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Firing Costs, Misallocation, and Aggregate Productivity*

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ABSTRACT

We study the impact of firing costs on aggregate total factor productivity (TFP) in a dynamic general-equilibrium framework where the evolution of establishment-level productivity is not invariant to the policy. Firing costs not only generate static factor misallocation, but also distort the selection of establishment's growth by size, contributing to larger aggregate TFP losses. Numerical experiments indicate that firing costs equivalent to 5 year's wages imply a reduction in TFP of more than 20 percent. Factor misallocation accounts for 20 percent of the productivity loss, whereas the remaining 80 percent arises from distorted selection in the productivity process.

Keywords: firing costs, inaction, misallocation, establishments, productivity.
JEL codes: O1, O4, E1, E6.

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1 Introduction

A fundamental issue in economic growth and development is identifying the policies and institutions that account for the large differences in total factor productivity (TFP) and output per capita across countries. A recent literature has emphasized factor misallocation across heterogeneous production units for aggregate TFP differences (Banerjee and Duflo, 2005; Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). While the empirical evidence of factor misallocation across countries is overwhelming, the connection with the specific policies and institutions that create the bulk of misallocation remains elusive.[1] In this paper, we study impact of a specific policy—firing costs—on aggregate TFP in a framework where the evolution of establishment-level productivity is not invariant to the policy. We focus on firing costs because unlike other specific policies, firing costs are easily measurable in the data, show substantial variation across countries, and have been studied extensively in the misallocation literature. While there is some uncertainty about the exact quantitative magnitude of the channel we emphasize, we show through a series of numerical experiments that empirically-plausible measures of firing costs can generate large aggregate TFP loses arising from distorted selection in the evolution of establishment's productivity, in contrast to the relatively small impact of firing costs on aggregate productivity found in earlier studies where the evolution of establishment-level productivity is invariant to the policy.

We consider an otherwise standard model of producer heterogeneity building on the seminal works of Hopenhayn (1992) and Hopenhayn and Rogerson (1993). For analytical tractability, the model is set up in continuous time. Establishments are heterogeneous in their TFP that follows a stochastic process over time. Crucially, and differently from the related previous literature, policies that distort the size of establishments such as firing costs, have an effect

[1] Some of the specific policies and institutions emphasized in accounting for factor misallocation and aggregate TFP losses include firing costs (Hopenhayn and Rogerson, 1993), size-dependent policies (Guner et al., 2008), financial frictions (Buera et al., 2011; Midrigan and Xu, 2014; Moll, 2014)), trade barriers (Mutreja et al., 2018), among many others. See also surveys of the literature in Restuccia and Rogerson (2013), Restuccia (2013), Hopenhayn (2014a), and Restuccia and Rogerson (2017).

on the evolution of productivity for individual establishments and, hence, on the stationary distribution of productivity and aggregate TFP. A well-known property of firing costs in the context of dynamic models of producer heterogeneity is that these policies generate an inaction zone in employment decisions whereby some small but productive establishments remain small and some large but relatively less productive establishments remain large. In other words, the inaction zone is a range of productivities for which even establishments with the same productivity have different employment levels. This type of inaction in employment decisions to changes in productivity generates factor misallocation, as the policy weakens the relationship between the allocation of employment and establishment productivity; but in particular, inaction creates rank reversals—the situation where some relatively more productive establishments are smaller than some relatively less productive establishments and vice versa—that is shown to be important in generating large output losses from misallocation (Hopenhayn, 2014b). In addition, in our framework, we emphasize that firing costs also alter the selection of establishment's productivity growth contributing to a substantial reduction in aggregate productivity that is beyond static factor misallocation.

Our model also differs from the literature on firing costs in that rather than a continuous choice of employment, we assume that establishments choose among a finite discrete set of employment levels. We make this assumption for tractability in order to be able to characterize analytically the distribution of establishment-level productivity in an environment where the employment demand is a correspondence. An implication of this assumption is that discrete employment levels together with production heterogeneity generate dispersion in the value of marginal products even in an undistorted economy with no firing costs. This dispersion can be interpreted as arising from "real" adjustment costs and is not treated as misallocation in our analysis since we emphasize the impact of firing costs on output and productivity relative to a benchmark economy with no firing costs. Importantly, firing costs create misallocation by inducing inaction in employment decisions but when interacted with discrete employment levels, they economize on adjustment costs which are present in the undistorted economy. As a result, to the extent that in real economies there are frictions to the adjustment of establishment employment that are not due to policies, our model with discrete employment levels can potentially provide a more conservative estimate of firing-cost policies than in a setting with continuous employment choices.

To study the importance of firing-cost policies, we conduct a series of numerical experiments in the context of a benchmark economy with no firing costs. In particular, we calibrate a benchmark economy with no firing costs to micro and macro data for the United States and consider quantitative experiments that increase the size of firing costs—the cost for an individual establishment to reduce employment—with a range from 6 months to 5 year's wages. Relative to the benchmark economy with no firing costs, aggregate TFP in the economy with a firing cost of 6 months' wages is 0.97 and in the economy with 5 year's wages is 0.79. These are large TFP losses when compared to the previous literature (Hopenhayn and Rogerson, 1993; Moscoso-Boedo and Mukoyama, 2012; Hopenhayn, 2014a). Interestingly, when we decompose the total effect of firing costs on aggregate TFP between static misallocation and dynamic misallocation (as changes in the productivity distribution of establishments), we find that dynamic misallocation accounts for around 80 percent of the total effect. This implies that the quantitative effect of static misallocation is of similar magnitude or even smaller than that in the existing literature. For example, in Hopenhayn (2014a) where there is no entry/exit of establishments as in our framework but instead features continuous employment choices, the TFP loss due to factor misallocation of a firing cost policy equivalent to 5 year's wages is 7.5 percent compared to 4.8 percent in our framework. But this policy translates into a 21.3 percent TFP loss in our framework when accounting for dynamic misallocation.

A desirable property of our framework is that we are able to provide direct analytical results on the main variables of interest. Following the seminal work of Dixit (1989), in the benchmark economy with no firing costs, there is only one threshold productivity for which large establishments whose productivity decline become small and small establishments whose productivity increase become large. In economies with firing costs there is an inaction zone. Additionally, the inaction zone becomes larger with higher firing costs and general equilibrium effects shift the inaction zone towards lower levels of productivity. These properties of establishment decisions entail a rich pattern of misallocation whereby employment is misallocated within establishments of the same productivity and across establishments of different productivity, by shifting employment from more to less productive establishments and generating rank reversals. Furthermore, we fully characterize the productivity distribution of establishments and show how changes in firing costs impact this distribution by making it flatter and reducing its average productivity. We solve the model using Laplace transforms techniques, which allows us to fully characterize how changes in firing costs impact the rate at which establishments adjust their employment size. In the model, higher firing costs reduce average productivity in the economy.

As discussed earlier, our paper relates closely to the literature assessing the aggregate productivity losses of firing costs. An important distinction of our framework is that firing costs affect the evolution of establishment productivity which can greatly contribute to amplify the negative impact of firing costs on aggregate productivity. In this respect, our paper is closer to the seminal work of Lagos (2006) that, in the context of a model of production heterogeneity and search frictions, studies the negative impact of firing costs on measured aggregate TFP, emphasizing the effect of this labor-market policy on the set of active producers in addition to resource misallocation. Also related to our work is Mukoyama and Osotimehin (2016) studying the effects of firing costs in a model of endogenous growth and Bentolila and Bertola (1990) emphasizing the effect of firing costs on establishment's labor demand in European countries in a partial equilibrium setting. More broadly, our paper shares with a growing literature emphasizing the dynamic effects of distortionary policies such as Hsieh and Klenow (2014), Da-Rocha et al. (2017), Guner et al. (2018), Bento and Restuccia (2017). We differ from this literature in quantifying the effect of a specific measurable policy. A key element of the distortionary effects of firing costs is the property of an inaction zone in employment decisions that generate misallocation. While firing costs share with many other policies and institutions that also produce inaction zones in economic decisions, the effects of firing costs are different because the inaction is with respect to the dynamic response of employment decisions, generating the rich patterns of misallocation discussed earlier. For instance, size-dependent policies such as those studied in Guner et al. (2008) imply inaction in employment decisions whereby inputs are constant for a range of establishment productivity. However, these policies do not generate rank reversals.[2] We also emphasize that our analysis of the productivity effects of firing costs has broader implications as we show an equivalence between the effects of firing costs policies and the effects of hiring costs policies. Overall, our analysis reveals the importance of considering the dynamic productivity effects of the policy for a more accurate assessment of the aggregate impacts of these policies.

The remainder of the paper is organized as follows. In the next section and Section 3, we describe the economic environment in detail and characterize its main properties. Section 4 calibrates a benchmark economy with no firing costs to data for the United States and perform a series of numerical experiments to study the aggregate implications of firing costs. We conclude in Section 5. Appendix A contains formal proofs of all the lemmas in the paper.

2 Model

Our framework builds from the work of Hopenhayn (1992), Hopenhayn and Rogerson (1993), and Dixit (1989). Establishments hire labor in a competitive market and their productivity follows a stochastic process. Time is continuous and the horizon is infinite. We focus on

[2] Gourio and Roys (2014) and Garicano et al. (2016) study misallocation from a specific form of size-dependent policy using micro data from France where a labor regulation only applies to establishments larger than a certain size. Also related is a trade model of inaction—the decision to export or not—in Impullitti et al. (2013).

a stationary equilibrium of this model and study the impact of firing costs on aggregate measures of TFP and output.

2.1 General description

The unit of production in the economy is the establishment. Establishments are heterogeneous in their productivity z. They are described by a production function $f(z, n)$ that uses labor to produce output. The function f is assumed to exhibit decreasing returns to scale in labor and to satisfy the usual Inada conditions. The production function is given by:

$$f(z, n) = zn^{\alpha}, \quad \alpha \in (0, 1).$$

For tractability, we assume that establishments can only hire two different amounts of labor n_1 and n_2, where n_2 is larger than n_1. This assumption implies that there will be dispersion in marginal products across establishments even when the economy features no firing costs. We will discuss in our results that this assumption of discrete employment levels makes the quantitative implications of firing costs on output and productivity conservative relative to a setting with continuous employment.

Establishment's productivity z follows a Geometric Brownian motion, that depends on the establishment size, the Geometric Brownian motions are given by:

$$dz = \mu_1 z dt + \sigma z dw_z \quad \text{and} \quad dz = \mu_2 z dt + \sigma z dw_z,$$

where the drift of the Brownian motion μ_i depends on the establishment's size and the standard deviation σ is the same for both sizes. There is a mass one of infinitely-lived households with preferences over consumption goods and labor supply described by the utility function,

$$\int_0^\infty e^{-\rho t}\left[u(c)-v(n)\right]dt,$$

where c is consumption, n is labor supply, and ρ is the discount rate. Households own equal shares of the establishments. We next introduce firing costs that distort the decision of establishments to adjust their size.

