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Uncertainty in cooperative interval games: How Hurwicz criterion compatibility leads to egalitarianism*

Lina Mallozzi† Juan Vidal-Puga‡

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Abstract

We study cooperative interval games. These are cooperative games where the value of a coalition is given by a closed real interval specifying a lower bound and an upper bound of the possible outcome. For interval cooperative games, several (interval) solution concepts have been introduced in the literature. We assume that each player has a different attitude towards uncertainty by means of the so-called Hurwicz coefficients. These coefficients specify the degree of optimism that each player has so that an interval becomes a specific payoff. We show that a classical cooperative game arises when applying the Hurwicz criterion to each interval game.

On the other hand, the same Hurwicz criterion can also be applied to any interval solution of the interval cooperative game. Given this, we say that a solution concept is Hurwicz compatible if the two procedures provide the same final payoff allocation. When such compatibility is possible, we characterize the class of compatible solutions, which reduces to the egalitarian solution when symmetry is required. The Shapley value and the core solution cases are also discussed.

Keywords: Cooperative interval games; Hurwicz criterion; Hurwicz compatibility

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†Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Claudio 21, 80125 Naples, Italy.

‡Department of Statistics and Operations Research, Universidade de Vigo, 36200 Vigo (Pontevedra), Spain.

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1 Introduction

Given a set of agents (or players), cooperative (or transferable utility) games assign to each coalition of agents a real number. This number represents the maximum utility that the members of this coalition can assure by themselves. Cooperative game theory has addressed these problems by proposing relevant solutions, or values, that suggest one or several payoff allocations satisfying certain desirable properties. Typically, the most standard property is efficiency, which implies that the worth of the grand coalition is shared. Some examples of efficient solutions are the core, the Weber set, the Shapley value, and the nucleolus.

Cooperative interval games generalize the idea of cooperative games by assigning to each coalition a closed interval. Analogously, efficient interval solutions propose an allocation of the interval generated by the grand coalition. Many of the classical solutions have been defined in the context of cooperative interval games by Alparslan-Gok et al. (2008); Alparslan Gök et al. (2009c). See Branzei et al. (2010) for a survey.

Interval games have been applied to bankruptcy problems (Branzei and Alparslan Gök, 2008), airport games (Alparslan Gök et al., 2009a), minimum cost spanning tree problems (Montemanni, 2006; Moretti et al., 2011), assignment problems (Pereira and Averbakh, 2011; van den Brink et al., 2017; Wu et al., 2018), and sequencing games (Alparslan-Gök et al., 2013).

A classical interpretation of the intervals is that each of them represents the possible worth range that a coalition can get by themselves. Examples are those that appear from the so-called games with externalities (Thrall and Lucas, 1963), where the worth of a coalition depends not only on the coalition itself but also on how the rest of the players cooperate.[1] van den Brink et al. (2017) propose a different motivation, where the worth of a coalition varies between the (classical) pessimistic assumption that the rest of players will try to harm them as much as possible and the most optimistic assumption given by the dual problem.

At this point, we have to distinguish between risk and uncertainty. In a risky situation, players are unsure of the final result of their cooperation, but they can assign a precise probability to each possible outcome. This kind of riskiness has been deeply studied in economic literature, from a cooperative and non-cooperative point of view. Frequently,

[1] For example, assume that players form an oligopoly that plans to create a cartel. The cartel can then anticipate their benefit as a monopoly. However, if two or more players are not present, the remaining players can not anticipate their exact benefit, as it would depend on whether the other players merge or not.

it is assumed that each player has some private information and precise knowledge of the probability distribution of how the others are. This approach does not fit into the model proposed by interval games, where each interval does not depend on the private information of the players. In other situations, the probability distribution is common knowledge, but their consequences are not homogeneous among players (Alparslan-Gök et al., 2013). Again, this approach neither fits interval games, where each interval is coalition-dependent.

As opposed, under uncertainty, players are not only unsure of the outcome of their potential cooperation but also the probability of these possible outcomes. When there is no private information, i.e. all the players agree on the uncertainty that lies behind the cooperation of each coalition, interval games provide a more realistic interpretation of uncertainty.

An interval solution is then a way to share the uncertainty of the grand coalition worth taking into account the uncertain worth of each coalition.

Given this, each player can have a different attitude towards uncertainty. For example, a pessimistic player would prefer to maximize the minimum possible outcome (*maximin criterion*), so that its preferred payoff is an interval with a high lower bound, whereas an optimistic (*maximax criterion*) player would prefer intervals with a high upper bound. An intermediate approach (*Laplace criterion* or *criterion of rationality*) is to assume all the possible outcomes are equally probable, in the sense that they follow a uniform distribution. Hence, players with a Laplace criterion prefer intervals with a high midpoint.

A generalization of these criteria is the so-called Hurwicz criterion (Hurwicz, 1951), which states that there exists a fixed coefficient between 0 and 1 that measures the degree of optimism. Hence, a pessimistic player would have a coefficient 0, an optimistic one 1, and a rational one (in the sense of Laplace) $\frac{1}{2}$.

Once the Hurwicz coefficients are stated for each player, uncertainty disappears, and players can uniquely assign a concrete value to each interval. The payoff allocation of the grand coalition interval becomes a payoff allocation for the grand coalition. We can then check the payoff allocation proposed by each interval solution.

In this paper, we study what happens when this statement is done on the interval game before applying any interval value. We prove (Proposition 3.1) that this operation generates a (classical) cooperative game so that we can compute its interval value.[2]

[2] Other applications of Hurwicz coefficients in interval games appear in Lardon (2017) and Li (2016), who also deduce a (classical) cooperative game by using a selection via degrees of optimism. However, these degrees are coalition-dependent, not individual. Hence, they cannot be identified as Hurwicz coefficients in the same way we do here.

Therefore, we can proceed in two ways. On the one hand, we can compute the interval value on the interval game, and then apply the Hurwicz criterion. On the other hand, we can apply the Hurwicz criterion in the interval game to get a (classical) cooperative game, then apply the interval solution.[3] We are interested in studying which values are compatible in the sense that both procedures provide the same final payoff allocation. This property has the potential of solving situations where players have uncertain needs for a resource when it has to be divided before uncertainty resolves (Xue, 2018).

Our second result (Proposition 3.2) implies that this property is incompatible with efficiency in the more general setting. However, we focus on the only two situations where they are compatible. On the one hand, we can assume that all the players have the same Hurwicz coefficient. On the other hand, we can assume that uncertainty disappears when the grand coalition forms. The latter is what happens in the motivating examples: both games with externalities and the optimistic-pessimistic approach given by van den Brink et al. (2017).

We prove (Theorem 4.1 and Theorem 5.1) that the unique compatible solutions are the proportional ones. These are solutions that have recently received increasing attention from the economic and social choice theory (Abe and Nakada, 2019; Béal et al., 2016; Koster and Boonen, 2019; Yokote et al., 2018). They state that the worth of the grand coalition should be proportionally distributed. As a direct corollary, we deduce (Theorem 4.2 and Theorem 5.2) that the unique anonymous (in the sense that symmetric agents are treated symmetrically) compatible solution is the egalitarian one. This states an equivalence between Hurwicz compatibility and the egalitarian notion given by the Dutta and Ray (1989) interpretation ideally yielding Lorenz-dominant allocations but without core-like participation constraints. Experimental evidence also backs egalitarian outcomes (Bolton and Ockenfels, 2000). It is worthy to note that usual characterizations of the egalitarian solution heavily rely on the properties of either additivity (Béal et al., 2016, 2019; Bergantiños and Vidal-Puga, 2004; Casajus and Huettner, 2014a; Hougaard and Moulin, 2018; van den Brink, 2007; van den Brink et al., 2015) or monotonicity (Bergantiños and Vidal-Puga, 2009; Casajus and Huettner, 2013, 2014b), which are not required here.

