

# Evaluation of the uncertainty of phase-difference measurements in (quasi-)Fourier transform digital holographic interferometry

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## ABSTRACT

Estimation of the uncertainty is an essential requisite for high-end measurement systems. In this communication we derive an expression to evaluate the standard uncertainty of the phase-difference measurements resulting from Fourier and quasi-Fourier transform digital holographic interferometry. We apply the law of propagation of uncertainty, as defined in the “Guide to the expression of uncertainty in measurement” (GUM), to the digital reconstruction of two holograms by Fourier transformation and to the subsequent calculation of the phase change between the holographic reconstructions. The resulting expression allows the evaluation of the uncertainty of the phase difference at every pixel in the reconstruction plane in terms of the measured hologram brightness values and their uncertainty at the whole of the pixels of the original digital holograms. This expression is simplified by assuming a linear dependence between the uncertainty and the local value of the original holograms; in that case, the local uncertainty of the phase difference can be evaluated from the local complex values of the reconstructed holograms. We assess the behavior of the method by comparing the predicted standard uncertainty with the sample variance obtained from experiments conducted under repeatability conditions, and found a good correlation between both quantities. This experimental procedure can be also used to calibrate the parameters of the linear function relating the uncertainty with the local value of the digital holograms, for a given set of operational conditions of the acquisition device.

**Keywords:** Digital holography, interferometry, uncertainty, phase measurement

## 1. INTRODUCTION

High-performance measurement techniques —and, among them, digital holographic interferometry— require a method to estimate the uncertainty for each measured value they yield. The “Guide to the expression of uncertainty in measurement” (GUM)<sup>1</sup> specifies two ways to evaluate the uncertainty of measurement. On the one hand, in type A evaluation, the measurement uncertainty is estimated by a statistical analysis of multiple values of the measured quantity obtained under repeatability or other well defined measurement conditions. On the other hand, in type B evaluation, the uncertainty is determined by means other than direct statistical analysis, such as using a measurement model and prior knowledge, specifications, calibration data, etc.

Though type A evaluation of uncertainty can be applied to digital holographic interferometry, in many practical situations —such as in industrial or non-controlled environments, when measuring dynamic or transient events, etc.— it is not possible to get repeated measurements in the same conditions and type B evaluation of uncertainty becomes necessary.

Digital Fourier-transform holograms, including quasi- and lensless Fourier-transform holograms, are reconstructed by simply calculating their Fourier transforms.<sup>2</sup> This simplicity make them particularly suitable for our first approach to type B uncertainty evaluation in digital holographic interferometry.

The goal of this work is to derive an expression to get a type B evaluation of the local standard uncertainty of the phase-change maps resulting from the application of single-exposure digital holographic interferometry techniques to Fourier-transform holograms, as well as to verify that, under repeatability conditions, the estimations of the uncertainty calculated with the resulting expression match those resulting of type A evaluation.

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## 2. THEORY

### 2.1 Propagation of uncertainty in Fourier-transform digital hologram reconstruction

A digital Fourier-transform hologram, recorded using a camera with  $N \times M$  pixels of size  $\Delta x \times \Delta y$ , is a matrix of positive real values

$$h = h(q, p) = h(q\Delta x, p\Delta y) \quad ; \quad 0 \leq q < N, \quad 0 \leq p < M \quad (1)$$

It is numerically reconstructed with a two-dimensional discrete Fourier transform, which can be defined as

$$\begin{aligned} \mathcal{F}(h) = H(n, m) = H(n\Delta f_x, m\Delta f_y) &= \Delta x \Delta y \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(q\Delta x, p\Delta y) \exp\left(i 2\pi \frac{qn}{N}\right) \exp\left(i 2\pi \frac{pm}{M}\right) \\ &= \Delta x \Delta y \text{FFT}(h) = \Delta x \Delta y \hat{H} \end{aligned} \quad (2)$$

where  $\Delta f_x = \frac{1}{N\Delta x}$ ,  $\Delta f_y = \frac{1}{M\Delta y}$  and the fast Fourier transform (FFT) of the hologram is defined as

$$\hat{H} = \hat{H}(n, m) = \hat{H}(n\Delta f_x, m\Delta f_y) = \text{FFT}(h) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(q, p) \exp\left[i 2\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \quad (3)$$

with real and imaginary parts, respectively,

$$\text{Re } \hat{H} = \text{Re } \hat{H}(n, m) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(q, p) \cos\left[2\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \quad (4)$$

$$\text{Im } \hat{H} = \text{Im } \hat{H}(n, m) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(q, p) \sin\left[2\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \quad (5)$$