2.2 Policy distortions

In a distorted economy, we assume that establishments have to pay a firing cost τ in units of labor per worker in order to reduce employment from large n_2 to small n_1. The firing costs policy creates inertia in employment decisions because establishments delay their decisions of firing and hiring workers and consequently adjusting their employment size. We assume that the revenue from firing costs paid by establishments is redistributed to households in the form of a lump-sum transfer T.

2.3 Incumbents' problem

Incumbents maximize the present value of profits. At each point in time, establishments observe their current TFP shock z and their employment size n_i, where $i \in \{1, 2\}$, and decide to keep their current size or to adjust by hiring or firing workers. This is a standard optimal switching problem described by Dixit (1989). The problem is characterized by the value function at the current state and by the value matching condition at the switching points. We first describe the dynamic problem of a small incumbent n_1 and then we describe the dynamic problem of a large incumbent n_2.

Small establishments observe their productivity and choose to keep their current size n_1 or to hire workers and become large n_2. They receive revenue from selling output and pay a wage bill at every point in time. Formally, the dynamic problem of a small establishment is defined by:

$$\rho V_1(z) \quad = \quad z n_1^\alpha - w n_1 + E_z \frac{dV_1(z)}{dt}, \qquad (1)$$
$$s.t. \quad dz = \mu_1 z dt + \sigma z dw_z,$$

and by the value matching condition at the switching point z_1 where small establishments hire workers and become larger, $V_1(z_1) = V_2(z_1)$, and the smooth pasting condition at the switching, $V_1'(z_1) = V_2'(z_1)$.

Large establishments observe their current productivity z and choose to keep their current size n_2 or to fire workers and become small n_1, paying firing costs $\tau w(n_2 - n_1)$. Large establishments receive revenue from selling output and pay a wage bill. Formally, the dynamic problem of a large establishment is defined by:

$$\rho V_2(z) \quad = \quad z n_2^\alpha - w n_2 + E_z \frac{dV_2(z)}{dt}, \qquad (2)$$
$$s.t. \quad dz = \mu_2 z dt + \sigma z dw_z,$$

and by the value matching condition at the switching point z_2, where large establishments are indifferent between paying the firing costs τ and become small or being large, $V_1(z_2) - \tau w(n_2 - n_1) = V_2(z_2)$, and the smooth pasting condition at the switching point, $V_1'(z_2) = V_2'(z_2)$. In Lemma 1 we characterize the value function of small $V_1(z)$ and large $V_2(z)$ establishments and the two policy functions $\{z_1, z_2\}$.

Lemma 1. *Given a wage rate w, interest rates ρ, and firing costs τ, the value function of small establishments $V_1(z)$ and the value function of large establishments $V_2(z)$ that solve the small establishment's problem (1) and the large establishment's problem (2) are given by*

$$V_i(z) = \frac{{n_i}^\alpha}{\rho - \mu_i} z - \frac{w n_i}{\rho} + B_i z^{\beta_i},$$

where $\beta_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$ *for* $i \in \{1, 2\}$, *and the constants* $\{B_1, B_2\}$ *and the policy functions* $\{z_1, z_2\}$ *solve the two value matching conditions and the two smooth pasting conditions together.*

Proof See Appendix A.1.

In order to fully characterize the impact of firing costs on the optimal decision of establishments, we choose the positive root β_1 for small establishments and the negative root β_2 for large establishments. The positive root for small establishment has the desirable property that the option value of becoming larger increases when the productivity increases, while the negative root for large establishment has the desirable property that the option of becoming smaller decreases when the productivity increases. In Lemma 2, we show that B_1 and B_2 are positive.

Lemma 2. *If* β_1 *is the positive root and* β_2 *is the negative root, then* B_1 *and* B_2 *are positive.*

Proof See Appendix A.2.

The value functions of large and small establishments have an intuitive interpretation, where the first two terms are the present value of being a small or a large establishment when switching is not allowed and the last term is the present value of the switching option. Changes in firing costs have two effects on the incumbents' problem. It has a direct effect on the present value of being large and small through the constants B_1 and B_2 and a general equilibrium effect through changes in wages. In the next Lemma 3 we characterize these two effects.

Lemma 3. *Given a wage rate* w, *interest rates* ρ, *and firing costs* τ.

1) The inaction rate, $\theta = z_2/z_1$, *is the solution of the following non-linear equation:*

$$\varphi(\theta) = \frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = \frac{1}{1 - \rho\tau}, \tag{3}$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1}\frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$.

2) The policy functions z_1 *and* z_2 *are given by:*

$$z_1 = \kappa\Omega_2(\theta)w \quad \textit{and} \quad z_2 = \theta z_1, \tag{4}$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

Proof See Appendix A.3.

Lemma 3 is key to understand the model dynamics. In the first part of Lemma 3, we characterize the inaction rate θ, which is a function of the productivity process, summarized by β_1 and β_2, the interest rates ρ, and firing costs τ. Overall, an increase in firing costs generates a decrease in the inaction rate. In an economy without firing costs, the inaction rate is equal to one and establishments do not delay their decision of firing and hiring workers. In this case, there is a unique switching point and no inaction zone.

An important result from Lemma 3 is that the inaction rate is independent of prices, but the policy functions are linear in prices. A reduction in the wage rate moves the policy functions to the left, reducing the average productivity in the economy. The final impact of an increase in firing costs on the inaction zone depends on the combination of the impact on the inaction rate, summarized by θ, and on the general equilibrium impact on the wage rate w.

In Figure 1, we illustrate these two mechanisms. In the left panel, we illustrate how a decrease in the inaction rate θ^*, increases the inaction zone measured by the area between z_1 and z_2. In the right panel, we illustrate how an increase in firing costs τ reduces the inaction rate θ^*.

The results in Lemma 3 are not restricted to firing costs. Instead, these results can be easily extended to the case where the costs are on hiring workers instead of firing and this is relevant

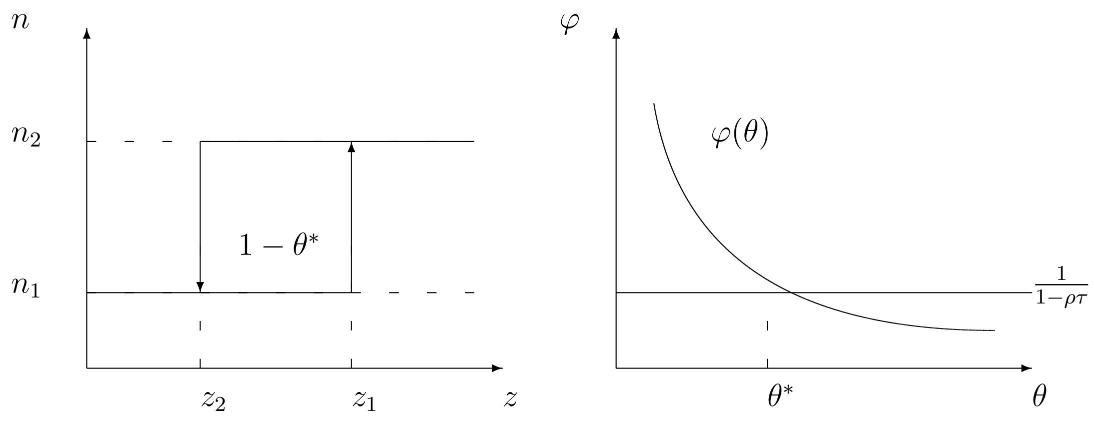


Figure 1: Inaction Zone and Inaction Zone Rate

empirically as many labor market policies generate costs associated with hiring workers above a certain threshold size. We establish an equivalence between firing costs and hiring costs. We can rewrite the small and large establishment's problems where establishments face hiring costs τ_h instead of firing costs τ. The new value-matching conditions are:

$$\begin{aligned} V_1(z_1) &= V_2(z_1) - \tau_h w(n_2 - n_1), \\ V_1(z_2) &= V_2(z_2), \end{aligned}$$

and the new smooth pasting conditions are the same in both the firing costs and hiring costs problem. In Lemma 4 we show that solving the model with hiring costs is equivalent to solving the model with firing costs.

Lemma 4. *Given hiring costs τ_h, there is firing costs τ that generates the same inaction zone rate, given by:*

$$\frac{1}{1-\rho\tau} = 1 + \rho\tau_h,$$

where ρ is the interest rate.

Proof See Appendix A.4.

Lemma 4 demonstrates that there is a simple relationship between firing costs τ and hiring costs τ_h. Given firing costs τ, we can find hiring costs τ_h that generates the same inaction zone, and consequently the same equilibrium in both economies. In the next section, we characterize the stationary distribution.

2.4 Stationary distribution

We first characterize the stationary distribution in a distorted economy with firing costs and then we solve for the stationary distribution of the undistorted economy.

2.4.1 Distorted economy

We characterize the solution of the stationary distribution of large establishments, leaving the solution of the stationary distribution of small establishments to Appendix A.5. In order to solve for the stationary distribution, it is easier to work in the logarithm of the establishment productivity z instead of levels. Let x be the logarithm of an establishment with productivity z and size i relative to the switching point z_i, that is $x = \log(z/z_i)$. The variable x is equal to zero at the switching point and has domain in $[0, +\infty)$.

Let $m_2(x,t)$ denote the number density function of large establishments with productivity x at time t. At time t, the total number of large establishment is equal to the integral from zero to plus infinity of the number density function, $M_2(t) = \int_0^{+\infty} m_2(x,t)dx$. The large establishments' productivity process can be characterized by a *modified* Kolmogorov-Fokker-Planck equation of the form:

$$\frac{\partial m_2(x,t)}{\partial t} = -\hat{\mu}_2 \frac{\partial m_2(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 m_2(x,t)}{\partial x^2} + B_2(x,t), \tag{5}$$

where the drift $\hat{\mu}_i$ is equal to $\mu_i - \frac{\sigma^2}{2}$ and the function $B_2(x,t)$ is the mass of new large establishments that arrives with productivity x at time t. We can rewrite the Geometric Brownian motion of large and small establishments as a Brownian motion in the logarithm of the establishment productivity z_i as $dx_i = \hat{\mu}_i dt + \sigma dw_z$. The *modified* Kolmogorov-Fokker-Planck equation (5) is supplemented by two boundary conditions:

$$\lim_{x \to +\infty} m_2(x,t) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{\partial m_2(x,t)}{\partial x} = 0.$$

The two boundary conditions guarantee that there is no establishments' mass at the upper limit and that the function is smooth at the upper limit. We are interested in the stationary distribution of the number density function and consequently we are looking for a solution that is separable in time, that is $m_2(x,t) = M_2(t)f_2(x)$ and $B_2(x,t) = M_2(t)b_2(x)$; where $f_2(\cdot)$ is the large establishments' probability density function and $b_2(x)$ is a Dirac delta function that describes the arrival of new establishments. After making this restriction, we can rewrite Kolmogorov-Fokker-Planck equation (5) as:

$$\frac{M_2'(t)}{M_2(t)} f_2(x) = -\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + b_2(x), \tag{6}$$

where $M_2'(t)/M_2(t)$ is the change in the mass of establishments over time denoted by η_2; and two boundary conditions are:

$$\lim_{x \to +\infty} f_2(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} f_2'(x) = 0. \tag{7}$$

These two conditions are supplemented by additional conditions that guarantee that $f_2(\cdot)$ is a pdf:

$$f_2(x) \geq 0 \quad \text{and} \quad \int_0^{+\infty} f_2(x)dx = 1. \tag{8}$$

Note that we can rewrite $M_2(t)$ as $M_2(t) = e^{\eta_2 t} M_2(0)$ and when η_2 is equal to zero the mass of large establishments does not change over time.