The rest of the paper is organized as follows. In Section 2, we present the notation. In Section 3, we define the Hurwicz criterion and prove that it can only be applied in two particular situations, which are analyzed in Section 4 and Section 5, respectively. In Section 6, we focus on the core. In Section 7, we present some concluding remarks.

[3] This is always possible to do since interval cooperative games generalize classical ones.

2 Notation

Let $\mathbb{I} := \{\mathbf{a} = [\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$ be the set of closed intervals in $\mathbb{R}$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{I}$, we say that $\mathbf{a} \preccurlyeq \mathbf{b}$ when $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. Notice that if $\mathbf{a} = [a, a]$ and $\mathbf{b} = [b, b]$ for some $a, b \in \mathbb{R}$, we have that $\mathbf{a} \preccurlyeq \mathbf{b}$ iff $a \leq b$. Remark that $\preccurlyeq$ is a partial order relation in $\mathbb{I}$; for example $[-1, 2] \preccurlyeq [0, 2]$ but $[-2, 3]$ and $[-1, 2]$ are not comparable with respect to $\preccurlyeq$.

Given $x \in \mathbb{R}_+$ and $\mathbf{a} \in \mathbb{I}$, we define $x \cdot \mathbf{a} := [x \cdot \underline{a}, x \cdot \overline{a}] \in \mathbb{I}$. Given $\alpha_i \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{I}$, we define

$$\alpha_i \circ \mathbf{a} := \alpha_i \cdot \overline{a} + (1 - \alpha_i) \cdot \underline{a} = \underline{a} + \alpha_i \cdot (\overline{a} - \underline{a}) \in \mathbb{R}.$$

Given $\mathbf{a}, \mathbf{b} \in \mathbb{I}$, we define $\mathbf{a} \oplus \mathbf{b} := \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right] \in \mathbb{I}$, and $\mathbf{a} \oslash \mathbf{b} := \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right] \in \mathbb{I}$; moreover, when $\underline{a} - \underline{b} \leq \overline{a} - \overline{b}$, we define $\mathbf{a} \ominus \mathbf{b} := \left[\underline{a} - \underline{b}, \overline{a} - \overline{b}\right] \in \mathbb{I}$. When $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subset \mathbb{I}$, we define

$$\sum_{\mathbf{a} \in \mathbf{A}} \mathbf{a} := \mathbf{a}_1 \oplus \cdots \oplus \mathbf{a}_k \in \mathbb{I}$$

with the convention that $\sum_{\mathbf{a} \in \emptyset} \mathbf{a} := [0, 0]$.

Let $N = \{1, \ldots, n\}$ be a finite set of players. A *coalitional interval game* is a pair $(N, \mathbf{v})$ where $\mathbf{v} : 2^N \to \mathbb{I}$ is a function that assigns a closed interval $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$ to each coalition $S \subseteq N$ with the property that $\mathbf{v}(\emptyset) = [0, 0]$, i.e. $\underline{v}(\emptyset) = \overline{v}(\emptyset) = 0$. Let $\mathcal{IG}^N$ denote the class of all coalitional interval games with N as set of players. Since N is fixed, from now on we write $\mathbf{v}$ instead of $(N, \mathbf{v})$ and $\mathcal{IG}$ instead of $\mathcal{IG}^N$.

Notice that coalitional interval games generalize classical coalitional (*transferable utility*, or TU) games. Just take $\underline{v}(S) = \overline{v}(S)$ for all $S \subseteq N$. There are three trivial TU games associated with any $\mathbf{v} \in \mathcal{IG}$. These are the border games $\overline{v}$ and $\underline{v}$, and the length game $|\mathbf{v}|$ given by $|\mathbf{v}|(S) = \overline{v}(S) - \underline{v}(S)$ for all $S \subseteq N$. Let $\mathcal{G}$ denote the set of TU games with N as the player set. With some abuse of notation, we assume $\mathcal{G} \subset \mathcal{IG}$.

Given $S \subseteq N$, let $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space with generic element $\hat{x} = (x_i)_{i \in S}$. Given $S \subseteq N$ and $\hat{x}, \hat{y} \in \mathbb{R}^S$, we define $\hat{z} = \hat{x} - \hat{y} \in \mathbb{R}^S$ as $z_i = x_i - y_i$ for all $i \in S$. Moreover, given $X, Y \subset \mathbb{R}^S$, we define $X - Y = \{\hat{x} - \hat{y} : \hat{x} \in X, \hat{y} \in Y\} \subseteq \mathbb{R}^S$.

Apart from coalitional interval games, another generalization of TU games is *non-transferable utility games* or NTU games. An NTU game V with player set N is given by a characteristic function $V : 2^N \to \bigcup_{S \subseteq N} \mathbb{R}^S$, with the convention $\mathbb{R}^\emptyset = \{0\}$, satisfying, for all $S \subseteq N$:

1. $V(S) \subseteq \mathbb{R}^S$
2. $V(S) \neq \emptyset$
3. $V(S)$ closed

4. $V(S)$ bounded from above, i.e. $\exists \hat{x} \in \mathbb{R}^S$ such that $V(S) \cap \{\hat{y} \in \mathbb{R}^S : \hat{x} \leq \hat{y}\} = \emptyset$, where $\hat{x} \leq \hat{y}$ means $x_i \leq y_i$ for all $i \in S$.

5. $V(S)$ comprehensive, i.e. $\hat{x} \in V(S), \hat{y} \leq \hat{x} \implies \hat{y} \in V(S)$.

It is well known that any TU game v can be written as an NTU game V with $V(S) = \left\{\hat{x} \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$ for all $S \subseteq N$. Clearly, $V(S)$ can be also be written as $V(S) = \left\{\hat{x} \in \mathbb{R}^S : \sum_{i \in S} x_i = v(S)\right\} - \mathbb{R}^S_+$ for all $S \subseteq N$. Notice that, in general, $X - \mathbb{R}^S_+$ builds the comprehensive cover of $X \subset \mathbb{R}^S$.

Analogously, we can write any interval game $\mathbf{v} \in \mathcal{IG}$ applying natural generalization of NTU games as follows:

$$\mathbf{V}(S) = \left\{\mathbf{a} \in \mathbb{I}^S : \sum_{j \in S} \mathbf{a}_j \preccurlyeq \mathbf{v}(S)\right\} \tag{1}$$

for all $S \subseteq N$. We then say that $\mathbf{V}$ is an *interval game written in NTU form.*

A relevant class of coalitional interval games is the following Alparslan Gök et al. (2009b). A coalitional interval game $\mathbf{v}$ is *size monotonic* if $|\mathbf{v}|(S) \leq |\mathbf{v}|(T)$ for all $S \subseteq T \subseteq N$. We denote as $\mathcal{SMIG}$ the set of size monotonic interval game with N as the player set. Clearly, all TU games are size monotonic interval games, i.e. $\mathcal{G} \subset \mathcal{SMIG}$.

A *solution in the set of TU games* is a function $\sigma : \mathcal{G} \to \mathbb{R}^N$ that assigns to each TU game $v \in \mathcal{G}$ a payoff allocation $\sigma(v) \in \mathbb{R}^N$. A solution σ in the set of TU games is *efficient* if $\sum_{j \in N} \sigma_j(v) = v(N)$ for all $v \in \mathcal{G}$. A well-known efficient solution in the set of TU games is the *Shapley value* (Shapley, 1953).