The standard uncertainty of the real and imaginary parts of the reconstructed holographic field can be calculated from the standard uncertainty of the digital hologram,  $u[h(q, p)]$ , by using the *law of propagation of uncertainty*<sup>1,3</sup> as follows<sup>4</sup>

$$\begin{aligned} u^2(\text{Re } \hat{H}) &= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \left[ \frac{\partial \text{Re } \hat{H}}{\partial h}(q, p) \right]^2 u^2[h(q, p)] = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \frac{1}{2} \left\{ 1 + \cos\left[4\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \right\} \\ &= \frac{1}{2} \left\{ \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] + \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \cos\left[4\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \right\} \end{aligned} \quad (6)$$

$$\begin{aligned} u^2(\text{Im } \hat{H}) &= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \left[ \frac{\partial \text{Im } \hat{H}}{\partial h}(q, p) \right]^2 u^2[h(q, p)] = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \frac{1}{2} \left\{ 1 - \cos\left[4\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \right\} \\ &= \frac{1}{2} \left\{ \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] - \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \cos\left[4\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \right\} \end{aligned} \quad (7)$$

$$\begin{aligned} u(\text{Re } \hat{H}, \text{Im } \hat{H}) &= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \frac{\partial \text{Re } \hat{H}}{\partial h}(q, p) \frac{\partial \text{Im } \hat{H}}{\partial h}(q, p) u^2[h(q, p)] \\ &= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \cos\left[2\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \sin\left[2\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \\ &= \frac{1}{2} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \sin\left[4\pi \left(\frac{qn}{N} + \frac{pm}{M}\right)\right] \end{aligned} \quad (8)$$

Taking into account that the fast Fourier transform of  $u^2[h(q, p)]$  is

$$\begin{aligned}\hat{U}_s(n, m) &= \text{FFT}\{u^2[h(q, p)]\} = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \exp \left[ i 2\pi \left( \frac{qn}{N} + \frac{pm}{M} \right) \right] \\ &= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \cos \left[ 2\pi \left( \frac{qn}{N} + \frac{pm}{M} \right) \right] + i \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} u^2[h(q, p)] \sin \left[ 2\pi \left( \frac{qn}{N} + \frac{pm}{M} \right) \right]\end{aligned}\quad (9)$$

the uncertainties can be rewritten as

$$u^2[\text{Re } \hat{H}(n, m)] = \frac{1}{2} [\hat{U}_s(0, 0) + \text{Re } \hat{U}_s(2n, 2m)] \quad (10)$$

$$u^2[\text{Im } \hat{H}(n, m)] = \frac{1}{2} [\hat{U}_s(0, 0) - \text{Re } \hat{U}_s(2n, 2m)] \quad (11)$$

$$u[\text{Re } \hat{H}(n, m), \text{Im } \hat{H}(n, m)] = \frac{1}{2} \text{Im } \hat{U}_s(2n, 2m) \quad (12)$$

## 2.2 Propagation of uncertainty in the calculation of the phase difference

Let us consider the digital reconstructions of two Fourier-transform holograms

$$H_i = H_i(n, m) = H_i(n\Delta f_x, m\Delta f_y) = \Delta x \Delta y \hat{H}_i(n\Delta f_x, m\Delta f_y) = \Delta x \Delta y \hat{H}_i(n, m) = \Delta x \Delta y \hat{H}_i \quad ; \quad i \in \{1, 2\} \quad (13)$$

### 2.2.1 Calculation of the phases and their difference

The most straightforward approach to the calculation of the phase difference between the reconstructed holograms consists in obtaining their respective random-distributed phases

$$\psi_i = \psi_i(n, m) = \psi_i[H_i(n, m)] = \arg H_i(n, m) = \arg \hat{H}_i = \arctan \frac{\text{Im } \hat{H}_i(n, m)}{\text{Re } \hat{H}_i(n, m)} \in (-\pi, \pi] \quad (14)$$

subtracting them and eventually reducing the phase difference to the principal interval  $(-\pi, \pi]$  to remove the phase-wrap arising from the random components of the phases

$$\phi_{12} = \phi_{12}(n, m) = \begin{cases} \psi_2 - \psi_1 + 2\pi & \text{if } -2\pi < \psi_2 - \psi_1 \leq -\pi \\ \psi_2 - \psi_1 & \text{if } -\pi < \psi_2 - \psi_1 \leq \pi \\ \psi_2 - \psi_1 - 2\pi & \text{if } \pi < \psi_2 - \psi_1 \leq 2\pi \end{cases} \quad (15)$$