In our model, arrival at the large establishment's distribution occurs at the switching point $x_1 = \log(z_1/z_2)$. In this point, small establishments choose to hire more workers and become large. Mathematically, we can express the arrival using a Dirac delta function:

$$\delta(x - x_1) = \begin{cases} +\infty & \text{if} \quad x = x_1 \\ 0 & \text{if} \quad x \neq x_1, \end{cases} \tag{9}$$

the function is equal to infinity at the point in which new establishments enter x_1 and zero otherwise. Let $\hat{b}_2$ be the arrival rate at point x, the function $b_2(x)$ can be described as:

$$b_2(x) = \hat{b}_2 \delta(x - x_1). \tag{10}$$

The Dirac delta function has two desirable properties: First, the integral over the domain is the arrival rate $\hat{b}_2$, i.e. $\int_0^\infty \hat{b}_2 \delta(x - x_1) dx = \hat{b}_2$, and second, the integral weighted by the density function is the mass of establishments at the switching point $\int_0^\infty \delta(x - x_1) f_2(x) dx = f_2(x_1)$.

The constraints in equations (7) to (10) restrict η_2, after integrating the Kolmogorov-Fokker-Planck equation (6), applying the boundary conditions, and using the Dirac delta function's properties, we find that the change in the mass of large establishments is given by:

$$\eta_2 = \hat{b}_2 - \hat{\mu}_2 f_2(0) + \frac{\sigma^2}{2} f_2'(0). \tag{11}$$

The expression for η_2 has a very intuitive interpretation, it states that the growth rate of the number (mass) of large establishments η_2 is equal to the rate at which the number of small establishments decide to hire workers and become large $\hat{b}_2$, minus the rate at which large establishments decide to fire workers and become small, $\hat{\mu}_2 f_2(0) - \frac{\sigma^2}{2} f_2'(0)$. As a result, the large establishments' mass grows when the hiring rate is larger than the firing rate, and it is constant when both are equal. Since we are solving for the stationary equilibrium, we restrict the solution to an economy with a constant mass of establishments of each size, that is η_2 equal to zero. We can now solve for the stationary distribution $f_2(\cdot)$.

After substituting η_2 equal to zero into the Kolmogorov-Fokker-Planck equation (6), we find the following second-order differential equation:

$$-\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + \hat{b}_2 \delta(x - x_1) = 0, \tag{12}$$

which supplemented by the boundary conditions and $f_2(\cdot)$ being a pdf completely characterize the problem. The presence of the Dirac delta functions in (12) indicates that we cannot expect classical solutions to the problem in $C^2[0, +\infty)$. We thus look for solutions $f_2(x)$ such that $f_2(x) \in C^2[0, +\infty) \cup L^1[0, \infty)$ and $f_2(x)$ has continuous second derivatives for all $x \in [0, +\infty)$ except perhaps at $x = x_1$.

We solve the second-order differential equation using Laplace transforms (see Appendix A.6 for details). In order for the probability density function be bounded we need to impose a boundary condition at 0. This boundary condition guarantees that the mass of establishments that are indifferent between switching or not is equal to zero, $f_2(0) = 0$. In Lemma 5 we characterize the small and large establishments' stationary distributions.

Lemma 5. *Let $x_1 = \log(z_1/z_2)$ be the switching point, where small establishments that become large enter the large establishment distribution, and let $x_2 = \log(z_2/z_1)$ be the switching point, where large establishments that become small enter the small establishment distribution. The stationary distribution of small establishments $f_1(\cdot)$ and the stationary distribution* *of large establishments* $f_2(\cdot)$ *are given by:*

$$f_1(x) = \begin{cases} \frac{-1}{x_2}(e^{\alpha_1(x-x_2)} - e^{\alpha_1 x}) & \forall x \in (-\infty, x_2] \\ \\ \frac{-1}{x_2}(1 - e^{\alpha_1 x}) & \forall x \in (x_2, 0) \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{x_1}(1 - e^{\gamma_1 x}) & \forall x \in [0, x_1) \\ \\ \frac{1}{x_1}(e^{\gamma_1(x-x_1)} - e^{\gamma_1 x}) & \forall x \in (x_1, +\infty) \end{cases}$$

where $\gamma_1 = \dfrac{2\hat{\mu}_2}{\sigma^2}$ *and* $\alpha_1 = \dfrac{2\hat{\mu}_1}{\sigma^2}$.

Proof See Appendix A.7.

In Figure 2, we illustrate the stationary distribution of small and large establishments and the stationary equilibrium dynamics. A small establishment hires workers at x equal to zero. At this point the establishment leaves the small establishments' distribution and enters the large establishments' distribution at point x_2. This establishment stays at the large establishments' distribution until its productivity reaches the point x equal to zero. At this point, the establishment fires workers becoming small again, and a new cycle starts. The rectangular area between the switching points and zero is the inaction zone caused by the firing costs policy. Inside the inaction zone there is rank reversal, there are small establishments that are more productive than large establishments. In the absent of firing costs, large and small establishments switch at the same point and there is no inaction zone. In the next section, we examine the stationary distribution of an undistorted economy.

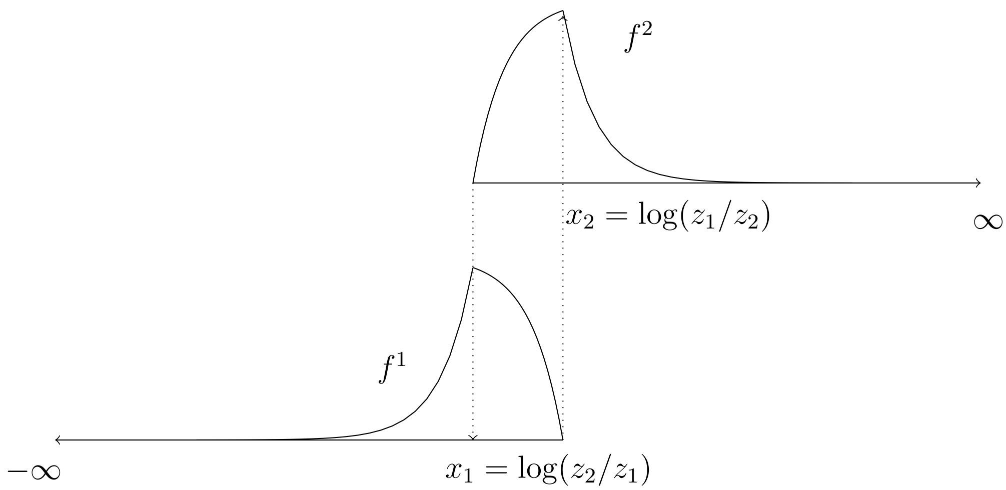


Figure 2: Stationary Distribution Dynamics

2.4.2 Undistorted economy

We characterize the stationary distribution of the undistorted economy following the same methodology as in the distorted economy. In the undistorted economy, there is an unique switching point z_*. We solve the economy again in logs and after the renormalization, the switching point is equal to zero, $x_* = \log(z_*/z_*) = 0$. The stationary distribution of large establishments in the undistorted economy is the solution of a *modified* Kolmogorov-Fokker-Planck equation that after following the same steps as in the distorted economy we obtain the following second-order differential equation:

$$-\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + \hat{b}_2 \delta(x) = 0, \tag{13}$$

subject to the respective boundary conditions and $f_2(\cdot)$ being a pdf. The main difference between the distorted and the undistorted economy is that in the undistorted economy there is no inaction zone and establishments hire and fire workers in the same point x_*. We solve the boundary-value problem using Laplace transforms (see again Appendix A.6). After applying the Laplace transforms, we find that the large establishments' boundary conditions are satisfied only when $\hat{\mu}_2$ is negative, and following the same methodology to small establishments, we find that the small establishments' boundary conditions are satisfied only when $\hat{\mu}_1$ is positive. These two conditions imply that the small establishments' growth rate μ_1 is larger than the large establishments' growth rate μ_2. In Lemma 6 we formalize this result.

Lemma 6. *If γ_1 is negative and α_1 is positive, the stationary distribution of small establishments $f_1(\cdot)$ and the stationary distribution of large establishments $f_2(\cdot)$ are given by:*

$$\begin{aligned} f_1(x) &= \alpha_1 e^{\alpha_1 x}, \quad x \leq 0, \\ f_2(x) &= -\gamma_1 e^{\gamma_1 x}, \quad x \geq 0, \end{aligned}$$

where $\gamma_1 = \frac{2\hat{\mu}_2}{\sigma^2}$ and $\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2}$.

Proof See Appendix A.8.

In the next section, we discuss the necessary equilibrium conditions to find a stationary equilibrium.

2.5 Flow condition

Our focus is on the stationary equilibrium where the distribution of small establishments $f_1(\cdot)$ and large establishments $f_2(\cdot)$ are constant, and the mass of small M_1 and large establishments M_2 are also constant. In order to guarantee that in the stationary equilibrium, the mass of small and large establishments are constant, it is necessary that the mass of establishments that leaves an establishment's size distribution is the same as the mass of establishments that enter on the other establishment's size distribution.

The mass of establishments that leaves a distribution is equal to the total mass of establishments multiplied the rate at which establishments reach the switching points, normalized to zero, and the mass of establishments that arrives is equal to the born rate multiplied by the mass. This condition gives rise to the following two equilibrium conditions one to small and another one to large establishments:

$$M_1 \left(-\hat{\mu}_1 f_1(0) + \frac{\sigma^2}{2} f_1'(0) \right) = -M_2 \hat{b}_2, \quad (14)$$

$$-M_2 \left(-\hat{\mu}_2 f_2(0) + \frac{\sigma^2}{2} f_2'(0) \right) = M_1 \hat{b}_1. \quad (15)$$

These two conditions guarantee that the total of mass of establishment is constant in the stationary equilibrium.

2.6 Household's problem

Households solve a static consumption-leisure maximization problem:

$$\max_{c,n} \left[u(c) - v(n) \right],$$

subject to the budget constraint $c = wn + \Pi + T$, where the right-hand side of the budget constraint is given by the wage income wn, the lump-sum transfer given by the government T, and the total profits of operating establishments Π. Now, we are ready to define the equilibrium.

