sum_{j \in N} c_j$ In this case,

$$nuc^1(v) = CEA \left(N, E, \frac{1}{2}c \right)$$

and $nuc_1^1(v) = \min \left\{ \frac{c_1}{2}, \frac{v(N)}{n} \right\}$. Besides, the game $(N \setminus \{1\}, v_{1,nuc_1^1(N,v)})$ is given by

$$v_{1,nuc_1^1(N,v)}(S) = \begin{cases} v(N) - nuc_1^1(v) & \text{if } S = N \setminus \{1\}, \\ v(S \cup \{1\}) - nuc_1^1(v) & \text{if } |S| = n - 2, \\ v(S) & \text{otherwise.} \end{cases}$$

Let $\tilde{E} = v(N) - nuc_1^1(v)$ and $\tilde{c} \in \mathbb{R}^{N \setminus \{1\}}$ with

$$\tilde{c}_j = v_{1,nuc_1^1(N,v)}(N \setminus \{1\}) - v_{1,nuc_1^1(N,v)}(N \setminus \{1, j\}) = v(N) - v(N \setminus \{j\}) = c_j$$

for every $j \in N \setminus \{1\}$. First, if $nuc_1^1(v) = \frac{c_1}{2}$, then, $v(N) = E \leq \frac{1}{2} \sum_{j=1}^n c_j$ implies

$$\tilde{E} = v(N) - \frac{c_1}{2} \leq \sum_{j=2}^n \frac{c_j}{2} = \sum_{j=2}^n \frac{\tilde{c}_j}{2}.$$

Second, if $nuc_1^1(v) = \frac{v(N)}{n}$, then, $\frac{c_1}{2} \leq \frac{v(N)}{n}$ and

$$\tilde{E} = \frac{(n-1)v(N)}{n} \leq (n-1) \frac{c_1}{2} \leq \sum_{j=2}^n \frac{c_j}{2} = \sum_{j=2}^n \frac{\tilde{c}_j}{2}$$

where the second inequality is a direct consequence of $c_1 \leq c_2 \leq \dots \leq c_n$. Therefore,

$$nuc^1(v_{1,nuc_1^1(N,v)}) = CEA \left(N \setminus \{1\}, \tilde{E}, \frac{1}{2}c_{N \setminus \{1\}} \right)$$

and

$$\begin{aligned} nuc_2^1(v_{1,nuc_1^1(N,v)}) &= \min \left\{ \frac{c_2}{2}, \frac{v(N) - nuc_1^1(v)}{n-1} \right\} \\ &= \min \left\{ \frac{c_2}{2}, \frac{v(N) - CEA_1 \left(N, E, \frac{1}{2}c \right)}{n-1} \right\} \\ &= CEA_2 \left(N, E, \frac{1}{2}c \right) = nuc_2^1(v). \end{aligned}$$

Next, assume that $nuc_j^1(v_{1,nuc_1^1(N,v)}) = nuc_j^1(v)$ for every $j = 2, \dots, i - 1$, with $i \in N \setminus \{1, 2\}$. We show that $nuc_i^1(v_{1,nuc_1^1(N,v)}) = nuc_i^1(N, v)$. Note that

$CEA_j \left(N, \tilde{E}, \frac{1}{2}c \right) = nuc_j^1(v_{1,nuc_1^1(N,v)}) = nuc_j^1(v) = CEA_j \left(N, E, \frac{1}{2}c \right)$ for $j = 2, \dots, i - 1$ and

$$\begin{aligned} nuc_i^1(v_{1,nuc_1^1(N,v)}) &= \min \left\{ \frac{c_j}{2}, \frac{\tilde{E} - \sum_{j=2}^{i-1} CEA_j \left(N, \tilde{E}, \frac{1}{2}c \right)}{n - i + 1} \right\} \\ &= \min \left\{ \frac{c_j}{2}, \frac{v(N) - nuc_1^1(v) - \sum_{j=2}^{i-1} CEA_j \left(N, \tilde{E}, \frac{1}{2}c \right)}{n - i + 1} \right\} \\ &= \min \left\{ \frac{c_j}{2}, \frac{v(N) - \sum_{j=1}^{i-1} CEA_j \left(N, E, \frac{1}{2}c \right)}{n - i + 1} \right\} \\ &= CEA_i \left(N, E, \frac{1}{2}c \right) = nuc_i^1(v). \end{aligned}$$

Case 2 $E > \frac{1}{2} \sum_{j \in N} c_j$

In this case,

$$nuc^1(v) = c - CEA \left(N, \sum_{j \in N} c_j - E, \frac{1}{2}c \right)$$

$nuc_1^1(v) = c_1 - \min \left\{ \frac{c_1}{2}, \frac{\sum_{j=1}^n c_j - v(N)}{n} \right\}$. Besides, the game $(N \setminus \{1\}, v_{1,nuc_1^1(N,v)})$ is given by

$$v_{1,nuc_1^1(N,v)}(S) = \begin{cases} v(N) - nuc_1^1(v) & \text{if } S = N \setminus \{1\}, \\ v(S \cup \{1\}) - nuc_1^1(v) & \text{if } |S| = n - 2, \\ v(S) & \text{otherwise.} \end{cases}$$