Let $\mathcal{IB}$ be a subset of $\mathcal{IG}$ such that it contains all the TU games, i.e. $\mathcal{G} \subseteq \mathcal{IB}$. An *efficient solution for* $\mathcal{IB}$ is a function $\boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{I}^N$ that assigns to each $\mathbf{v} \in \mathcal{IB}$ a payoff allocation $\boldsymbol{\sigma}(\mathbf{v}) \in \mathbb{I}^N$ such that $\sum_{i \in N} \boldsymbol{\sigma}_i(\mathbf{v}) = \mathbf{v}(N)$ for all $\mathbf{v} \in \mathcal{IB}$.

3 Hurwicz criterion

Hurwicz (1951) first stated the most well-known criterion to deal with uncertainty. Assume that each player $i \in N$ has a coefficient $\alpha_i \in [0, 1]$ which determines its degree of optimism. This means that if player i is assigned to an interval $\mathbf{a} \in \mathbb{I}$, its valuation of it would be $\alpha_i \circ \mathbf{a}$.

Given $\hat{\alpha} \in [0, 1]^N$ and an interval game solution $\boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{I}^N$, we define $\hat{\alpha} \circ \boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{R}^N$ as the function given by applying the Hurwicz criterion to $\boldsymbol{\sigma}$ with coefficients in $\hat{\alpha}$, i.e.

$$(\hat{\alpha} \circ \boldsymbol{\sigma})_i(\mathbf{v}) := \alpha_i \circ \boldsymbol{\sigma}_i(\mathbf{v}) \in \mathbb{R} \tag{2}$$

for all $\mathbf{v} \in \mathcal{IB}$ and all $i \in N$.

Analogously, given $\hat{\alpha} \in [0,1]^N$ and $\mathbf{v} \in \mathcal{IG}$, we define $\hat{\alpha} \circ \mathbf{V}$ as the NTU game given by applying the Hurwicz criterion to each $\mathbf{v}(S)$ with coefficients in $\hat{\alpha}$. Formally,

$$(\hat{\alpha} \circ \mathbf{V})(S) := \left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbf{V}(S)\right\} - \mathbb{R}^S_+ \tag{3}$$

for all $S \subseteq N$, where $\mathbf{V}$ is defined as in (1). As usual, $-\mathbb{R}^S_+$ allows us to assure comprehensiveness (despite the incompleteness of the partial order $\preccurlyeq$).

Proposition 3.1 *Given $\hat{\alpha} \in [0,1]^N$ and $\mathbf{v} \in \mathcal{IG}$, the associated NTU game $\hat{\alpha} \circ \mathbf{V}$ is equivalent to the TU game $\hat{\alpha} \circ \mathbf{v}$ defined as follows:*

$$(\hat{\alpha} \circ \mathbf{v})(S) := \left(\max_{i \in S} \alpha_i\right) \circ \mathbf{v}(S) \in \mathbb{R} \tag{4}$$

for all $S \subseteq N$, $S \neq \emptyset$.

Proof. Fix $S \subseteq N$, $S \neq \emptyset$. We first prove that, given $\mathbf{b} \in \mathbb{I}$,

$$\begin{aligned} &\left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbb{I}^S, \sum_{i \in S} \mathbf{a}_i \preccurlyeq \mathbf{b}\right\} - \mathbb{R}^S_+ \\ = &\left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbb{I}^S, \sum_{i \in S} \mathbf{a}_i = \mathbf{b}\right\} - \mathbb{R}^S_+. \end{aligned} \tag{5}$$

The "$\supseteq$" part follows from the fact that $\sum_{i \in S} \mathbf{a}_i = \mathbf{b}$ implies $\sum_{i \in S} \mathbf{a}_i \preccurlyeq \mathbf{b}$. For the "$\subseteq$" part, let $\hat{\mathbf{a}} \in \mathbb{I}^S$ such that $\sum_{i \in S} \mathbf{a}_i \preccurlyeq \mathbf{b}$, i.e. $\sum_{i \in S} \underline{a}_i \leq \underline{b}$ and $\sum_{i \in S} \overline{a}_i \leq \overline{b}$. Then, fix $i_0 \in S$ and define $\hat{\mathbf{c}} \in \mathbb{I}^S$ as

$$\underline{c}_{i_0} = \min\left\{\underline{b} - \sum_{i \in S} \underline{a}_i, \overline{b} - \sum_{i \in S} \overline{a}_i\right\},$$

$$\overline{c}_{i_0} = \max\left\{\underline{b} - \sum_{i \in S} \underline{a}_i, \overline{b} - \sum_{i \in S} \overline{a}_i\right\},$$

and $\mathbf{c}_i = [0,0]$ otherwise. Hence, $(\alpha_i \circ \mathbf{c})_{i \in S} \in \mathbb{R}^S_+$. Now, it is straightforward to check that $(\alpha_i \circ \mathbf{a})_{i \in S} = (\alpha_i \circ (\mathbf{a} \oplus \mathbf{c}))_{i \in S} - (\alpha_i \circ \mathbf{c})_{i \in S}$, $\sum_{i \in S} (\mathbf{a} \oplus \mathbf{c})_i = \mathbf{b}$, and hence the following statement (equivalent to the "$\subseteq$" part in (5)) holds:

$$\begin{aligned} &\left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbb{I}^S, \sum_{i \in S} \mathbf{a}_i \preccurlyeq \mathbf{b}\right\} \subseteq \\ &\left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbb{I}^S, \sum_{i \in S} \mathbf{a}_i = \mathbf{b}\right\} - \mathbb{R}^S_+. \end{aligned}$$

If $S = \{j\}$, then

$$
\begin{aligned}
(\hat{\alpha} \circ \mathbf{V})(S) = (\hat{\alpha} \circ \mathbf{V})(\{j\}) &\overset{(3)(1)}{=} \{\alpha_j \circ \mathbf{a}_j : \mathbf{a}_j \preccurlyeq \mathbf{v}(\{j\})\} - \mathbb{R}_+^{\{j\}} \\
&\overset{(5)}{=} \{\alpha_i \circ \mathbf{v}(\{j\})\} - \mathbb{R}_+^{\{j\}} = \left\{x \in \mathbb{R}^{\{j\}} : x \leq \alpha_i \circ \mathbf{v}(\{j\})\right\} \\
&\equiv \alpha_j \circ \mathbf{v}(\{j\}) = \left(\max_{i \in S} \alpha_i\right) \circ \mathbf{v}(S).
\end{aligned}
$$

Take $S \subseteq N$, $|S| > 1$, and $j \in \arg\max_{i \in S} \alpha_i$. In case of more than one possible j, we take any one of them. Let $T = S \setminus \{j\} \neq \emptyset$. For each $\mathbf{a} \in \mathbb{I}$, let $|\mathbf{a}| := \overline{a} - \underline{a}$. Then,

$$
\begin{aligned}
&(\hat{\alpha} \circ \mathbf{V})(S) \\
&\overset{(3)}{=} \left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \hat{\mathbf{a}} \in \mathbf{V}(S)\right\} - \mathbb{R}_+^S \\
&\overset{(1)}{=} \left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \sum_{i \in S} \mathbf{a}_i \preccurlyeq \mathbf{v}(S)\right\} - \mathbb{R}_+^S \\
&\overset{(5)}{=} \left\{(\alpha_i \circ \mathbf{a}_i)_{i \in S} \in \mathbb{R}^S : \sum_{i \in S} \mathbf{a}_i = \mathbf{v}(S)\right\} - \mathbb{R}_+^S \\
&= \left\{(x_i + \alpha_i \cdot y_i)_{i \in S} \in \mathbb{R}^S : \sum_{i \in S} x_i = \underline{v}(S), \hat{x} \in \mathbb{R}^S, \sum_{i \in S} y_i = |\mathbf{v}|(S), \hat{y} \in \mathbb{R}_+^S\right\} \\
&- \mathbb{R}_+^S.
\end{aligned}
$$