2.7 Equilibrium definition

Definition The stationary equilibrium for this economy is an stationary distribution for small and large establishments $\{f_1(\cdot), f_2(\cdot)\}$, a value function for small and large establishments $\{V_1(\cdot), V_2(\cdot)\}$, a mass of small and large establishments $\{M_1, M_2\}$, a policy function for small and large establishments $\{z_1, z_2\}$, prices $\{w, \rho\}$, profits Π, transfer T, and household allocations $\{c, n\}$, such that:

i) Given prices and profits, the allocations $\{c, n\}$ solve the household's problem.

ii) Given prices, incumbents' policy functions $\{z_1, z_2\}$ and value functions $\{V_1(\cdot), V_2(\cdot)\}$ solve the incumbents' problem.

iii) The stationary distributions $\{f_1(\cdot), f_2(\cdot)\}$ solve the Kolmogorov-Fokker-Planck equations and determine aggregate profits.

iv) Labor market clears.

v) The flow conditions are satisfied.

vi) The government budget constraint is satisfied, $T = \tau(n_2 - n_1)wM_2\hat{b}_2$.

vii) Mass condition $M = M_1 + M_2$.

Conditions (i) and (ii) are standard. Condition (iii) is key to find the stationary distribution. Condition (iv) is the labor market clearing. Condition (v) guarantees that the total mass of establishments is constant. Condition (vi) guarantees that the government budget constraint is satisfied and condition (vii) that the mass of establishments clears. In the next section, we characterize the stationary equilibrium.

3 Solution and characterization

The model is very tractable and we characterize key stationary equilibrium properties in more detail. From the flow conditions, we can characterize the mass of small and large establishments. After substituting the stationary distributions from Lemma 5 into equations (14) and (15) we obtain the following condition:

$$M_1 \frac{\hat{\mu}_1}{\sigma^2} = -M_2 \frac{\hat{\mu}_2}{\sigma^2}, \tag{16}$$

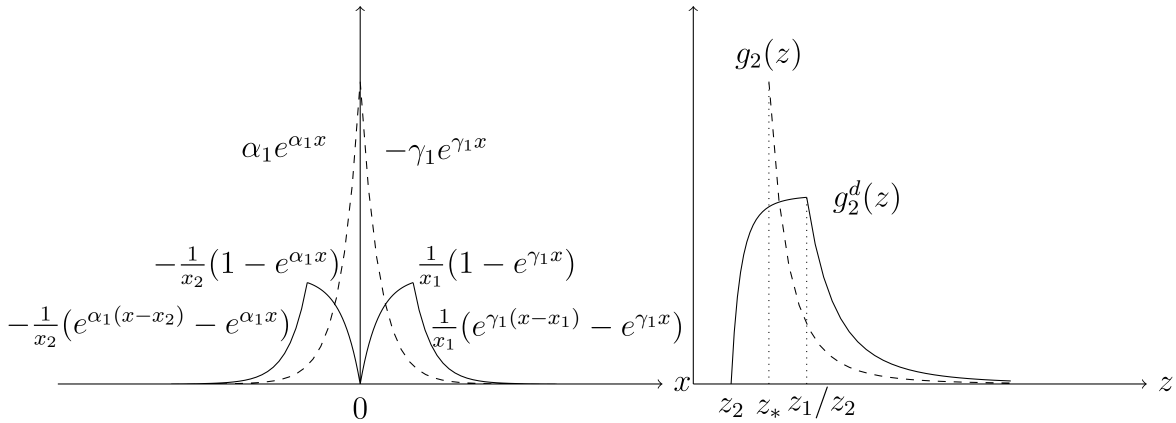
which relates the mass of establishments with the stochastic process of productivity. It provides a condition for the mass of establishments and drifts that guarantees that the flows of large and small establishments is constant in the stationary equilibrium.

We normalize the total mass of establishments to be equal to one and focus on a symmetric equilibrium where the mass of large and small establishments is the same, i.e. $M_1 = 0.5$. Given these assumptions, solving the stationary equilibrium is very simple. The parameters determining the stochastic process for productivity must satisfy the flow condition. Given firing costs τ, from Lemma 3, we find the inaction rate θ. Since profits are linear in wages, the equilibrium wage can be found by solving the labor market clearing condition, and then after applying the second part of Lemma 3, we find policy functions and the stationary distribution (we leave the formal solution of the stationary equilibrium in Appendix A.9).

In Figure **??**, we illustrate the impact of firing costs on the stationary distribution. Panel (a) illustrates the impact of firing costs on the entire distribution, whereas panel (b) focus on the distribution of large establishments.

The distribution of TFP in the benchmark economy denoted by f is a combination of the TFP distribution of large and small establishments. An increase in firing costs makes the distribution flatter because of inaction. In panel (b), the effect of firing costs is decomposed into two effects: (i) static misallocation, caused by large establishments that wait longer to switch—reducing the average productivity of large establishments—represented by the area between z_2 and z_*; and (ii) dynamic misallocation, caused by the impact firing costs on the selection of establishments that enter the large establishments' distribution.

In an economy with firing costs, small establishments wait for a higher productivity in order to hire workers and become large. As result, the flow of small establishments that enter in the large establishment's distribution with higher productivity is lower than in the case with no firing costs. This effect is represented by the area after z_1/z_2 between the undistorted economy's density function and the distorted economy's density function. In



(a) Distribution of (log) TFP

(b) Misallocation Decomposition

Figure 3: Distribution Dynamics

Notes: The figure reports the stationary distribution of establishment TFP for undistorted and distorted economies. Dashed lines represent the distribution of the undistorted benchmark economy and solid lines for the distorted economy. Panel (a) reports the distributions of log establishment TFP for the entire economy. Panel (b) reports the distribution of TFP levels of large establishments. The TFP level, z_* is the switching point in the benchmark economy, whereas (z_2, z_1) is the inaction zone in the distorted economy.

addition, in an economy with firing costs, large establishments wait for a lower productivity in order to fire workers and become small. As result, large establishments exit the large establishment's distribution with lower productivity than in the case with no firing costs. This effect is represented by the area between z_2 and z_* (which is zero in the non distorted economy). Note that increasing the cumulative density before z_* in the economy with firing costs reduces the cumulative density after z_*, reducing aggregate TFP.

Since an objective of the paper is to understand the impact of firing costs on aggregate TFP, we now characterize aggregate TFP in the model as:

$$\text{TFP} = (1 - s)E_1 z + sE_2 z,$$

which is the weighted average of the productivity of large and small establishments, with the weight s given by $s = \frac{n_1^\alpha}{n_1^\alpha + n_2^\alpha}$. Note that this weight is constant across economies with different firing costs so their effect on aggregate TFP is reflected in the change of productivity of small and large establishments.

4 Quantitative analysis

We consider a benchmark economy with no firing costs and calibrate this economy to data for the United States. We then illustrate the possible quantitative impact of firing costs.

4.1 Calibration

We calibrate a benchmark economy with no firing costs to data for the United States. Our main objective is to study the quantitative impact of firing costs on the distribution of establishments and on aggregate outcomes relative to the undistorted economy in the same spirit of Restuccia and Rogerson (2008). We start by defining briefly our benchmark undistorted economy.

To calibrate this economy, we start by selecting a set of parameters that are standard in the literature, these parameters have either well-known targets which we match or the parameter values have been well discussed in the literature. Our calibration follows closely Hopenhayn and Rogerson (1993). A model period is set to 5 years. Preferences are given by the following utility function:

$$u(c) - v(n) = \frac{c^{1-\eta}}{1-\eta} - An.$$

We select η to be equal to 0.5 and normalize A to be equal to 1. We select M_1 equal to 0.5 and we focus on a symmetric stationary equilibrium where the share of large and small establishments is the same. We normalize the size of small establishments n_1 to be equal 1 and we choose the size of large establishments n_2 to be equal to 122.4. to match the average size of establishments equal to 61.7 from Hopenhayn and Rogerson (1993). We note that while the relative size of small establishments matters for the level of output and TFP in the benchmark economy, it does not affect the relative outcomes across distorted and undistorted economies. This finding in our framework is important as it implies that the relative size of expanding and contracting establishments does not matter for the quantitative effect of firing costs on relative TFP. We formalize this normalization result in Appendix A.10.

Following the literature we assume decreasing returns in the establishment-level production function and set α equal to 0.64 (Hopenhayn and Rogerson, 1993). We select the discount rate ρ to generate an annual real interest rate of 4 percent.

We calibrate the remaining two parameters $\{\mu_1, \sigma^2\}$ by solving the stationary equilibrium where two moments in the model match two corresponding targets in the data. We construct the following two statistics in the model and match with the corresponding targets in the data:

(1) Turnover rate of jobs. We calculate the turnover rate of jobs as the amount of employment in contracting firms divided by total employment:

$$\text{Turnover rate} = \frac{(n_2 - n_1)}{n_1 + n_2}\hat{b}_2,$$

where in equilibrium the endogenous flow is given by $\hat{b}_2 = \hat{\mu}_2 f_2(0) - \frac{\sigma^2}{2} f_2'(0)$.

(2) Gini coefficient of establishment's size distribution:

$$\text{Gini} = \frac{1}{\gamma_1 - 2} + \frac{1}{\alpha_1 - 2}.$$

The two parameters that define the productivity's stochastic process $\{\mu_1, \sigma^2\}$ are selected simultaneously to match a turnover rate of jobs of 0.30 for the calibrated economy to the United States in Hopenhayn and Rogerson (1993) and a the Gini coefficient of the establishment size distribution of 0.89 reported by Luttmer (2010). This procedure implies that

$\mu_1 = 0.0703$ and $\sigma = 0.1895$. Since there is a simple mapping between μ_1 and μ_2 given by equation (16), the calibration implies $\mu_2 = -0.0344$. Table 1 summarizes the calibrated parameter values.

Table 1: Benchmark Calibration to U.S. Data

Parameter	Value	Target
A	1	Normalization
α	0.64	Literature
ρ	0.217	Literature
η	0.50	Literature
n_1	1.00	Normalization
n_2	122.40	Average establishment size
M_1	0.50	Normalization
μ_1	0.0703	Turnover rate
σ	0.1895	Gini size distribution

4.2 Firing costs

We quantify the impact of firing costs on aggregate TFP and output by comparing these statistics in each distorted economy relative to the undistorted benchmark economy.

We study the impact of different firing-cost policies by changing τ and report statistics relative to the benchmark economy. Firing cost τ has a direct interpretation with other values in the model. Since a period in the model is equal to 5 years, a value of τ equal to 0.1 corresponds to firing costs equivalent to 6 months' wages, a value of τ equal to 0.2 corresponds to firing costs of 1 year's wages, and a value of τ equal to 1 corresponds to firing costs of 5 year's wages. We report the results of these experiments in Table 2 for a number of statistics such as aggregate output, aggregate TFP, and wages. All statistics are reported relative to the benchmark economy in percent.

In Table 2 firing costs have a substantial negative impact on aggregate output and TFP.

Table 2: Aggregate Effects of Changes in Firing Costs τ

	τ		
	0.1	0.2	1.0
Relative Y	97.8	95.6	78.7
Relative TFP	97.8	95.6	78.7
Relative wages	98.9	97.8	88.7

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 year's wages. Statistics reported relative to the benchmark economy in percent.