Applying the law of propagation of uncertainty to Eq. (15) and assuming that the two measurements of the phase are statistically independent —i.e.  $u(\psi_1, \psi_2) = 0$ — yields

$$u^2(\phi_{12}) = u^2(\psi_1) + u^2(\psi_2) \quad (16)$$

where the standard uncertainties of the phase measurements are estimated by further applying the law of propagation to Eq. (14). Let us take, for simplicity, the generic expression

$$\psi_i = \arg z_i = \arctan \frac{b_i}{a_i} \quad (17)$$

with  $z_i = a_i + i b_i = \text{Re } \hat{H}_i + i \text{Im } \hat{H}_i = \hat{H}_i$ . The square of the combined standard uncertainty of  $\psi_i$  is

$$u^2(\psi_i) = \left( \frac{\partial \psi_i}{\partial a_i} \right)^2 u^2(a_i) + \left( \frac{\partial \psi_i}{\partial b_i} \right)^2 u^2(b_i) + 2 \frac{\partial \psi_i}{\partial a_i} \frac{\partial \psi_i}{\partial b_i} u(a_i, b_i) = \frac{b_i^2}{|z_i|^4} u^2(a_i) + \frac{a_i^2}{|z_i|^4} u^2(b_i) - \frac{2a_i b_i}{|z_i|^4} u(a_i, b_i) \quad (18)$$

Incorporating Eq. (18) into Eq. (16) eventually results

$$u^2(\phi_{12}) = \frac{b_1^2}{|z_1|^4} u^2(a_1) + \frac{a_1^2}{|z_1|^4} u^2(b_1) + \frac{b_2^2}{|z_2|^4} u^2(a_2) + \frac{a_2^2}{|z_2|^4} u^2(b_2) - 2 \frac{a_1 b_1}{|z_1|^4} u(a_1, b_1) - 2 \frac{a_2 b_2}{|z_2|^4} u(a_2, b_2) \quad (19)$$

### 2.2.2 Direct calculation of the phase difference with Stetson and Brohinsky's algorithm

The Stetson-Brohinsky differential algorithm<sup>5,6</sup> is a widely used alternative which directly yields the value of phase difference between the two reconstructed holograms constrained to the principal interval  $(-\pi, \pi]$

$$\phi_{12} = \psi_2 - \psi_1 = \arg H_{12} = \arg(H_1^* H_2) = \arg(\Delta x \Delta y \hat{H}_1^* \Delta x \Delta y \hat{H}_2) = \arg[(\Delta x)^2 (\Delta y)^2 \hat{H}_{12}] \quad (20)$$

Since it can be reasonably assumed that  $\Delta x$  and  $\Delta y$  are exactly the same at each pixel for both holograms,

$$\phi_{12} = \arctan \frac{(\Delta x)^2 (\Delta y)^2 \operatorname{Im} \hat{H}_{12}}{(\Delta x)^2 (\Delta y)^2 \operatorname{Re} \hat{H}_{12}} = \arctan \frac{\operatorname{Im} \hat{H}_{12}}{\operatorname{Re} \hat{H}_{12}} = \arctan \frac{\operatorname{Re} \hat{H}_1 \operatorname{Im} \hat{H}_2 - \operatorname{Re} \hat{H}_2 \operatorname{Im} \hat{H}_1}{\operatorname{Re} \hat{H}_1 \operatorname{Re} \hat{H}_2 + \operatorname{Im} \hat{H}_1 \operatorname{Im} \hat{H}_2} \quad (21)$$

Let us simplify the notation in this expression to apply the law of propagation of uncertainty:

$$\phi_{12} = \phi_{12}(z_1, z_2) = \phi_{12}(a_1 + i b_1, a_2 + i b_2) = \phi_{12}(a_1, b_1, a_2, b_2) = \arctan \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \quad (22)$$

where, again,  $z_i = a_i + i b_i = \operatorname{Re} \hat{H}_i + i \operatorname{Im} \hat{H}_i = \hat{H}_i$ . The square of the standard uncertainty of the phase difference at a given point is, accordingly,