Let $\tilde{E} = v(N) - nuc_1^1(v)$ and $\tilde{c} \in \mathbb{R}^{N \setminus \{1\}}$ with

$$\tilde{c}_j = v_{1,nuc_1^1(N,v)}(N \setminus \{1\}) - v_{1,nuc_1^1(N,v)}(N \setminus \{1, j\}) = c_j$$

for every $j \in N \setminus \{1\}$. First, if $nuc_1^1(v) = \frac{c_1}{2}$, $v(N) > \frac{1}{2} \sum_{j=1}^n \frac{c_j}{2}$ implies

$$\tilde{E} = v(N) - \frac{c_1}{2} > \frac{1}{2} \sum_{j=2}^n c_j = \frac{1}{2} \sum_{j=2}^n \tilde{c}_j.$$

Second, if $nuc_1^1(v) = c_1 - \frac{\sum_{j=1}^n c_j - v(N)}{n}$, then,

$$\begin{aligned} \tilde{E} &= v(N) - c_1 + \frac{\sum_{j=1}^n c_j - v(N)}{n} = \sum_{j=2}^n c_j - \frac{n-1}{n} \left(\sum_{j=1}^n c_j - v(N) \right) \\ &= \sum_{j=2}^n \left(c_j - \frac{\sum_{k=1}^n c_k - v(N)}{n} \right) \geq \sum_{j=2}^n \frac{c_j}{2} = \frac{1}{2} \sum_{j=2}^n \tilde{c}_j. \end{aligned}$$

where the inequality is a direct consequence of $c_j - \frac{\sum_{k=1}^n c_k - v(N)}{n} \geq \frac{c_j}{2}$ for every $j \in \{2, \dots, n\}$. To see this, note that $c_1 - \frac{\sum_{j=1}^n c_j - v(N)}{n} \geq \frac{c_1}{2}$ implies $\frac{c_1}{2} - \frac{\sum_{j=1}^n c_j - v(N)}{n} \geq 0$. Then, since $c_j \geq c_1$ for $j \in \{2, \dots, n\}$, we have $\frac{c_j}{2} - \frac{\sum_{k=1}^n c_k - v(N)}{n} \geq \frac{c_1}{2} - \frac{\sum_{k=1}^n c_k - v(N)}{n} \geq 0$ and $c_j - \frac{\sum_{k=1}^n c_k - v(N)}{n} \geq \frac{c_j}{2}$ for every $j \in \{2, \dots, n\}$. Therefore,

$$nuc^1(v_{1, nuc_1^1(N,v)}) = c - CEA \left(N \setminus \{1\}, \sum_{j=2}^n c_j - \tilde{E}, \frac{1}{2} c_{N \setminus \{1\}} \right)$$

and

$$\begin{aligned} nuc_2^1(v_{1, nuc_1^1(N,v)}) &= c_2 - CEA_2 \left(N \setminus \{1\}, \sum_{j=2}^n c_j - \tilde{E}, \frac{1}{2} c_{N \setminus \{1\}} \right) \\ &= c_2 - \min \left\{ \frac{c_2}{2}, \frac{\sum_{j=2}^n c_j - \tilde{E}}{n-1} \right\} \\ &= c_2 - \min \left\{ \frac{c_2}{2}, \frac{\sum_{j=2}^n c_j - (v(N) - nuc_1^1(v))}{n-1} \right\} \\ &= c_2 - \min \left\{ \frac{c_2}{2}, \frac{\sum_{j=1}^n c_j - v(N) - (c_1 - nuc_1^1(v))}{n-1} \right\} \\ &= c_2 - \min \left\{ \frac{c_2}{2}, \frac{\sum_{j=1}^n c_j - E - \min \left\{ \frac{c_1}{2}, \frac{\sum_{j=1}^n c_j - E}{n} \right\}}{n-1} \right\} \\ &= c_2 - \min \left\{ \frac{c_2}{2}, \frac{\sum_{j=1}^n c_j - E - CEA_1 \left(N, \sum_{j=1}^n c_j - E, \frac{1}{2} c \right)}{n-1} \right\} \\ &= c_2 - CEA_2 \left(N, \sum_{j=1}^n c_j - E, \frac{1}{2} c \right) = nuc_2^1(v). \end{aligned}$$

Following the same lines as in the case above, one can see that $nuc_j^1(v_{1, nuc_1^1(N,v)}) = nuc_j^1(N, v)$ for $j = 2, \dots, n$.

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