An economy with 6 months' wages of firing costs has aggregate output and TFP that is 97.8 percent of the benchmark economy. While an economy with 1 year's wage of firing costs has aggregate output and TFP that is 95.6 percent of the benchmark economy, and an economy with 5 years of firing costs has aggregate output and TFP of 78.7 percent of the undistorted economy. The negative impact on TFP is much larger than estimates from the related literature. For example, Hopenhayn and Rogerson (1993) estimate an impact on average productivity relative to a benchmark economy of 99.2 percent for 6 months' wages and 97.9 percent to 1 year's wage and Hopenhayn (2014a) estimates the TFP loss of a firing cost of 5 year's wages at 7.5 percent.

Overall, differences in output and TFP relative to the benchmark economy are increasing in the amount of firing costs, more firing costs implies lower TFP and consequently output. Firing costs distort the employment decisions of establishments. Table 3 contains all this information relative to the benchmark economy. The first result from Table 3 is that increases in firing costs decrease the inaction zone rate, increasing the inaction zone, measured by the difference between z_1 and z_2. Distortions in employment decisions by establishments are also reflected in the turnover rate of job which decreases from 30 percent in the benchmark economy to 14, 11, and 6 percent in the three distorted economies.

As emphasized by Hopenhayn (2014b), policy distortions that generate rank reversals are important in generating substantial negative effects in aggregate output and TFP. We dis-

Table 3: Effects of Changes in Firing Costs τ

	τ		
	0.1	0.2	1.0
Relative θ	0.69	0.62	0.39
Relative z_1	1.17	1.21	1.21
Relative z_2	0.81	0.74	0.48

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 years' wages. Statistics reported relative to the benchmark economy in percent.

cuss how firing costs generate rank reversals in our framework. Figure ?? documents the impact of firing costs on the inaction zone, the measure of establishments whose decisions are distorted and the amount of rank reversals. In the benchmark economy, there is a threshold productivity z_* whereby establishments with productivity above this level are large and below are small. With firing costs in the distorted economy, there is an inaction zone defined by the range (z_2, z_1) whereby establishments in this range do not change their employment decisions generating rank reversals. For instance, establishments with productivity between z_2 and z_* are large and remain large even though in the benchmark economy would become small; and establishments with productivity between z_* and z_1 are small and remain small even though without firing costs would become large. This situation entails misallocation as resources are not reallocated to best uses, but note that it also entails rank reversals since some less productive establishments (those in the shaded area between z_2 and z_*) are larger than other establishments that are more productive yet small (those in the shaded area between z_* and z_1). Moreover, the amount of rank reversals increases with firing costs. As discussed earlier, the inaction zone is increasing in firing costs but note that because of the change in the stationary distribution of small and large establishments, it also changes the amount of rank reversals in the economy. Quantitatively this effect is significant in our framework, for example, the mass of firms with relatively low productivity that remain large increases from zero in the benchmark economy to 20 percent and almost 50 percent in the economies with 1 years' wages and 5 year's wages respectively; whereas the mass of establishments with relatively high productivity that remain small increase from zero to 9 and 5 percent.

We now discuss the implications for the results of our assumption of discrete levels of employment relative to a model with continuous employment such as that in Hopenhayn and Rogerson (1993) and Hopenhayn (2014a). We first note that our model features dispersion in marginal products of labor across establishments even in the undistorted economy as there is dispersion in productivity within large and small establishments due to the discrete nature of employment levels. One interpretation of the dispersion in marginal products in the benchmark economy is "real" factor adjustment costs. This dispersion reduces aggregate output and productivity relative to an economy with continuous employment, but in our assessment of firing costs we quantify their impact relative to the benchmark economy with discrete levels of employment. More importantly, the impact of firing costs on aggregate output and productivity is smaller in a framework with discrete levels of employment than in an economy with continuous employment. The reason is that firing costs mitigate the cost associated with adjusting employment for other reasons. To the extent that firms in the real world face frictions in the adjustment of labor that are not due to policy distortions, our model with discrete levels of employment potentially provides a more conservative estimate of the impact of firing costs on aggregate productivity across countries than in a setting with continuous employment choices. Despite this, our numerical results indicate a substantial amplification effect of firing costs policies due to distorted selection in establishment's productivity growth.

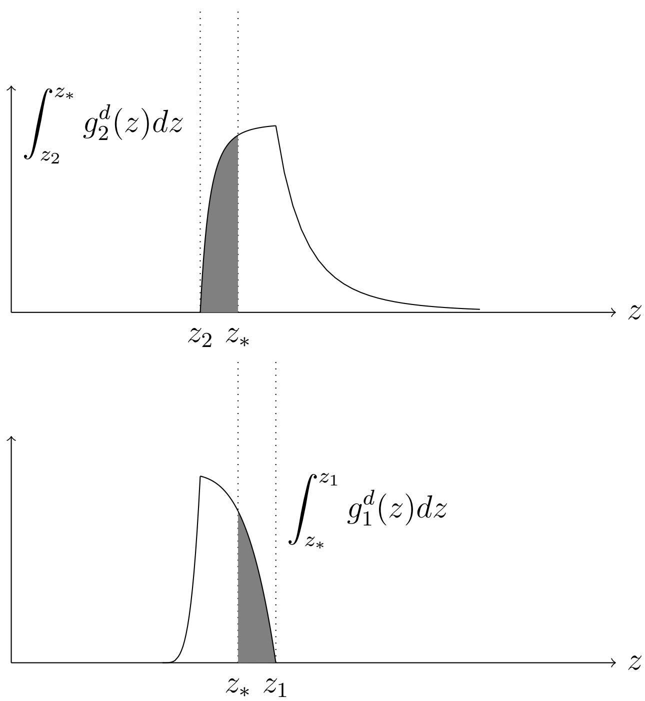
4.3 Static and dynamic misallocation

The impact of firing costs on misallocation can be decomposed in two effects: a static and a dynamic. The static effect is the classic impact on the policy function of small and large

Figure 4: Firing Costs and Rank Reversals

Benchmark Economy		
	$z_2 = z^* = z_1$	
	n_1	n_2
$z < z_*$	$g_1(.) \in (0, z_*)$	0
$z > z_*$	0	$g_2(.) \in (z_*, \infty)$

Distorted Economy		
	$z_2 < z^* < z_1$	
	n_1	n_2
$z < z_*$	$g_1^d(.) \in (0, z_*)$	$g_2^d(.) \in (z_2, z_*)$
$z > z_*$	$g_1^d(.) \in (z_*, z_1)$	$g_2^d(.) \in (z_*, \infty)$



Notes: TFP level z_* is the switching point in the benchmark economy, whereas (z_2, z_1) is the inaction zone in the distorted economy. The area defined by $\int_{z_*}^{z_1} g_1^d(z)dz$ represent small establishments (n_1) with high productivity ($z > z_*$) in the distorted economy that in the benchmark economy would be large; and the area defined by $\int_{z_2}^{z_*} g_2^d(z)dz$ represent large establishments (n_2) with low productivity ($z < z_*$). These areas represent the extent of rank reversals.

establishments, which causes large establishments to switch at lower levels of productivity and small establishments to switch at higher levels productivity. The dynamic effect is due to changes in the average productivity of switching establishments. Since in our model the productivity distribution of establishments is endogenous, when large firms start switching at lower levels of productivity, this also impacts on the distribution of productivity of small establishments as switchers to low employment have on average lower productivity than before, and the opposite is true for large establishments as small firms start switching at higher level of productivity, the average productivity of switchers to high employment is higher than before. In Table 4 we quantify these two effects of misallocation on aggregate TFP.

Table 4: Static and Dynamic Misallocation

	τ		
	0.1	0.2	1.0
Relative TFP	97.8	95.6	78.7
Decomposition:			
Static misallocation	10.3	12.8	23.0
Dynamic misallocation	89.7	87.2	77.0

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 years' wages. TFP is reported relative to the benchmark economy in percent.

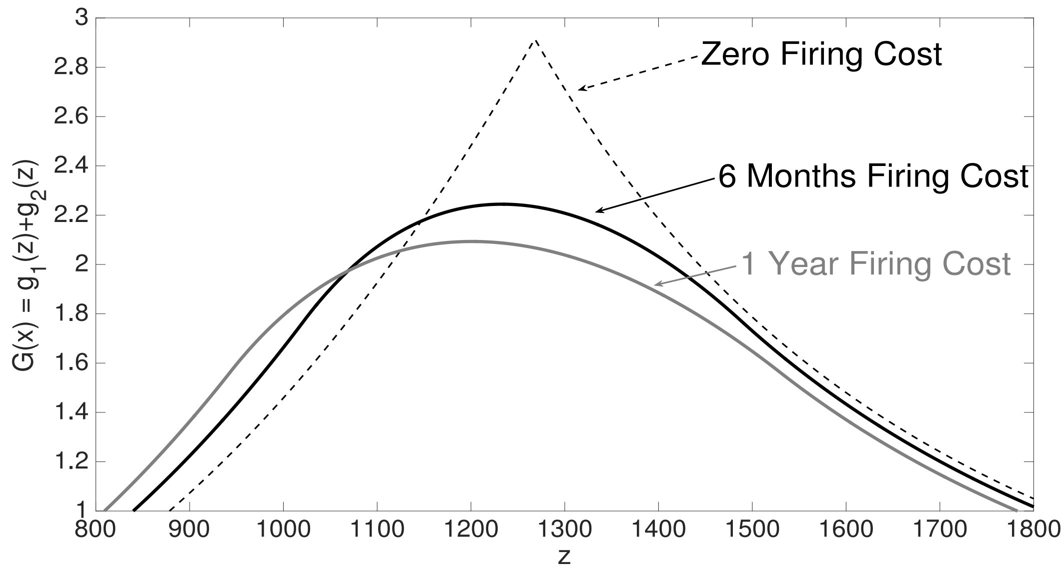
From Table 4 we observe that the bulk of the negative effects of firing costs on aggregate TFP are due to dynamic misallocation. When firing costs are the equivalent of 6 months' wages, the TFP loss is around 2.2 percent: 90 percent due to dynamic misallocation (2 percent TFP loss) and 10 percent due to static misallocation (0.2 percent TFP loss). When firing costs are the equivalent of 5 year's wages, the TFP loss is 21.3 percent: static misallocation accounts for 23 percent (4.8 percent TFP loss) and dynamic misallocation accounts for 77 percent (16.4 percent TFP loss). As firing costs increase static misallocation becomes more relevant, but not nearly as relevant as dynamic misallocation. As discussed earlier, because of our assumption of discrete employment levels, the static effect of misallocation on

TFP is smaller in our model than in models with continuous employment. For instance, in Hopenhayn and Rogerson (1993), firing costs equivalent to 6 months' wages reduce average productivity by 0.8 percent compared to 0.2 percent in our model. In Hopenhayn (2014a), firing costs equivalent to 5 year's wages reduce TFP by 7.5 percent compared to 4.8 percent in our model. Nevertheless, in all these cases, the total effect of misallocation on TFP is much larger in our model because of distorted selection of firing costs on establishment's productivity growth.