Given that $\hat{x} \in \mathbb{R}^S$ is only restricted by $\sum_{i \in S} x_i = \underline{v}(S)$, this amount $\underline{v}(S)$ can be freely transferable among the players, so that the Pareto frontier is reached when $\sum_{i \in S} \alpha_i \cdot y_i$ is maximum, given that $\hat{y} \in \mathbb{R}_+^N$ and $\sum_{i \in S} y_i = |\mathbf{v}|(S)$. Since $\alpha_j \geq \alpha_i$ for all $i \in S$, this maximum is $\alpha_j \cdot |\mathbf{v}|(S)$, reached at least when $y_j = |\mathbf{v}|(S)$ and $y_i = 0$ otherwise. Hence,

$$
\begin{aligned}
(\hat{\alpha} \circ \mathbf{V})(S) &= \left\{(x_i)_{i \in T} \times (x_j + \alpha_j \cdot |\mathbf{v}|(S)) \in \mathbb{R}^S : \sum_{i \in S} x_i = \underline{v}(S)\right\} - \mathbb{R}_+^S \\
&= \left\{\hat{x} \in \mathbb{R}^S : \sum_{i \in S} x_i = \underline{v}(S) + \alpha_j \cdot |\mathbf{v}|(S)\right\} - \mathbb{R}_+^S \\
&\equiv \alpha_j \circ \mathbf{v}(S) = \left(\max_{i \in S} \alpha_i\right) \circ \mathbf{v}(S).
\end{aligned}
$$

■

Under Proposition 3.1, any coalitional interval game $\mathbf{v} \in \mathcal{IG}$ turns into a unique TU game $\hat{\alpha} \circ \mathbf{v} \in \mathcal{G}$ by applying the Hurwicz criterion with coefficients in $\hat{\alpha}$.

Given $\hat{\alpha} \in [0,1]^N$ and an interval game solution $\boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{I}^N$, we define $\boldsymbol{\sigma} \circ \hat{\alpha} : \mathcal{IB} \to \mathbb{R}^N$ as the function given by applying $\boldsymbol{\sigma}$ to $\hat{\alpha} \circ \mathbf{v}$ for each $\mathbf{v} \in \mathcal{IB}$, i.e.

$$(\boldsymbol{\sigma} \circ \hat{\alpha})_i(\mathbf{v}) := \boldsymbol{\sigma}_i(\hat{\alpha} \circ \mathbf{v}) \in \mathbb{R} \tag{6}$$

for all $\mathbf{v} \in \mathcal{IB}$ and $i \in N$.

Notice that both (2) and (6) apply an interval solution and the Hurwicz criterion with some coefficients. The difference between both approaches is the order in which they do so.

It is then natural to require this order to be irrelevant, i.e. both $\hat{\alpha} \circ \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} \circ \hat{\alpha}$ should coincide. We call this property *Hurwicz compatibility.*

Definition 3.1 *An interval game solution* $\boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{I}^N$ *is* Hurwicz compatible in $\mathcal{IB}$ *if* $(\boldsymbol{\sigma} \circ \hat{\alpha})(\mathbf{v}) = (\hat{\alpha} \circ \boldsymbol{\sigma})(\mathbf{v})$ *for all* $\mathbf{v} \in \mathcal{IB}$ *and all* $\hat{\alpha} \in [0,1]^N$.

Proposition 3.2 *For all* $\hat{\alpha} \in [0,1]^N$ *and all* $\mathbf{v} \in \mathcal{IG}$,

$$\sum_{i \in N} (\hat{\alpha} \circ \boldsymbol{\sigma})_i(\mathbf{v}) \leq \sum_{i \in N} (\boldsymbol{\sigma} \circ \hat{\alpha})_i(\mathbf{v})$$

for all efficient solution $\boldsymbol{\sigma} : \mathcal{IG} \to \mathbb{I}^N$. *Moreover,*

- *Equality holds for all* $\mathbf{v} \in \mathcal{IG}$ *and all efficient solution* $\boldsymbol{\sigma} : \mathcal{IG} \to \mathbb{I}^N$ *if and only if* $\alpha_i = \alpha_j$ *for all* $i, j \in N$.
- *Equality holds for all* $\hat{\alpha} \in [0,1]^N$ *if and only if* $|\mathbf{v}|(N) = 0$.

Proof. Under efficiency,

$$\begin{aligned}
\sum_{i \in N} (\hat{\alpha} \circ \boldsymbol{\sigma})_i(\mathbf{v}) &\overset{(2)}{=} \sum_{i \in N} \alpha_i \circ \boldsymbol{\sigma}_i(\mathbf{v}) = \sum_{i \in N} \left(\underline{\boldsymbol{\sigma}}_i(\mathbf{v}) + \alpha_i \cdot |\boldsymbol{\sigma}_i(\mathbf{v})| \right) \\
&= \underline{v}(N) + \sum_{i \in N} \alpha_i \cdot |\boldsymbol{\sigma}_i(\mathbf{v})| \leq \underline{v}(N) + \left(\max_{i \in N} \alpha_i \right) \cdot \sum_{i \in N} |\boldsymbol{\sigma}_i(\mathbf{v})| \\
&= \underline{v}(N) + \left(\max_{i \in N} \alpha_i \right) \cdot |\mathbf{v}|(N) = \left(\max_{i \in N} \alpha_i \right) \circ \mathbf{v}(N) \\
&\overset{(4)}{=} \left(\hat{\alpha} \circ \mathbf{v} \right)(N) = \sum_{i \in N} \boldsymbol{\sigma}_i(\hat{\alpha} \circ \mathbf{v}) \overset{(6)}{=} \sum_{i \in N} (\boldsymbol{\sigma} \circ \hat{\alpha})_i(\mathbf{v}).
\end{aligned}$$

Notice that this statement has a unique inequality. Hence, equality holds when this inequality is not strict. That is, equality holds iff

$$\sum_{i \in N} \alpha_i \cdot |\boldsymbol{\sigma}_i(\mathbf{v})| = \left(\max_{i \in N} \alpha_i \right) \cdot \sum_{i \in N} |\boldsymbol{\sigma}_i(\mathbf{v})|$$

which is equivalent to either $\alpha_i = \alpha_j$ for all $i, j \in N$ or $|\boldsymbol{\sigma}_i(\mathbf{v})| = 0$ for all $i \in N$. Since $|x|$ is always nonnegative, this second condition is equivalent to $|\mathbf{v}|(N) = 0$. ■

Corollary 3.1 *No efficient solution in the set of (size monotonic) coalitional interval games is Hurwicz-compatible.*

In view of Proposition 3.2, we can only find Hurwicz compatibility in efficient solutions when we restrict ourselves to two possible situations:

1. All the Hurwicz coefficients coincide (equal degree of optimism).
2. There is no uncertainty when all players cooperate (grand coalition certainty).

The second situation arises in particular for some economic situations such as oligopolies, as commented in the Introduction as a particular case of games with externalities. On the other hand, the first situation completes the class of conditions under which Hurwicz compatibility is possible, and it is a reasonable requirement to study its implications under the symmetry of the players (apart from the differences derived from the characteristic function).

4 Uniform degree of optimism

In this Section, we study which interval solutions are Hurwicz compatible when all coefficients coincide.