$$\begin{aligned} u^2(\phi_{12}) = & \left( \frac{\partial \phi_{12}}{\partial a_1} \right)^2 u^2(a_1) + \left( \frac{\partial \phi_{12}}{\partial b_1} \right)^2 u^2(b_1) + \left( \frac{\partial \phi_{12}}{\partial a_2} \right)^2 u^2(a_2) + \left( \frac{\partial \phi_{12}}{\partial b_2} \right)^2 u^2(b_2) \\ & + 2 \frac{\partial \phi_{12}}{\partial a_1} \frac{\partial \phi_{12}}{\partial b_1} u(a_1, b_1) + 2 \frac{\partial \phi_{12}}{\partial a_1} \frac{\partial \phi_{12}}{\partial a_2} u(a_1, a_2) + 2 \frac{\partial \phi_{12}}{\partial a_1} \frac{\partial \phi_{12}}{\partial b_2} u(a_1, b_2) \\ & + 2 \frac{\partial \phi_{12}}{\partial b_1} \frac{\partial \phi_{12}}{\partial a_2} u(b_1, a_2) + 2 \frac{\partial \phi_{12}}{\partial b_1} \frac{\partial \phi_{12}}{\partial b_2} u(b_1, b_2) + 2 \frac{\partial \phi_{12}}{\partial a_2} \frac{\partial \phi_{12}}{\partial b_2} u(a_2, b_2) \end{aligned} \quad (23)$$

which, assuming that  $u(a_i, b_j) = u(a_i, a_j) = u(b_i, b_j) = 0 \forall i \neq j$ , i.e.  $z_1 = \hat{H}_1$  and  $z_2 = \hat{H}_2$  are statistically uncorrelated, results

$$u^2(\phi_{12}) = \frac{b_1^2}{|z_1|^4} u^2(a_1) + \frac{a_1^2}{|z_1|^4} u^2(b_1) + \frac{b_2^2}{|z_2|^4} u^2(a_2) + \frac{a_2^2}{|z_2|^4} u^2(b_2) - 2 \frac{a_1 b_1}{|z_1|^4} u(a_1, b_1) - 2 \frac{a_2 b_2}{|z_2|^4} u(a_2, b_2) \quad (24)$$

### 2.2.3 Application to the digital reconstructions of Fourier-transform holograms

Eqs. (24) and (19) are the same and, therefore, the uncertainty of the measured phase difference is the same regardless of which of the two methods is used to calculate it. The expression is particularized for the digital reconstructions of two holograms by substituting  $\hat{H}_i = \operatorname{Re} \hat{H}_i + i \operatorname{Im} \hat{H}_i$  for  $z_i = a_i + i b_i$  and Eqs. (10) to (12) for the uncertainties into Eq. (24), resulting

$$\begin{aligned} u^2(\phi_{12}) = & \frac{(\operatorname{Im} \hat{H}_1)^2}{2 |\hat{H}_1|^4} [\hat{U}_{1s}(0, 0) + \operatorname{Re} \hat{U}_{1s}(2n, 2m)] + \frac{(\operatorname{Re} \hat{H}_1)^2}{2 |\hat{H}_1|^4} [\hat{U}_{1s}(0, 0) - \operatorname{Re} \hat{U}_{1s}(2n, 2m)] \\ & + \frac{(\operatorname{Im} \hat{H}_2)^2}{2 |\hat{H}_2|^4} [\hat{U}_{2s}(0, 0) + \operatorname{Re} \hat{U}_{2s}(2n, 2m)] + \frac{(\operatorname{Re} \hat{H}_2)^2}{2 |\hat{H}_2|^4} [\hat{U}_{2s}(0, 0) - \operatorname{Re} \hat{U}_{2s}(2n, 2m)] \\ & - \frac{\operatorname{Re} \hat{H}_1 \operatorname{Im} \hat{H}_1}{|\hat{H}_1|^4} \operatorname{Im} \hat{U}_{1s}(2n, 2m) - \frac{\operatorname{Re} \hat{H}_2 \operatorname{Im} \hat{H}_2}{|\hat{H}_2|^4} \operatorname{Im} \hat{U}_{2s}(2n, 2m) \end{aligned} \quad (25)$$

and, since  $(\operatorname{Im} \hat{H}_i)^2 + (\operatorname{Re} \hat{H}_i)^2 = |\hat{H}_i|^2$ , eventually