4.4 Discussion

We now discuss our results with regards to the relative size of small and large establishments. A crucial assumption in models of firing costs is the extent of mean reversion in the establishment's productivity process, as the implied changes in productivity determine the expansion or contraction in establishment sizes that is potentially distorted by firing costs. Our framework assumes a form of mean reversion that in addition implies a change in the selection of establishments' productivity growth that is systematic across small and large establishments. As a result, it is important to provide some empirical evidence on the size-productivity growth relationship and to discuss the importance of relative sizes across establishments. We start by noting that while there is some disagreement about the exact size-productivity growth relationship, the empirical evidence leans towards the case where productivity growth is larger for small than for large firms, especially when panel data of firms is available. For example, using a panel data of firms from the Amadeus database for a set of European countries, Da-Rocha et al. (2018) find that productivity growth is systematically higher in small compared to large establishments in every country in the sample. We also emphasize that the relative size of establishments is not crucial for our results, in fact we can easily prove that relative TFP between the distorted and undistorted economies is independent of the normalized value n_1, see Appendix A.10 for a formal proof. This result

Figure 5: Stationary Distribution of Establishment TFP for Different Firing Costs



suggest that it is not the size-productivity growth relationship per se that is key for the amplification effect of firing costs on productivity but rather the distortion in the selection of productivity growth that is generated by the firing cost policy which generates a stronger reduction on the turnover rate of jobs.

An implication of our framework is that firing costs change the stationary distribution of establishment's productivity in the economy. In Figure 5, we plot the distribution of establishment TFP of the undistorted economy, the distribution of an economy with 6 months of firing costs, and an economy with a year of firing costs. The increase in the inaction zone makes the TFP distribution flatter, increasing its variance. Since wages also decrease, from Table 2, the overall effect on the distribution of TFP is both a move to the left of the distribution, caused by lower wages, and a flatter distribution from the increase in the inaction zone. Although it is difficult to precisely assess from micro data of establishments across countries whether the distributions in poor countries have these properties we think that at least the direction of these changes seems plausible. Our numerical exercises at least illustrate that more research is desired and needed in order to identify the exact magnitude of the productivity-size channel.

5 Conclusions

We developed a tractable framework with heterogeneous production units that builds on the work of Dixit (1989), Hopenhayn (1992), and Hopenhayn and Rogerson (1993). We showed that in this framework firing costs not only impact the employment decision of establishments of firing and hiring workers as emphasized by the existing literature, but also impact the distribution of productivity across establishments. We showed through a series of numerical experiments that firing costs have a substantial negative impact on aggregate TFP, a quantitative effect that is orders of magnitude larger than in the earlier literature.

We decomposed the effect of firing costs on aggregate TFP in two channels: a static misallocation effect and a dynamic effect through changes in establishment's productivity. This decomposition allows us to relate the quantitative impact of the static misallocation channel with previous estimates in the literature and to show that the bulk of the effect of firing costs on aggregate TFP in our framework is due to the dynamic channel.

We discussed the implications of firing costs in our framework with respect to potentially observable outcomes such as the turnover rate of jobs and the characteristics of the distribution of productivity which we hope can lay the foundation for the empirical identification of the size-productivity growth channel in future research. We also established an equivalence between firing costs and hiring costs allowing us to broaden the implications of our analysis. In future work, this equivalence could help connect our framework with micro data in specific policy contexts to obtain empirical estimates of the impact of these policies on aggregate TFP.

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A Appendix

A.1 Proof Lemma 1

The proof is by guessing and verifying. We guess the following functional form for the value function $W_i(z) = a_i + A_i z + B_i z^{\beta_i}$ and solving the Hamilton-Jacobi-Bellman equation we find that $a_i = -\frac{wn_i}{\rho}$, $A_i = \frac{{n_i}^{\alpha}}{\rho - \mu_i}$ and β_i is equal to

$$\beta_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{17}$$

Finally from the boundary and smooth pasting conditions we find B_1, B_2, z_1, and z_2 solving the following system of nonlinear equations:

$$\begin{aligned}
(1-\beta_1)B_1 z_1^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_1^{\beta_2}, &(18)\\
(1-\beta_1)B_1 z_2^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_2^{\beta_2} + \tau(n_2 - n_1)w, &(19)\\
\beta_1 B_1 z_1^{\beta_1} &= (A_2 - A_1)z_1 + \beta_2 B_2 z_1^{\beta_2}, &(20)\\
\beta_1 B_1 z_2^{\beta_1} &= (A_2 - A_1)z_2 + \beta_2 B_2 z_2^{\beta_2}. &(21)
\end{aligned}$$

And this conclude the proof. ■

A.2 Proof Lemma 2

First we need to show that the positive root of (17) is greater than one. First note that this polynomial is convex, since $\Omega''(\beta) = 1$, and at zero $\Omega(0) = -\frac{\rho}{\sigma^2}$, which is negative. So, $\Omega(\beta)$ has a positive and a negative root. At one $\Omega(1) = \frac{\mu_1}{\sigma^2} - \frac{\rho}{\sigma^2}$, since ρ is greater than μ_1, $\Omega(1)$ is also negative. Consequently, the positive root must be greater than one. Now, we can prove that B_1 and B_2 are positive.

From Lemma 1 and the two equations on the smoothing pasting conditions (21) and (21), we can write B_1 and B_2 as a function of parameters, z_1 and z_2, the constant B_2 is equal to:

$$B_2 = \frac{(A_2 - A_1)(z_2^{1-\beta_1} - z_1^{1-\beta_1})}{\beta_2(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})}. \tag{22}$$

Note that the numerator is positive (negative), because $A_2 - A_1$ is positive and $(z_2^{1-\beta_1} - z_1^{1-\beta_1})$ is positive, since $z_1 > z_2$ and $\beta_1 > 1$, as we can see from ($\beta_1 < 1$). The denominator is also positive, because β_2 is negative and $(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})$ is algo negative. As a result B_2 is positive. Substituting the expression for B_2 into equation (21), we find B_1 equal to:

$$B_1 = \frac{(A_2 - A_1)(z_1 z_2)^{-\beta_1}(z_1^{\beta_2} z_2 - z_1 z_2^{\beta_2})}{\beta_1(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})}. \tag{23}$$

Note that the numerator is negative, because $z_1 > z_2$ and $z_1^{\beta_2} < z_2^{\beta_2}$ since $\beta_2 < 0$. In addition, the denominator is also negative, because $(z_1^{\beta_2-\beta_1} - z_1^{\beta_2-\beta_1})$ is negative and β_1 is positive, as result B_1 is positive. ■

A.3 Proof Lemma 3

In order to characterize the inaction zone rate, we first assume that $z_2 = \theta z_1$ for a $\theta \in (0, 1]$, substituting $z_2 = \theta z_1$ into the expression of B_1 in equation (23) and B_2 into equation (22), we find:

$$B_1 = \frac{(A_2 - A_1) z_1^{1-\beta_1} (\theta^{1-\beta_2} - 1)}{\beta_1 (\theta^{\beta_1-\beta_2} - 1)}, \tag{24}$$

$$B_2 = \frac{(A_2 - A_1) z_1^{1-\beta_2} (\theta^{1-\beta_1} - 1)}{-\beta_2 (\theta^{\beta_2-\beta_1} - 1)}. \tag{25}$$

Second we divide both sides of of the boundary condition (18) and the boundary condition (20) by $(1 - \beta_2) B_2$ and we substitute $z_2 = \theta z_1$ to obtain:

$$\frac{(1-\beta_1) B_1}{(1-\beta_2) B_2} z_1^{\beta_1} - z_1^{\beta_2} = \frac{(a_2 - a_1)}{(1-\beta_2) B_2}, \tag{26}$$

$$\frac{(1-\beta_1) B_1}{(1-\beta_2) B_2} (\theta z_1)^{\beta_1} - (\theta z_1)^{\beta_2} = \frac{(a_2 - a_1) + \tau (n_2 - n_1) w}{(1-\beta_2) B_2}. \tag{27}$$

We can substitute B_1 and B_2 from equation (24) and (25) into equation (26) and (27) and divide the two equations to obtain:

$$\varphi(\theta) = \frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = \frac{1}{1 - \rho\tau},$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1} \frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1-1})}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$.

To find z_1 we can substitute B_1 and B_2 from equations (26) and (27) into the boundary condition (18), after some algebraic manipulation we find z_1 equal to:

$$z_1 = \frac{-\left[\frac{n_2}{\rho} - \frac{n_1}{\rho}\right]}{\left[\frac{n_2^\alpha}{\rho-\mu_2} - \frac{n_1^\alpha}{\rho-\mu_1}\right]} \Omega_2(\theta) w,$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$. ■

A.4 Proof Lemma 4

Consider the optimal switching problem of small and large establishments facing a hiring cost. This problem is characterized by the two value matching conditions one for small and one for large establishments and by the two smoothing pasting conditions:

$$\begin{aligned} V_1(z_1) &= V_2(z_1) - \tau_h w(n_2 - n_1), \\ V_1(z_2) &= V_2(z_2), \\ V_1'(z_1) &= V_2'(z_1), \\ V_1'(z_2) &= V_2'(z_2). \end{aligned}$$

From Lemma 1 we know that the solution of this problem is given by two roots that solve the polynomial $\beta_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$ for $i \in \{1, 2\}$, and by the constants $\{B_1, B_2\}$ and the policy functions $\{z_1, z_2\}$ solve the two value matching conditions and the two smoothing pasting conditions together. Note that for a given stochastic process for small and large establishments the two roots are the same in the case of firing or hiring costs. Following the same methodology as Lemma 3, we find:

$$\varphi(\theta) = \frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = 1 + \rho\tau_h,$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1} \frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$. Since $\varphi(\theta)$ only depends on β_1 and β_2 that are independent of the costs we find that for a given hiring costs τ_h there exist a firing costsτ that solves

$$\frac{1}{1 - \rho\tau} = 1 + \rho\tau_h$$

and generate the same inaction zone rate θ. ■

A.5 Small establishment's stationary distribution

In order to find the stationary distribution of small establishments we apply the same methodology as in the distribution of large establishments. First, let $m_1(x, t)$ denote the number density function of small establishments. At time t, the small establishments productivity process follows the modified Kolmogorov-Fokker-Planck equation below:

$$\frac{\partial m_1(x,t)}{\partial t} = -\hat{\mu}_1 \frac{\partial m_1(x,t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 m_1(x,t)}{\partial x^2} + B_1(x,t), \tag{28}$$

where the function $B_1(x, t)$ are the new small establishment that arrival with productivity x at time t. The partial differential equation (28) is supplement by the two boundary conditions

$$\lim_{x \to -\infty} m_1(x,t) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{\partial m_1(x,t)}{\partial x} = 0$$

where in the case of small establishments the boundary conditions are at the bottom of the distribution. We are interested in solving for the steady state productivity distribution, as in the case of large establishments, we look for a solutions with the form $m_1(x, t) = M_1(t)f_1(x)$ and $B_1(x, t) = M_1(t)b_1(x)$, where substituting in the Kolmogorov-Fokker-Planck equation we find:

$$\frac{M_1'(t)}{M_1(t)} f_1(x) = -\hat{\mu}_1 f_1'(x) + \frac{\sigma^2}{2} f_1''(x) + b_1(x - x_2), \tag{29}$$

and the new boundary conditions:

$$\begin{aligned}\lim_{x \to -\infty} f_1(x) &= 0,\\ \lim_{x \to -\infty} f_1'(x) &= 0,\end{aligned}$$

and the additional requirement that $f_1(x)$ is also a pdf leads to the conditions:

$$\begin{aligned}f_1(x) &\geq 0,\\ \int_{-\infty}^{0} f_1(x)dx &= 1,\end{aligned}$$

where now differently from large establishments, small establishment domain is from minus infinity to zero. As in the large establishment problem, we calculate the separation rate of small establishments by integrating equation (29) from minus infinity to zero, where small establishment decide to become large. The growth rate for small establishment η_1, is given by

$$\eta_1 \int_{-\infty}^{0} f_1(x)dx = \left(-\hat{\mu}_1 f_1(x) + \frac{\sigma^2}{2} f_1'(x)\right)\Big|_{-\infty}^{x=0} + \int_{0}^{+\infty} \hat{b}_1 \delta(u)du,$$

and applying the boundary conditions and using the delta function, we find that growth rate of small firms is equal to:

$$\eta_1 = -\hat{\mu}_1 f_1(0) + \frac{\sigma^2}{2} f_1'(0) + \hat{b}_1,$$

which has the same interpretation as in the large establishment case. As in the large establishments, we look for the stationary equilibrium in which the number of small establishments does not grow over time, which restricts η_1 to be equal to zero. Now, we can rewrite the Kolmogorov-Fokker-Planck equation by substituting η_1 we obtain:

$$-\hat{\mu}_1 f_1'(x) + \frac{\sigma^2}{2} f_1''(x) + \hat{b}_1 \delta(x - x_2) = 0,$$

subject to the boundary conditions $f_1(0) \geq 0$ and $\int_{-\infty}^{0} f_1(x)dx = 1$. Therefore, the stationary pdf is the solution of the boundary-value problem that consists of solving the equation:

$$f_1''(x) - \alpha_1 f_1'(x) + \alpha_2 \delta(x - x_2) = 0,$$

and the boundary conditions $f_1(0) \geq 0$ and $\int_{-\infty}^{0} f_1(x)dx = 1$, where the constants α_1 and α_2 are given by

$$\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2} \qquad \text{and} \qquad \alpha_2 = \frac{2\hat{b}_1}{\sigma^2}.$$

A.6 Laplace Transforms

Laplace transforms are given by

$$\begin{aligned}\mathscr{L}[f'(x)] &= s\mathscr{L}[f(x)] - f(0),\\ \mathscr{L}[f''(x)] &= s^2\mathscr{L}[f(x)] - sf(0) - f'(0),\\ \mathscr{L}[\delta(x-x_*)] &= e^{-sx_*}\end{aligned}$$

After applying Laplace transforms to equation (12), we find the following Laplace transforms equation:

$$(s^2 - \gamma_1 s)Y(s) = f_2'(0) + (s-\gamma_1)f_2(0) - \gamma_2 e^{-sx_1}.$$

After applying the Laplace inverse and some algebraic manipulation, we find the following solution to the boundary value problem:

$$f_2(x) = \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \gamma_2\frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right] + f_2(0),$$

where $H(x_1)$ is Heaviside step function given by:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x \le x_1, \\ 1 & \text{if} \quad x > x_1. \end{cases}$$

After applying Laplace transforms on the large establishments' second order differential equation (13) we obtain the following differential equation:

$$(s^2 - \gamma_1 s)Y_2(s) = (f_2'(0) - \gamma_1 f_2(0) - \gamma_2) + sf_2(0),$$

because the growth rate of large establishments is zero in equilibrium, the second term between parenthesis must be zero, leading to the following solution to the stationary distribution of large establishments:

$$Y_2(s) = \frac{f_2(0)}{(s-\gamma_1)},$$

and after solving for the Laplace inverse we obtain the following pdf $f_2(x) = f_2(0)e^{\gamma_1 x}$. We solve for the constants $f_2(0)$ to guarantee that the integral over the domain is equal to one.

A.7 Proof Lemma 5

The stationary pdf is the solution of the boundary-value problem that consists of solving the equation:

$$f_2''(x) - \gamma_1 f_2'(x) = -\gamma_2\delta(x - x_1),$$

subject to the boundary conditions $f_2(0) \ge 0$ and $\int_0^{+\infty} f_2(x)dx = 1$. As in the undistorted economy case, Lemma 6, we are going to use Laplace transforms. After some algebraic manipulation we obtain the following equation:

$$(s^2 - \gamma_1 s)Y(s) = f_2'(0) + (s - \gamma_1)f_2(0) - \gamma_2 e^{-sx_1},$$

and

$$Y(s) = \frac{f_2'(0) + (s - \gamma_1)f_2(0) - \gamma_2 e^{-sx_1}}{(s - \gamma_1)s}.$$

Note that Laplace inverses of each component is equal to:

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s(s-\gamma_1)}\right] &= \frac{1}{\gamma_1}\left(e^{\gamma_1 x} - 1\right), \\
\mathscr{L}^{-1}\left[\frac{1}{s}\right] &= 1, \\
\mathscr{L}^{-1}\left[\frac{e^{-sx_1}}{s(s-\gamma_1)}\right] &= \frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right],
\end{aligned}$$

where $H(x_1)$ is Heaviside step function given by:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x \leq x_1, \\ 1 & \text{if} \quad x > x_1. \end{cases}$$

Substituting the Laplace inverses in the second oder differential equation give us the stationary distribution for large firms:

$$f_2(x) = \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \gamma_2 \frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right] + f_2'(0).$$

Note that $f_2(0)$ must be equal to zero in order to the integral to be bounded. This boundary condition guarantee the mass of firms that are indifferent between switching or not is equal to zero. Now substituting the Heaviside step function $H(x_1)$ we find:

$$f_2(x) = \begin{cases} \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) & \text{if} \quad x \leq x_1, \\ \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \frac{\gamma_2}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right] & \text{if} \quad x > x_1. \end{cases}$$

We only need to solve for $f_2'(0)$ to have the complete characterization of the large establishments distributions. We find $f_2'(0)$ to guarantee that that the integral of $f_2(\cdot)$ is one.

$$\frac{f_2'(0)}{\gamma_1}\int_0^{+\infty}\left(e^{\gamma_1 x} - 1\right)dx - \frac{\gamma_2}{\gamma_1}\int_{x_1}^{+\infty}\left(e^{\gamma_1(x-x_1)} - 1\right)dx = 1,$$

where we can rewrite the integral above as:

$$\frac{f_2'(0)}{\gamma_1}\int_0^{+\infty} e^{\gamma_1 x}dx - \frac{\gamma_2}{\gamma_1}\int_{x_1}^{+\infty} e^{\gamma_1(x-x_1)}dx - \frac{f_2'(0)}{\gamma_1}\int_0^{x_1} 1dx - \left(\frac{f_2'(0)}{\gamma_1} - \frac{\gamma_2}{\gamma_1}\right)\int_{x_1}^{+\infty} 1dx = 1.$$

The last term of the integral is zero, so the mass of establishments is constant in equilibrium, by integrating all the other terms we find the following expression:

$$-\frac{f_2'(0)}{\gamma_1^2}+\frac{\gamma_2}{\gamma_1^2}-\frac{f_2'(0)}{\gamma_1}x_1=1.$$

Substituting γ_2 gives

$$-\frac{f_2'(0)}{\gamma_1^2}+\frac{f_2'(0)}{\gamma_1^2}-\frac{f_2'(0)}{\gamma_1}x_1=1,$$

as a result, $f_2'(0)=-\frac{\gamma_1}{x_1}=-\frac{2\hat{\mu}_2}{x_1\sigma^2}$.

Now we can solve for the small establishment's distribution, first we are going to change variables. Let $f_1(x)=g(-y)$ and $\delta(x)=\delta(-y)$, we can rewrite the second order differential equation for small establishment as:

$$g''(-y)+\alpha_1 g'(-y) \quad = \quad -\alpha_2\delta(-y-x_2).$$

After some algebraic manipulation we obtain the following equation:

$$(s^2-\alpha_1 s)Y(s)=-g'(0)+(s+\alpha_1)g(0)-\alpha_2 e^{-sx_2}$$

and

$$Y(s)=\frac{-g'(0)+(s+\alpha_1)g(0)-\alpha_2 e^{-sx_2}}{(s+\alpha_1)s}.$$

Applying the Laplace inverse as in the large establishment case we obtain the following equation:

$$g(-y)=\frac{-g'(0)}{\alpha_1}\left(1-e^{-\alpha_1 y}\right)-\frac{\alpha_2}{\alpha_1}H(x_2)\left[1-(y-x_2)e^{-\alpha_1(y+x_2)}\right]+g(0),$$

using that

$$f_1(x)=\frac{f_1'(0)}{\alpha_1}\left(1-e^{\alpha_1 x}\right)+\frac{\alpha_2}{\alpha_1}\left[1-H(x_2)\right]\left[e^{\alpha_1(x-x_2)}-1\right]+f_1(0).$$

Note that again as in the large establishment case we need to impose the boundary condition $f_1(0)=0$ to guarantee that integral is bounded and we use again the symmetric Heavside function $H(x_2)$ equal to

$$1-H(x_2)=\begin{cases}0 & \text{if} \quad x\geq x_2,\\ 1 & \text{if} \quad x<x_2.\end{cases}$$

Last we obtain the following stationary distribution,

$$f_1(x)=\begin{cases}\frac{f_1'(0)}{\alpha_1}\left(1-e^{\alpha_1 x}\right)+\frac{\alpha_2}{\alpha_1}\left(1-e^{\alpha_1(x-x_2)}\right) & \text{if} \quad x<x_2,\\[2ex] \frac{f_1'(0)}{\alpha_1}\left(1-e^{\alpha_1 x}\right) & \text{if} \quad x\geq x_2.\end{cases}$$

We only need to solve for $f_1'(0)$ to have the complete characterization of the small establishments distribution. We find $f_1'(0)$ to guarantee that the integral of $f_1(x)$ is one.

$$\frac{f_1'(0)}{\alpha_1}\int_{-\infty}^{0}(1-e^{\alpha_1 x})\,dx+\frac{\alpha_2}{\alpha_1}\int_{-\infty}^{x_2}\left(1-e^{\alpha_1(x-x_2)}\right)dx=1,$$

where we can rewrite the integral above as:

$$-\frac{f_1'(0)}{\alpha_1}\int_{-\infty}^{0}e^{\alpha_1 x}dx-\frac{\alpha_2}{\alpha_1}\int_{-\infty}^{x_2}e^{\alpha_1(x-x_2)}dx-\frac{f_1'(0)}{\alpha_1}\int_{0}^{x_1}1dx-\left(\frac{f_1'(0)}{\alpha_1}+\frac{\alpha_2}{\alpha_1}\right)\int_{-\infty}^{x_2}1dx=1.$$

From the labor market clearing we find that the last term of the integral is zero, by integrating all the other terms we find the following expression:

$$-\frac{f_1'(0)}{\alpha_1^2}-\frac{\alpha_2}{\alpha_1^2}-\frac{f_1'(0)}{\alpha_1}x_2=1.$$

Substituting α_2 we obtain

$$-\frac{f_1'(0)}{\alpha_1^2}+\frac{f_1'(0)}{\alpha_1^2}-\frac{f_1'(0)}{\alpha_1}x_2=1,$$

as a result, $f_1'(0)=-\dfrac{\alpha_1}{x_2}=-\dfrac{2\hat{\mu}_1}{x_2\sigma^2}$. ■

A.8 Proof Lemma 6

The stationary large establishment's pdf is the solution of the following second order differential equation

$$f_2''(x)-\gamma_1 f_2'(x)=-\gamma_2\delta(x),$$

where the constants γ_1 and γ_2 are given by $\gamma_1=\dfrac{2\hat{\mu}_2}{\sigma^2}$ and $\gamma_2=\dfrac{2\hat{b}_2}{\sigma^2}$, subject to the boundary conditions $f_2(0)\geq 0$ and $\displaystyle\int_0^{+\infty}f_2(x)dx=1$.