Definition 4.1 *Given* $\mathcal{IB} \subseteq IG$*, an interval game solution* $\boldsymbol{\sigma} : \mathcal{IB} \to \mathbb{I}^N$ *is* uniform Hurwicz compatible in $\mathcal{IB}$ *if, for all* $\alpha_0 \in [0,1]$*, we have* $(\boldsymbol{\sigma} \circ \hat{\alpha})(\mathbf{v}) = (\hat{\alpha} \circ \boldsymbol{\sigma})(\mathbf{v})$ *for all* $\mathbf{v} \in \mathcal{IB}$*, where* $\alpha_i = \alpha_0$ *for all* $i \in N$.

Let us consider a *coalitional interval game* $\mathbf{v}$ and $\hat{\alpha} \in [0,1]^N$ the vector of players' Hurwicz coefficients. We denote $\alpha_S := \max_{i \in S} \alpha_i$ and

$$n_S := \frac{|S|!(n-1-|S|)!}{n!}$$

for each $S \subseteq N$. By convention, $\alpha_\emptyset := 0$.

We consider here the Shapley value of the TU game $\hat{\alpha} \circ \mathbf{v}$, i.e.

$$Sh_i(\hat{\alpha} \circ \mathbf{v}) := \sum_{S \subseteq N \setminus \{i\}} n_S \cdot \big(\alpha_{S \cup \{i\}} \circ \mathbf{v}(S \cup \{i\}) - \alpha_S \circ \mathbf{v}(S)\big) \tag{7}$$

for each $i \in N$. On the class of size monotonic coalitional interval games, Alparslan Gök, Brânzei and Tijs (Alparslan Gök et al., 2010) define a generalization of the Shapley value, which we call the *interval Alparslan Gök-Brânzei-Tijs (ABT) solution*, as follows:

$$\mathbf{ABT}_i(\mathbf{v}) := \sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\mathbf{v}(S \cup \{i\}) \ominus \mathbf{v}(S))$$

for each $\mathbf{v} \in \mathcal{SMIG}$ and each $i \in N$. Analogously, Han, Sun and Xu (Han et al., 2012) define another generalization of the Shapley value, which we call the *interval Han-Sun-Xu (HSX) solution*, as follows:

$$\mathbf{HSX}_i(\mathbf{v}) := \sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\mathbf{v}(S \cup \{i\}) \oslash \mathbf{v}(S))$$

for each $\mathbf{v} \in \mathcal{IG}$ and each $i \in N$.

Proposition 4.1 *The interval ABT solution is uniform Hurwicz compatible in $\mathcal{SMIG}$.*

Proof. The interval ABT solution can be written as

$$\begin{aligned}&\mathbf{ABT}_i(\mathbf{v})\\ &= \left[\sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\underline{v}(S \cup \{i\}) - \underline{v}(S)), \sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\overline{v}(S \cup \{i\}) - \overline{v}(S))\right]\end{aligned}$$

for each $\mathbf{v} \in \mathcal{SMIG}$ and each $i \in N$. Since all the Hurwicz coefficients coincide ($\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha_0$), then $\hat{\alpha} \circ \mathbf{ABT}(\mathbf{v}) = Sh(\hat{\alpha} \circ \mathbf{v})$. In particular,

$$\begin{aligned}&\hat{\alpha} \circ \mathbf{ABT}_i(\mathbf{v})\\ &= (1-\alpha_0) \cdot \left(\sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\underline{v}(S \cup \{i\}) - \underline{v}(S))\right)\\ &+ \alpha_0 \cdot \left(\sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\overline{v}(S \cup \{i\}) - \overline{v}(S))\right)\\ &= \sum_{S \subseteq N \setminus \{i\}} n_S \cdot ((1-\alpha_0) \cdot (\underline{v}(S \cup \{i\}) - \underline{v}(S)) + \alpha_0 \cdot (\overline{v}(S \cup \{i\}) - \overline{v}(S)))\\ &= \sum_{S \subseteq N \setminus \{i\}} n_S\\ &\cdot (((1-\alpha_0) \cdot \underline{v}(S \cup \{i\}) + \alpha_0 \cdot \overline{v}(S \cup \{i\})) - ((1-\alpha_0) \cdot \underline{v}(S) + \alpha_0 \cdot \overline{v}(S)))\\ &= \sum_{S \subseteq N \setminus \{i\}} n_S \cdot (\alpha_0 \circ \mathbf{v}(S \cup \{i\}) - \alpha_0 \circ \mathbf{v}(S)) = Sh_i(\hat{\alpha} \circ \mathbf{v})\\ &= \mathbf{ABT}_i(\hat{\alpha} \circ \mathbf{v}) = (\mathbf{ABT} \circ \hat{\alpha})_{\mathbf{i}}(\mathbf{v})\end{aligned}$$

for each $\mathbf{v} \in \mathcal{SMIG}$ and each $i \in N$. ■

Theorem 4.1 *An efficient solution $\boldsymbol{\sigma} : \mathcal{IG} \to \mathbb{R}^N$ is uniform Hurwicz compatible in $\mathcal{IG}$ if and only if there exist two vectors $\hat{\delta} \in \mathbb{R}^N$ and $\hat{\gamma} \in \mathbb{R}^N_+$ with $\sum_{i \in N} \delta_i = 0$ and $\sum_{i \in N} \gamma_i = 1$, and such that*

$$\boldsymbol{\sigma}_i(\mathbf{v}) = \delta_i \cdot [1,1] + \gamma_i \cdot \mathbf{v}(N) \tag{8}$$

for all $\mathbf{v} \in \mathcal{IG}$ *and all* $i \in N$.

Proof. ($\Leftarrow$) Assume $\boldsymbol{\sigma}$ is defined as in (8). It is clear that such a solution is efficient. We now check that it is uniform Hurwicz compatible. Fix $\mathbf{v} \in \mathcal{IG}$. Let $\alpha_0 \in [0,1]$ and let $\hat{\alpha} \in [0,1]^N$ defined as $\alpha_i = \alpha_0$ for all $i \in N$. We have to prove that $(\boldsymbol{\sigma} \circ \hat{\alpha})(\mathbf{v}) = (\hat{\alpha} \circ \boldsymbol{\sigma})(\mathbf{v})$. For each $i \in N$,

$$\begin{aligned}(\boldsymbol{\sigma} \circ \hat{\alpha})_i(\mathbf{v}) &= \boldsymbol{\sigma}_i(\hat{\alpha} \circ \mathbf{v}) \\ &= \delta_i \cdot [1,1] + \gamma_i \cdot (\hat{\alpha} \circ \mathbf{v}(N)) \\ &\equiv \delta_i + \gamma_i \cdot (\hat{\alpha} \circ \mathbf{v}(N))\end{aligned}$$

and

$$\begin{aligned}(\hat{\alpha} \circ \boldsymbol{\sigma})_i(\mathbf{v}) &= \hat{\alpha} \circ \boldsymbol{\sigma}_i(\mathbf{v}) \\ &= \hat{\alpha} \circ (\delta_i \cdot [1,1] + \gamma_i \cdot \mathbf{v}(N)) \\ &= \delta_i + \gamma_i \cdot (\hat{\alpha} \circ \mathbf{v}(N)).\end{aligned}$$