$$\begin{aligned} u^2(\phi_{12}) = & \frac{1}{2} \left[ \frac{1}{|\hat{H}_1|^2} \hat{U}_{1s}(0, 0) + \frac{1}{|\hat{H}_2|^2} \hat{U}_{2s}(0, 0) \right. \\ & + \frac{(\operatorname{Im} \hat{H}_1)^2 - (\operatorname{Re} \hat{H}_1)^2}{|\hat{H}_1|^4} \operatorname{Re} \hat{U}_{1s}(2n, 2m) + \frac{(\operatorname{Im} \hat{H}_2)^2 - (\operatorname{Re} \hat{H}_2)^2}{|\hat{H}_2|^4} \operatorname{Re} \hat{U}_{2s}(2n, 2m) \\ & \left. - 2 \frac{\operatorname{Re} \hat{H}_1 \operatorname{Im} \hat{H}_1}{|\hat{H}_1|^4} \operatorname{Im} \hat{U}_{1s}(2n, 2m) - 2 \frac{\operatorname{Re} \hat{H}_2 \operatorname{Im} \hat{H}_2}{|\hat{H}_2|^4} \operatorname{Im} \hat{U}_{2s}(2n, 2m) \right] \end{aligned} \quad (26)$$

with  $\phi_{12} = \phi_{12}(n, m)$  and  $\hat{H}_i = \hat{H}_i(n, m)$

### 2.2.4 Linear hologram uncertainty

If we assume that the square of the standard uncertainty is linearly dependent with the local values of the digital holograms  $h_i(q, p)$

$$u^2[h_i(q, p)] = k h_i(q, p) + u_0^2 \quad ; \quad i \in \{1, 2\} \quad (27)$$

Here,  $u_0$  is a component of the standard uncertainty which takes the same value for all of the pixels in the hologram. It will typically model the uncertainty arising from quantization and dark noise. On the other hand,  $k h_i(q, p)$  is proportional to the local value of the hologram, with the same value of the proportionality constant  $k$ —the ADC gain (counts/electron)—for all of the pixels. This will typically model the uncertainty deriving from shot noise.

The fast Fourier transform of  $u_i^2[h_i(q, p)]$  is, in this case,

$$\hat{U}_{is}(n, m) = k \hat{H}_i(n, m) + \text{FFT}(u_0^2) = \begin{cases} k NM \langle h_i \rangle + NM u_0^2 & \text{if } n = m = 0 \\ k \hat{H}_i(n, m) & \text{otherwise} \end{cases} \quad (28)$$

with

$$\langle h_i \rangle = \frac{1}{NM} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h_i(q, p) = \frac{\hat{H}_i(0, 0)}{NM} \quad (29)$$

equation (28) can be substituted into Eq. (26) resulting

$$u^2[\phi_{12}] = \frac{1}{2} \left[ \frac{1}{|\hat{H}_1|^2} NM (k \langle h_1 \rangle + u_0^2) + \frac{1}{|\hat{H}_2|^2} NM (k \langle h_2 \rangle + u_0^2) \right. \\ \left. + \frac{[\text{Im } \hat{H}_1]^2 - [\text{Re } \hat{H}_1]^2}{|\hat{H}_1|^4} k \text{Re } \hat{H}_1(2n, 2m) + \frac{[\text{Im } \hat{H}_2]^2 - [\text{Re } \hat{H}_2]^2}{|\hat{H}_2|^4} k \text{Re } \hat{H}_2(2n, 2m) \right. \\ \left. - 2 \frac{\text{Re } \hat{H}_1 \text{Im } \hat{H}_1}{|\hat{H}_1|^4} k \text{Im } \hat{H}_1(2n, 2m) - 2 \frac{\text{Re } \hat{H}_2 \text{Im } \hat{H}_2}{|\hat{H}_2|^4} k \text{Im } \hat{H}_2(2n, 2m) \right] \quad (30)$$

**Simplification for holograms of speckle patterns** In the fast Fourier transform of an hologram generated by the interference of a speckle pattern with an uniform (or nearly uniform) reference beam there are, typically,

$$\left. \begin{aligned} \text{Re } \hat{H}_i(2n, 2m) &\ll \hat{H}_i(0, 0) = NM \langle h_i \rangle \\ \text{Im } \hat{H}_i(2n, 2m) &\ll \hat{H}_i(0, 0) = NM \langle h_i \rangle \end{aligned} \right\} \text{for } n \neq 0 \text{ and } m \neq 0 \quad (31)$$

and Eq. (30) can be approximated as

$$u^2[\phi_{12}] \approx \frac{NM}{2} \left[ k \left( \frac{\langle h_1 \rangle}{|\hat{H}_1|^2} + \frac{\langle h_2 \rangle}{|\hat{H}_2|^2} \right) + u_0^2 \left( \frac{1}{|\hat{H}_1|^2} + \frac{1}{|\hat{H}_2|^2} \right) \right] \quad (32)$$