We solve the boundary-value problem using Laplace transforms. By applying Laplace transforms in equation (A.8) we obtain:

$$(s^2-\gamma_1 s)\mathscr{L}[f(x)]-(s-\gamma_1)f_2(0)-f_2'(0)=-\gamma_2\mathscr{L}[\delta(x)],$$

and after some algebraic manipulation, we find:

$$(s^2-\gamma_1 s)Y(s)=(f_2'(0)-\gamma_1 f_2(0)-\gamma_2)+sf_2(0).$$

First note that the first term between parenthesis in the right hand side is equal to zero, because the growth rate of large firms η_2 is equal to zero. So, we can simplify the expression and above and obtain:

$$Y(s)=f_2(0)\frac{1}{(s-\gamma_1)},$$

and after solving for the Laplace inverse we obtain $f_2(x) = f_2(0)e^{\gamma_1 x}$. Now we need to find the constant $f_2(0) = -\gamma_1$ to guarantee that the integral over the domain is equal to one. In addition, note that the solution above only satisfy the boundary condition for γ_1 negative.

For the small establishment distribution, we are going to follow the same methodology. The stationary small establishment's pdf is the solution of the following second order differential equation

$$f_1''(x) - \alpha_1 f_1'(x) = -\alpha_2 \delta(x),$$

subject to the boundary conditions $f_1(0) \geq 0$ and $\int_{+\infty}^{0} f_2(x)dx = 1$. We solve the boundary-value problem using Laplace transforms and after some algebraic manipulation we obtain:

$$(s^2 - \alpha_1 s)Y(s) = (f_1'(0) - \alpha_1 f_1(0) - \alpha_2) + s f_1(0),$$

as in the large establishment's distribution, the term between the parenthesis is also zero, because small establishments do not grow $\eta_1 = 0$. We can simplify the expression and above and obtain:

$$Y(s) = f_1(0)\frac{1}{(s - \alpha_1)},$$

and after solving for the Laplace inverse we obtain $f_1(x) = f_1(0)e^{\alpha_1 x}$. Now we need to find the constant $f_1(0) = \alpha_1$ to guarantee that the integral over the domain is equal to one. In addition, note that the solution above only satisfy the boundary condition for α_1 positive. ■

A.9 Stationary equilibrium

We assume $M_1 + M_2 = 1$ and focus on a symmetric stationary equilibrium where $M_1 = M_2 = 0.5$. Given a value for μ_1 and σ, finding the stationary equilibrium involves finding θ, μ_2, z_1, z_2, and w to solve the following five equations:

(1) We find the inaction rate θ by solving the non-linear equation from Lemma 3:

$$\frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = \frac{1}{1 - \rho\tau},$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1}\frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$ and $\beta_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$ for $i \in \{1, 2\}$.

(2) We find $\hat{\mu}_2$ using the flow condition:

$$\hat{\mu}_1 = -\hat{\mu}_2,$$

where $\hat{\mu}_i = \mu_i - \sigma^2/2$.

(3) We find w using the labor market clearing condition:

$$w = A^{\frac{-1}{(\eta-1)}}\left(M_1 n_1^{\alpha}\frac{\alpha_1}{1+\alpha_1}\kappa\Omega_2(\theta) + M_2 n_2^{\alpha}\frac{\gamma_1}{1+\gamma_1}\kappa\Omega_2(\theta)\theta\right)^{\frac{-\eta}{(\eta-1)}},$$

where $\gamma_1 = \frac{2\hat{\mu}_2}{\sigma^2}$, $\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2}$, and $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

(4) We characterize policy functions z_1 and z_2 using the two equations in Lemma 3:

$$z_1 = \kappa\Omega_2(\theta)w \quad \text{and} \quad z_2 = \theta z_1,$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

Then, aggregate TFP in the model is defined by:

$$TFP = sE_1 z + (1-s)E_2 z,$$

where $s = \frac{M_1 n_1^\alpha}{M_1 n_1^\alpha + M_2 n_2^\alpha}$ and $M_1 = M_2 = 0.5$.

In the undistorted economy with no firing costs, aggregate TFP is given by:

$$TFP_u = s\frac{\alpha_1}{(1+\alpha_1)}z_* + (1-s)\frac{\gamma_1}{(1+\gamma_1)}z_*,$$

while in the distorted economy with firing costs, aggregate TFP is given by:

$$TFP_d = s\frac{\alpha_1}{(1+\alpha_1)}z_1\left(\frac{e^{x_2}}{x_2} - \frac{1}{x_2}\right) + (1-s)\frac{\gamma_1}{(1+\gamma_1)}z_2\left(\frac{e^{x_1}}{x_1} - \frac{1}{x_1}\right),$$

where $x_1 = \log(z_1/z_2)$ and $x_2 = \log(z_2/z_1)$.

A.10 The Relevance of n_1 Normalization

Lemma 7. *Changes in n_1 and n_2 do not affect the ratio of wages and total factor productivity between the distorted and undistorted economies:*

$$\frac{TFP_u}{TFP_d} \text{ and } \frac{w_u}{w_d} \text{ are constant}$$

for any n_1 and n_2.

The proof is as follows:

1. The inaction rate θ is independent of n_1 and n_2. This fact is straightforward from Lemma 3.

2. The ratio of wages between the distorted and undistorted economies is constant and independent of n_1 and n_2. The wage in the undistorted economy is given by:

$$w_u = A^{\frac{-1}{(\eta-1)}}\left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1}\kappa\Omega_2(\theta) + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1}\kappa\Omega_2(\theta)\theta\right)^{\frac{-\eta}{(\eta-1)}}.$$

Using the property in step 1. and $\theta = 1$, we can rewrite the wage in the undistorted economy as:

$$w^u = A^{\frac{-1}{(\eta-1)}} \left(\kappa\Omega_2(1)\right)^{\frac{-\eta}{(\eta-1)}} \left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1} + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1}\right)^{\frac{-\eta}{(\eta-1)}}.$$

The wage in the distorted economy is given by:

$$w^d = A^{\frac{-1}{(\eta-1)}} \left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1} \kappa\Omega_2(\theta) \left(\frac{e^{x_2}}{x_2} - \frac{1}{x_2}\right) + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1} \kappa\Omega_2(\theta)\theta \left(\frac{e^{x_1}}{x_1} - \frac{1}{x_1}\right)\right)^{\frac{-\eta}{(\eta-1)}}.$$

Using the property in 1. and $x_1 = -\log(\theta)$ and $x_2 = \log(\theta)$, we can rewrite the wage in an distorted economy as:

$$w^d = A^{\frac{-1}{(\eta-1)}} \left(\kappa\Omega_2(\theta)\right)^{\frac{-\eta}{(\eta-1)}} \left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1} \left(\frac{\theta-1}{\log(\theta)}\right) + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1} \theta \left(\frac{1}{-\theta\log(\theta)} - \frac{1}{-\log(\theta)}\right)\right)^{\frac{-\eta}{(\eta-1)}},$$

$$w^d = A^{\frac{-1}{(\eta-1)}} \left(\kappa\Omega_2(\theta)\right)^{\frac{-\eta}{(\eta-1)}} \left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1} \left(\frac{\theta-1}{\log(\theta)}\right) + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1} \left(\frac{\theta-1}{\log(\theta)}\right)\right)^{\frac{-\eta}{(\eta-1)}},$$

$$w^d = A^{\frac{-1}{(\eta-1)}} \left(\kappa\Omega_2(\theta)\frac{\theta-1}{\log(\theta)}\right)^{\frac{-\eta}{(\eta-1)}} \left(M_1 n_1^\alpha \frac{\alpha_1}{1+\alpha_1} + M_2 n_2^\alpha \frac{\gamma_1}{1+\gamma_1}\right)^{\frac{-\eta}{(\eta-1)}}.$$

As a result, the ratio of wages between the undistorted and distorted economies is given by:

$$\frac{w^d}{w^u} = \frac{\left(\Omega_2(1)\right)^{\frac{-\eta}{(\eta-1)}}}{\left(\Omega_2(\theta)\frac{\theta-1}{\log(\theta)}\right)^{\frac{-\eta}{(\eta-1)}}},$$

which is independent of n_1 and n_2.

3. Regarding TFP we can prove similarly that the ratio of TFP between the distorted and undistorted economies is independent of n_1 and n_2. TFP in the undistorted economy is given by:

$$TFP_u = \frac{M_1 n_1^\alpha \frac{\alpha_1}{(1+\alpha_1)} z_* + M_2 n_2^\alpha \frac{\gamma_1}{(1+\gamma_1)} z_*}{M_1 n_1^\alpha + M_2 n_2^\alpha},$$

where $z_* = \kappa\Omega_2(1)w^u$. TFP in the distorted economy is given by:

$$TFP_d = \frac{M_1 n_1^\alpha \frac{\alpha_1}{(1+\alpha_1)} z_1 \left(\frac{e^{x_2}}{x_2} - \frac{1}{x_2}\right) + M_2 n_2^\alpha \frac{\gamma_1}{(1+\gamma_1)} z_2 \left(\frac{e^{x_1}}{x_1} - \frac{1}{x_1}\right)}{M_1 n_1^\alpha + M_2 n_2^\alpha},$$

where $z_1 = \kappa\Omega_2(\theta)w^d$ and $z_2 = \theta z_1$. After following the same methodology as for the wage ratio, we find that the ratio of TFP between the distorted and undistorted economies is given by:

$$\frac{TFP^d}{TFP^u} = \frac{w^d}{w^u} \frac{\left(\Omega_2(\theta)\frac{\theta-1}{\log(\theta)}\right)}{(\Omega_2(1))},$$

which does not depend on n_1 and n_2 since we have shown that the ratio of wages between the distorted and undistorted economy is independent on n_1 and n_2.