($\Rightarrow$) Let $\boldsymbol{\sigma} : \mathcal{IG} \to \mathbb{I}^N$ be an efficient, Hurwicz compatible solution in $\mathcal{IG}$. Let $\mathbf{v} \in \mathcal{IG}$. Let $\hat{1} \in [0,1]^N$ defined as $1_i = 1$ for all $i \in N$, and let $\hat{0} \in [0,1]^N$ defined as $0_i = 0$ for all $i \in N$. Then, for each $\mathbf{v} \in \mathcal{IG}$ and $i \in N$,

$$\overline{\sigma}_i(\mathbf{v}) = \hat{1} \circ \boldsymbol{\sigma}_i(\mathbf{v}) = \boldsymbol{\sigma}_i(\hat{1} \circ \mathbf{v}) = \boldsymbol{\sigma}_i(\overline{v}) \tag{9}$$

and

$$\underline{\sigma}_i(\mathbf{v}) = \hat{0} \circ \boldsymbol{\sigma}_i(\mathbf{v}) = \boldsymbol{\sigma}_i(\hat{0} \circ \mathbf{v}) = \boldsymbol{\sigma}_i\left(\underline{v}\right). \tag{10}$$

Since both $\overline{v}$ and $\underline{v}$ are TU games, we deduce that $\boldsymbol{\sigma}$ only depends on its restriction on TU games, i.e., once we define $\boldsymbol{\sigma}(v)$ for each $v \in \mathcal{G}$, we can deduce $\boldsymbol{\sigma}(\mathbf{v})$ for any other $\mathbf{v} \in \mathcal{IG}$. Moreover, given $v, w \in \mathcal{G}$ with $v(S) \le w(S)$ for all $S \subseteq N$, it holds $\boldsymbol{\sigma}(v) \le \boldsymbol{\sigma}(w)$ (otherwise, $\boldsymbol{\sigma}(\mathbf{w})$ with $\mathbf{w}(S) = [v(S), w(S)]$ for all $S \subseteq N$ would not be well-defined). Now, for each $v \in \mathcal{G}$, define $v^-, v^+ \in \mathcal{G}$ as follows:

$$v^+(S) := \begin{cases} v(N) & \text{if } v(S) \le v(N) \\ v(S) & \text{if } v(S) > v(N) \end{cases}$$

$$v^-(S) := \begin{cases} v(S) & \text{if } v(S) \le v(N) \\ v(N) & \text{if } v(S) > v(N) \end{cases}$$

for all $S \subseteq N$. Clearly, $v^-(S) \le v(S) \le v^+(S)$ for all $S \subseteq N$. Hence, $\boldsymbol{\sigma}(v^-) \le \boldsymbol{\sigma}(v) \le \boldsymbol{\sigma}(v^+)$. Moreover, $v^-(N) = v(N) = v^+(N)$ and hence $\boldsymbol{\sigma}(v^-) = \boldsymbol{\sigma}(v) = \boldsymbol{\sigma}(v^+)$. For each

$x \in \mathbb{R}$, define $u^x \in \mathcal{G}$ as follows:

$$u^x(S) := \begin{cases} x & \text{if } S = N \\ 0 & \text{if } S \neq N \end{cases}$$

for all $S \subseteq N$. On the one hand, either $u^{v(N)}(S) \leq v^+(S)$ for all $S \subseteq N$ (when $v(N) > 0$) or $v^-(S) \leq u^{v(N)}(S)$ for all $S \subseteq N$ (when $v(N) \leq 0$). On the other hand, $u^{v(N)}(N) = v(N) = v^+(N) = v^-(N)$. Hence, either $\boldsymbol{\sigma}(u^{v(N)}) = \boldsymbol{\sigma}(v^+)$ (when $v(N) > 0$) or $\boldsymbol{\sigma}(u^{v(N)}) = \boldsymbol{\sigma}(v^-)$ (when $v(N) \leq 0$). In either case,

$$\boldsymbol{\sigma}(v) = \boldsymbol{\sigma}\left(u^{v(N)}\right) \tag{11}$$

i.e. the only relevant value is $v(N)$. Let

$$f(x) := \boldsymbol{\sigma}(u^x) \tag{12}$$

for all $x \in \mathbb{R}$. Now, we prove that

$$f_i(\alpha_0 \cdot x) = \alpha_0 \cdot f_i(x) + (1 - \alpha_0) \cdot f_i(0) \tag{13}$$

for all $i \in N$, all $\alpha_0 \in [0, 1]$, and all $x \in \mathbb{R}$. We assume $x > 0$. Case $x < 0$ is analogous and case $x = 0$ is trivial. Define $\mathbf{u}^{0x} \in \mathcal{IG}$ as follows:

$$\mathbf{u}^{0x}(S) := \begin{cases} [0, x] & \text{if } S = N \\ 0 & \text{if } S \neq N \end{cases}$$

for all $S \subseteq N$. Let $\hat{\alpha} \in [0, 1]^N$ defined by $\alpha_i = \alpha_0$ for all $i \in N$ for some $\alpha_0 \in [0, 1]$. Hence,

$$\begin{aligned} f_i(\alpha_0 \cdot x) = f_i(\alpha_0 \cdot x + (1 - \alpha_0) \cdot 0) &= \boldsymbol{\sigma}_i\left(u^{\alpha_0 \cdot x + (1 - \alpha_0) \cdot 0}\right) \\ &= \boldsymbol{\sigma}_i\left(\hat{\alpha} \circ \mathbf{u}^{0x}\right) = (\boldsymbol{\sigma} \circ \hat{\alpha})_i\left(\mathbf{u}^{0x}\right) \end{aligned}$$

for all $i \in N$. By uniform Hurwicz compatibility of $\boldsymbol{\sigma}$:

$$\begin{aligned} f_i(\alpha_0 \cdot x) &= (\hat{\alpha} \circ \boldsymbol{\sigma})_i\left(\mathbf{u}^{0x}\right) = \alpha_0 \circ \boldsymbol{\sigma}_i\left(\mathbf{u}^{0x}\right) \\ &= \alpha_0 \circ \left[\boldsymbol{\sigma}_i\left(u^0\right), \boldsymbol{\sigma}_i\left(u^x\right)\right] \\ &= \alpha_0 \cdot \boldsymbol{\sigma}_i\left(u^x\right) + (1 - \alpha_0) \cdot \boldsymbol{\sigma}_i\left(u^0\right) \\ &= \alpha_0 \cdot f_i(x) + (1 - \alpha_0) \cdot f_i(0) \end{aligned}$$

for all $i \in N$, and hence (13) holds. This implies that, for each $i \in N$, there exist $\delta_i, \gamma_i \in \mathbb{R}$ such that

$$f_i(x) = \delta_i + \gamma_i \cdot x \tag{14}$$

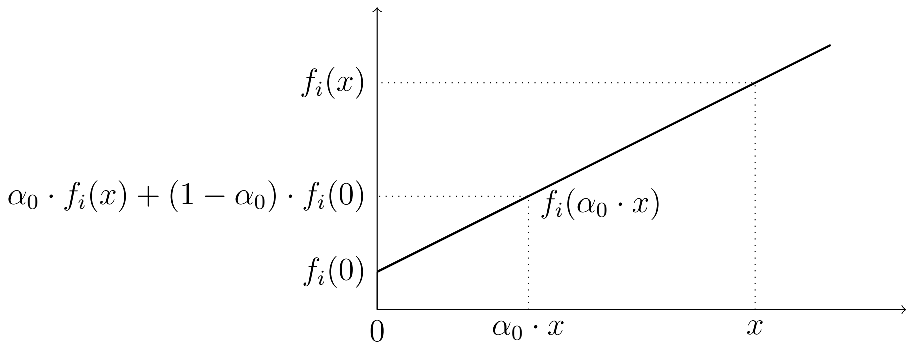

Figure 1: Visual proof that (13) implies (14).

for all $x \in \mathbb{R}$ (see Figure 1).