If, in addition, the illumination conditions and the object are the same for both holograms, it may be a reasonable assumption that

$$\langle h_1 \rangle \approx \langle h_2 \rangle \approx \frac{\langle h_1 \rangle + \langle h_2 \rangle}{2} = \langle h \rangle \quad (33)$$

the expression of the combined uncertainty squared is further simplified to

$$u^2[\phi_{12}] \approx \frac{NM}{2} \left( \frac{1}{|\hat{H}_1|^2} + \frac{1}{|\hat{H}_2|^2} \right) (k \langle h \rangle + u_0^2) = \eta (k \langle h \rangle + u_0^2) \quad (34)$$

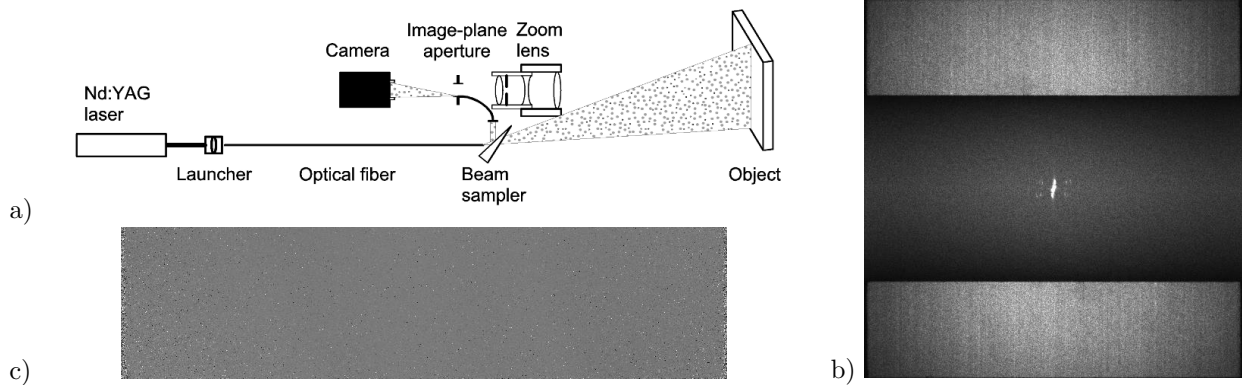


Figure 1. a) Layout of the quasi-Fourier transform experimental system. b) Full digital reconstruction of a hologram ( $\langle h \rangle \approx 4300$ ), showing the spectral separation of the two conjugate images of the object and the autocorrelations of the object and reference beams. c) Phase-difference map between b) and another hologram recorded within less than 100 ms.

### 3. EXPERIMENTAL

To verify the validity of the estimations of the uncertainty yielded by the expressions derived in section 2.2.4 we have conducted a set of experimental Fourier-transform digital holographic interferometry phase-difference measurements and compared the actual observed value of the variance of the measured phase-difference—which constitutes a type A evaluation of the standard uncertainty of the measurements—with the type B estimation of the square of the standard uncertainty provided by Eq. (34).

The experiments have been arranged to nominally get the same phase difference for all of the pixels of the digital reconstruction of the object, and thus the variance of the phase-difference corresponding to a given value of  $\eta$  in Eq. (34) can be calculated by comparing the phase-difference values corresponding to the pixels with such value of  $\eta$  with the average of the phase difference in the whole of the object.

The effect of the average value of the digital holograms  $\langle h \rangle$ —i.e., of the hologram’s illumination level—on the uncertainty of the phase-difference has been analyzed by repeating the measurements with 64 different values of the hologram exposure time, ranging from 0.02 ms to 1.28 ms.

#### 3.1 Experimental arrangement

The holograms have been acquired with a hybrid lensless-Fourier-transform digital holographic camera which has been fully described elsewhere.<sup>7</sup> As shown in Fig. 1-a, the object is illuminated with a frequency-doubled Nd:YAG laser and its image is projected with an objective lens on a plane where a rectangular aperture limits the extension of the object field. A lensless Fourier-transform hologram is eventually generated by adding a fiber-optic guided reference beam diverging from this plane. The relative positions of the aperture and the reference-beam source are carefully chosen to prevent the overlapping of the object image and the autocorrelation terms in the subsequent hologram reconstruction, as shown in Fig. 1-b.

The holograms are recorded as  $2048 \times 2048$  pixel 14 bit images using a camera equipped with the SONY ICX625A CCD sensor. To minimize the effects of air convection and thermal instability in the reference-beam optical fiber, the two holograms in each experiment are acquired with the minimum delay ( $< 100$  ms) allowed by the camera. Phase difference maps as the one in Fig. 1-c, comprising  $2048 \times 512$  pixels each, are eventually calculated by applying Stetson and Brohinky’s algorithm (see section 2.2.2) to the reconstructed object fields.

The test object is a  $160 \text{ mm} \times 40 \text{ mm}$  region of an uncoated  $250 \text{ mm} \times 250 \text{ mm} \times 10 \text{ mm}$  aluminum plate. The plate is fixed to the same table than the optical system and regarded rigid enough to assume that the phase difference due to its displacement is nominally  $\phi_{12} = 0$ .

#### 3.2 Data processing

An initial guess of  $k = 3$  and  $u_0^2 = 9000$  was made, based the reported characteristics<sup>8</sup> of the ICX625A sensor, for the parameters in Eq. 34, thus making it applicable to estimate a value for the uncertainty of phase differences.

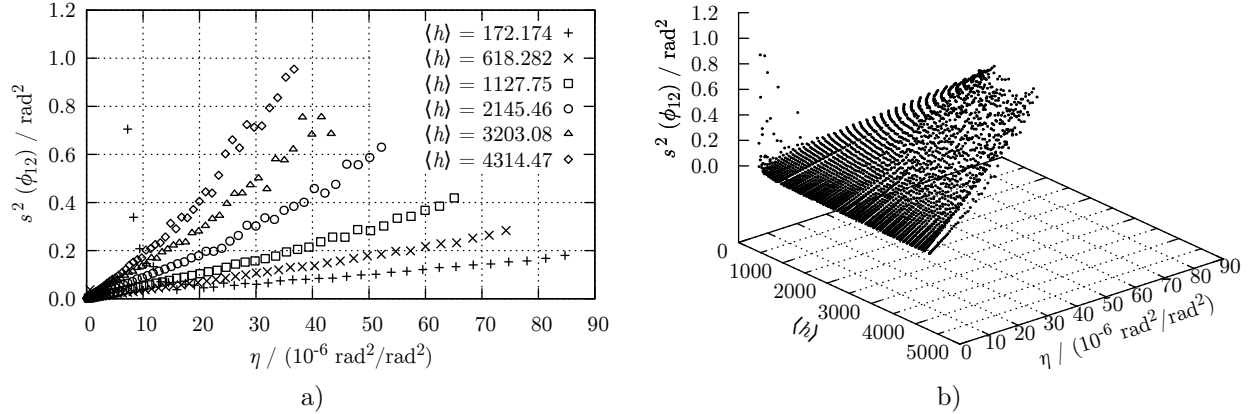


Figure 2. Experimentally observed variance of the phase difference vs.  $\eta = \frac{NM}{2} \left( \frac{1}{|H_1|^2} + \frac{1}{|H_2|^2} \right)$  a) for holograms with five different values of  $\langle h \rangle$  b) for the whole set of 64 different values of  $\langle h \rangle$  used to validate Eq. (34).

Then, the following procedure was applied to analyze the dependence of the variance of the measurements  $s^2(\phi_{12})$  with the values of  $\langle h \rangle$  and  $\eta$ —defined in Eqs. (33) and (34)—and find the values of  $k$  and  $u_0^2$  that provide a better agreement between  $s^2(\phi_{12})$  and  $u^2(\phi_{12})$ :

1. For each of the 64 phase-difference measurements with different exposure times repeat the following steps:
  - (a) Calculate  $\langle h \rangle$  using Eqs. (29) and (33); calculate the average phase-difference  $\langle \phi_{12} \rangle$ .
  - (b) Classify the pixels into 50 sets according to the standard uncertainty estimated with Eq. (34), discarding those pixels with  $u(\phi_{12}) > \frac{\pi}{\sqrt{12}}$ ; assign to each of the sets the corresponding value of  $\eta = \frac{NM}{2} \left( \frac{1}{|H_1|^2} + \frac{1}{|H_2|^2} \right)$ .
  - (c) For each of the 50 sets, find the variance of the phase-difference  $s^2(\phi_{12})_{\eta, \langle h \rangle}$ , taking  $\langle \phi_{12} \rangle$  as the expected value.
2. Find the values of  $k$  and  $u_0^2$  that optimize the agreement between  $s^2(\phi_{12})$  and  $u^2(\phi_{12})$  by non-linear fitting

$$s^2(\phi_{12})_{\eta, \langle h \rangle} = \eta (k \langle h \rangle + u_0^2) \quad (35)$$

(we have used in this step the *gnuplot*<sup>9</sup> program, which implements the Marquardt-Levenberg algorithm).