Clearly, $\delta_i = f_i(0)$ for all $i \in N$. Moreover, $\gamma_i \geq 0$ for all $i \in N$ because otherwise $\boldsymbol{\sigma}_i(\mathbf{v}) \notin \mathbb{I}$ when $|\mathbf{v}|(N) > 0$. By efficiency of $\boldsymbol{\sigma}$:

$$\sum_{i \in N} \delta_i = \sum_{i \in N} f_i(0) = \sum_{i \in N} \boldsymbol{\sigma}_i \left(u^0\right) = u^0(N) = 0$$

and

$$\sum_{i \in N} \gamma_i = \sum_{i \in N} (\gamma_i \cdot 1) = \sum_{i \in N} (f_i(1) - \delta_i) = \sum_{i \in N} \boldsymbol{\sigma}_i \left(u^1\right) - \sum_{i \in N} \delta_i = u^1(N) - 0 = 1.$$

Finally, for all $\mathbf{v} \in \mathcal{IG}$ and all $i \in N$,

$$\begin{aligned} \boldsymbol{\sigma}_i(\mathbf{v}) &\overset{(10)(9)}{=} \left[\boldsymbol{\sigma}_i\left(\underline{v}\right), \boldsymbol{\sigma}_i\left(\overline{v}\right)\right] \\ &\overset{(11)}{=} \left[\boldsymbol{\sigma}_i\left(u^{\underline{v}(N)}\right), \boldsymbol{\sigma}_i\left(u^{\overline{v}(N)}\right)\right] \\ &\overset{(12)}{=} \left[f_i(\underline{v}(N)), f_i\left(\overline{v}(N)\right)\right] \\ &\overset{(14)}{=} \left[\delta_i + \gamma_i \cdot \underline{v}(N), \delta_i + \gamma_i \cdot \overline{v}(N)\right] \\ &= \delta_i \cdot [1,1] + \gamma_i \cdot [\underline{v}(N), \overline{v}(N)] = \delta_i \cdot [1,1] + \gamma_i \cdot \mathbf{v}(N). \end{aligned}$$

■

In particular, when $\delta_i = \delta_j$ and $\gamma_i = \gamma_j$ for all $i, j \in N$, then $\delta_i = 0$ and $\gamma_i = \frac{1}{n}$ for all $i \in N$, and so we obtain the *interval egalitarian* solution:

$$\boldsymbol{e}_i\left(\mathbf{v}\right) := \frac{1}{n} \cdot \mathbf{v}(N)$$

for all $i \in N$.

An important implication of Theorem 4.1 is the following:

Theorem 4.2 *The interval egalitarian solution is the only efficient, symmetric, and uniform Hurwicz compatible solution in* $\mathcal{IG}$*.*

Proof. It follows from Theorem 4.1 and the fact that the interval egalitarian solution is the only symmetric one in the family of efficient, uniform Hurwicz-compatible solutions. ■

Corollary 4.1 *For $n > 1$, the interval HSX solution is not uniform Hurwicz compatible in $\mathcal{IG}$.*

Proof. It follows from Theorem 4.2 and the fact that the interval HSX solution is efficient, symmetric, defined in all $\mathcal{IG}$, and (for $n > 1$) different from the interval egalitarian solution. ■

5 Grand coalition certainty

Let $\mathcal{IC}$ denote the set of interval coalitional games $\mathbf{v} \in \mathcal{IG}$ satisfying $\underline{v}(N) = \overline{v}(N)$. Clearly, $\mathcal{G} \subset \mathcal{IC}$.

A simple family of efficient solutions in $\mathcal{IC}$ is given by

$$e_i^\omega(\mathbf{v}) := \omega_i(\mathbf{v}(N))$$

for all $i \in N$, where $\omega(x) \in \mathbb{R}^N$ satisfies $\sum_{i \in N} \omega_i(x) = x$ for all $x \in \mathbb{R}$. In particular, when $\omega_i(x) = \omega_j(x)$ for all $i, j \in N$ and all $x \in \mathbb{R}$, we obtain the *egalitarian* solution:

$$e_i(\mathbf{v}) := \frac{1}{n} \cdot \mathbf{v}(N)$$

for all $i \in N$.

Theorem 5.1 *An efficient solution $\sigma : \mathcal{IC} \to \mathbb{R}^N$ is Hurwicz-compatible in $\mathcal{IC}$ if and only if there exists a function ω that assigns to each $x \in \mathbb{R}$ a vector $\omega(x) \in \mathbb{R}^N$ with $\sum_{i \in N} \omega_i(x) = x$ and such that*

$$\sigma(\mathbf{v}) = e^\omega(\mathbf{v})$$

for all $\mathbf{v} \in \mathcal{IC}$.

Proof. Since we are in $\mathcal{IC}$, $\hat{\alpha} \circ \sigma = \sigma$ for all efficient solution σ and all $\hat{\alpha} \in [0,1]^N$. Hence, Hurwicz compatibility is equivalent to

$$\sigma(\mathbf{v}) = \sigma(\hat{\alpha} \circ \mathbf{v})$$

for all $\mathbf{v} \in \mathcal{IC}$ and all $\hat{\alpha} \in [0,1]^N$. It is not difficult to check that, given $\omega : \mathbb{R} \to \mathbb{R}^N$ with $\sum_{i \in N} \omega_i(x) = x$ for all $x \in \mathbb{R}$, e^ω is Hurwicz-compatible for $\mathcal{IC}$. Let $\sigma : \mathcal{IC} \to \mathbb{R}^N$

be an efficient Hurwicz compatible solution for $\mathcal{IC}$. For each $x \in \mathbb{R}$, let $\mathbf{e}^x \in \mathcal{IC}$ defined as $\mathbf{e}^x(N) = [x, x]$ and $\mathbf{e}^x(S) = [0, 0]$ otherwise. Let

$$\omega_i(x) = \sigma_i(\mathbf{e}^x)$$

for all $x \in \mathbb{R}$ and all $i \in N$. By efficiency of σ, we deduce $\sum_{i \in N} \omega_i(x) = x$ for all $x \in \mathbb{R}$. We prove that $\sigma(\mathbf{v}) = e^{\omega}(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{IC}$ by induction on the cardinality of

$$\Theta(\mathbf{v}) = \{S \subset N, S \neq \emptyset : \mathbf{v}(S) \neq [0, 0]\}.$$

Assume first $|\Theta(\mathbf{v})| = 0$. Then, $\mathbf{v} = \mathbf{e}^{\mathbf{v}(N)}$ and hence

$$\sigma(\mathbf{v}) = \sigma\left(\mathbf{e}^{\mathbf{v}(N)}\right) = \omega(\mathbf{v}(N)) = e^{\omega}(\mathbf{v}).$$

Assume now the result holds when the cardinality of $\Theta(\mathbf{v})$ is less than $\theta > 0$, and let $|\Theta(\mathbf{v})| = \theta$. Let $S \in \Theta(\mathbf{v})$. We have three cases:

Case 1: $0 \in \mathbf{v}(S)$. Since $S \in \Theta(\mathbf{v})$, we deduce that there exists some $\alpha_0 \in [0, 1]$ such that $0 = \alpha_0 \circ \mathbf{v}(S)$. Let $\alpha_i = \alpha_0$ for all $i \in N$ and let $\mathbf{v}^{-S} \in \mathcal{IC}$ defined as follows:

$$\mathbf{v}^{-S}(T) := \begin{cases} \mathbf{v}(T) & \text{if } T \neq S \\ [0, 0] & \text{if } T = S \end{cases}$$

for all $T \subseteq N$. It is straightforward to check that $\hat{\alpha} \circ \mathbf{v} = \hat{\alpha} \circ \mathbf{v}^{-S}$. Under Hurwicz compatibility of σ and the induction hypothesis,

$$\begin{aligned} \sigma(\mathbf{v}) &= \sigma(\hat{\alpha} \circ \mathbf{v}) = \sigma\left(\hat{\alpha} \circ \mathbf{v}^{-S}\right) = \sigma\left(\mathbf{v}^{-S}\right) \\ &= e^{\omega\left(\mathbf{v}^{-S}(N)\right)}\left(\mathbf{v}^{-S}\right) = e^{\omega(\mathbf{v}(N))}(\mathbf{v}). \end{aligned}$$