A first iteration of this procedure yielded  $k = 4.01$  and  $u_0^2 = 1150$ . Since the procedure relies on these parameters to estimate the uncertainty and classify the pixels, and their values resulted significantly different from our initial guess, a second iteration was carried out yielding, eventually,  $k = 3.92$  and  $u_0^2 = 1630$ .

## 4. DISCUSSION

After the first iteration of the procedure described in section 3.2, the results (Fig. 2) already evidence a linear dependence between the observed variance of the phase change and the value of  $\eta$ , with its slope increasing with  $\langle h \rangle$ , as expected from Eq. (34).

Once the best-fit values of the parameters,  $k$  and  $u_0^2$ , are found and incorporated into Eq. (34), the estimation of the uncertainty matches remarkably well the actual observed standard deviation, as shown in Fig. 3. The largest values of the uncertainty are, nevertheless, slightly underestimated. For the smallest values of  $\langle h \rangle$ , on the other hand, some estimations of low uncertainty are grossly mismatched with the corresponding experimental standard deviation. This seems to be due to the presence of electrical noise which, locally, has much higher spectral power than the optical signal. This noise is readily apparent as bright spots when such low-valued holograms are digitally reconstructed.

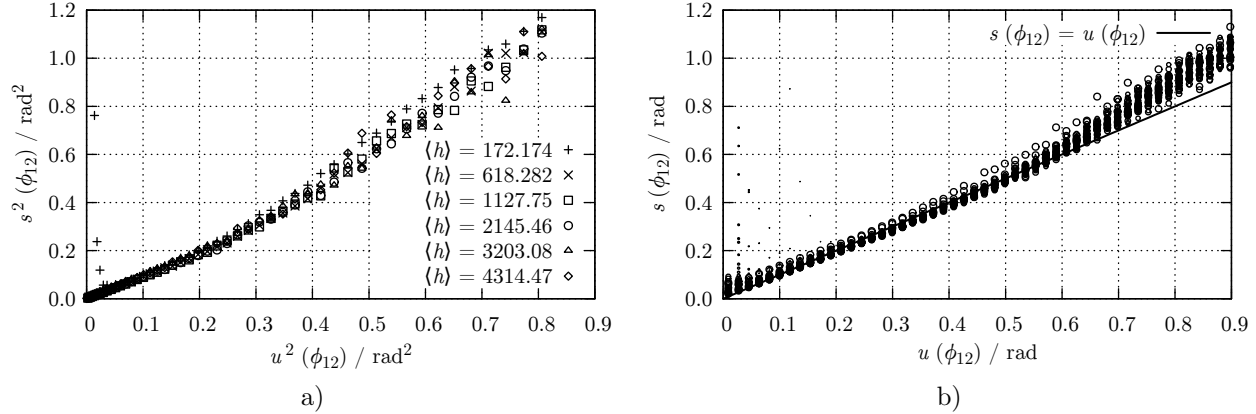


Figure 3. a) Experimentally observed variance of the phase difference vs. estimation of the square of the standard uncertainty, for holograms with five different values of  $\langle h \rangle$ . b) Experimentally observed standard deviation vs. estimation of the standard uncertainty, for the whole set of measurements. The diameters of the circles are proportional to the corresponding values of  $\langle h \rangle$ .  $u^2(\phi_{12})$  and  $u(\phi_{12})$  are calculated with Eq. (34) using the best-fit values of  $k$  and  $u_0^2$ .

## 5. CONCLUSIONS

We have derived a general expression, Eq. (26), which allows type B evaluation of the standard uncertainty of the measurements of the phase change between holographic reconstructions in Fourier and quasi-Fourier transform digital holographic interferometry. This expression is increasingly simplified by assuming a linear behavior of the sources of uncertainty in hologram recording, Eq. (30), a speckle pattern object beam, Eq. (32), and constant average illumination, Eq. (34). As an intermediate result, Eqs. (10), (11) and (12) provide general expressions to evaluate the uncertainties of the real and imaginary parts of individual holographic reconstructions.

A good correspondence between the estimations of the standard uncertainty provided by Eq. (34) and the actual values of the sample standard deviation of the measured phase-difference has been demonstrated in a set of experiments conducted under repeatability conditions. The procedure followed here to assess the aforementioned correspondence can be used to calibrate the parameters of the linear function relating the uncertainty with the local value of the digital holograms,  $k$  and  $u_0^2$ , for a given set of operational conditions of the acquisition device.

## ACKNOWLEDGMENTS

The authors acknowledge funding from the *Ministerio de Ciencia e Innovación* and the European Commission (ERDF) (project DPI2011-26163) as well as from the *Universidad de Vigo* (c.p. 14VI06).

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