Case 2: $0 < \underline{v}(S)$. Let $\alpha_i = 1$ for all $i \in N$. Let $\mathbf{v}^{-0,S} \in \mathcal{IC}$ defined as follows:

$$\mathbf{v}^{-0,S}(T) := \begin{cases} \mathbf{v}(T) & \text{if } T \neq S \\ [0, \overline{v}(S)] & \text{if } T = S \end{cases}$$

for all $T \subseteq N$. It is straightforward to check that $\hat{\alpha} \circ \mathbf{v} = \hat{\alpha} \circ \mathbf{v}^{-0,S}$. Under Hurwicz compatibility of σ,

$$\sigma(\mathbf{v}) = \sigma(\hat{\alpha} \circ \mathbf{v}) = \sigma\left(\hat{\alpha} \circ \mathbf{v}^{-0,S}\right) = \sigma\left(\mathbf{v}^{-0,S}\right)$$

and we proceed as in Case 1 with $\left(\mathbf{v}^{-0,S}\right)$.

Case 3: $\overline{v}(S) < 0$. Let $\alpha_i = 0$ for all $i \in N$. Let $\mathbf{v}^{+0,S} \in \mathcal{IC}$ defined as follows:

$$\mathbf{v}^{+0,S}(T) := \begin{cases} \mathbf{v}(T) & \text{if } T \neq S \\ [\underline{v}(S), 0] & \text{if } T = S \end{cases}$$

for all $T \subseteq N$. It is straightforward to check that $\hat{\alpha} \circ \mathbf{v} = \hat{\alpha} \circ \mathbf{v}^{+0,S}$. Under Hurwicz compatibility of σ,

$$\sigma(\mathbf{v}) = \sigma(\hat{\alpha} \circ \mathbf{v}) = \sigma\left(\hat{\alpha} \circ \mathbf{v}^{+0,S}\right) = \sigma\left(\mathbf{v}^{+0,S}\right)$$

and we proceed as in Case 1 with $\mathbf{v}^{+0,S}$.

■

An important implication of Theorem 5.1 is the following:

Theorem 5.2 *The egalitarian solution is the only efficient and symmetric Hurwicz compatible solution in $\mathcal{IC}$.*

Proof. It follows from Theorem 5.1 and the fact that the egalitarian solution is the only symmetric one in the family of efficient, Hurwicz-compatible solutions. ■

6 A comment on the core

Let us consider a coalitional interval game $\mathbf{v} \in \mathcal{IG}$ and $\hat{\alpha} \in [0,1]^N$. For any coalition $S \subseteq N$, we denote $\alpha_S = \max_{i \in S} \alpha_i$. The *core* of the TU game $\hat{\alpha} \circ \mathbf{v}$ defined as $(\hat{\alpha} \circ \mathbf{v})(S) = \alpha_S \circ \mathbf{v}(S)$, for all $S \subseteq N$, $S \neq \emptyset$ is given by the set:

$$C(\hat{\alpha} \circ \mathbf{v}) := \left\{ \hat{x} \in \mathbb{R}^N : \sum_{i \in N} x_i = \alpha_N \circ \mathbf{v}(N), \alpha_S \circ \mathbf{v}(S) \leq \sum_{i \in S} x_i, \forall S \subseteq N \right\}.$$

6.1 The interval core

Let us recall that the interval core with N players as defined in Alparslan Gök et al. (2009a) as the set

$$\mathcal{C}(\mathbf{v}) := \left\{ \mathbf{a} \in \mathbb{I}^N : \sum_{i \in N} \mathbf{a}_i = \mathbf{v}(N), \mathbf{v}(S) \preccurlyeq \sum_{i \in S} \mathbf{a}_i, \forall S \subseteq N \right\}$$

together with conditions that guarantee $\mathcal{C}(\mathbf{v}) \neq \emptyset$.

In this case, we can apply the Hurwicz criterion to $\mathcal{C}(\mathbf{v})$ and obtain the set

$$\hat{\alpha} \circ \mathcal{C}(\mathbf{v}) := \left\{(\hat{\alpha} \circ \mathbf{a}_i)_{i \in N} \in \mathbb{R}^N : \mathbf{a} \in \mathcal{C}(\mathbf{v})\right\}$$
$$= \left\{(\underline{a}_i + \alpha_i \cdot |\mathbf{a}_i|)_{i \in N} \in \mathbb{R}^N : \sum_{i \in N} \mathbf{a}_i = \mathbf{v}(N), \mathbf{v}(S) \preceq \sum_{i \in S} \mathbf{a}_i, \forall S \subseteq N\right\}$$

where

$$\sum_{i \in N} \mathbf{a}_i = \mathbf{v}(N) \Longleftrightarrow \sum_{i \in N} \underline{a}_i = \underline{v}(N) \text{ and } \sum_{i \in N} |\mathbf{a}_i| = |\mathbf{v}|(N)$$

and

$$\mathbf{v}(S) \preceq \sum_{i \in S} \mathbf{a}_i \Longleftrightarrow \underline{v}(S) \leq \sum_{i \in S} \underline{a}_i \text{ and } |\mathbf{v}|(S) \leq \sum_{i \in S} |\mathbf{a}_i|$$

for all $S \subseteq N$. Notice that $\preceq$ has a different formal meaning than $\preccurlyeq$. In particular, $\preceq$ implies $\preccurlyeq$ but the opposite does not hold. For example, $[0,1] \preccurlyeq [2,2]$ but $[0,1] \npreceq [2,2]$ because $|[0,1]| > |[2,2]|$.

Proposition 6.1 *If all the Hurwicz coefficients coincide, $\alpha_i = \alpha_0$ for all $i \in N$, then*

$$\hat{\alpha} \circ \mathcal{C}(\mathbf{v}) \subseteq C(\hat{\alpha} \circ \mathbf{v}). \tag{15}$$

Proof. Let us consider $\hat{x} \in \hat{\alpha} \circ \mathcal{C}(\mathbf{v})$. Then, $x_i = \underline{a}_i + \alpha_0 \cdot |\mathbf{a}_i|$ for any $i \in N$, $\sum_{i \in N} \underline{a}_i = \underline{v}(N)$ and $\sum_{i \in N} |\mathbf{a}_i| = |\mathbf{v}|(N)$, and $\underline{v}(S) \leq \sum_{i \in S} \underline{a}_i$ and $|\mathbf{v}|(S) \leq \sum_{i \in S} |\mathbf{a}_i|$ for any $S \subseteq N$. Then,

$$\sum_{i \in N} x_i = \sum_{i \in N} \underline{a}_i + \alpha_0 \cdot \sum_{i \in N} |\mathbf{a}_i| = \alpha_0 \circ \mathbf{v}(N)$$

and, for any $S \subseteq N$,

$$\alpha_0 \circ \mathbf{v}(S) \leq \sum_{i \in S} \underline{a}_i + \alpha_0 \cdot \sum_{i \in S} |\mathbf{a}_i| = \sum_{i \in S} x_i$$

so $\hat{x} \in C(\hat{\alpha} \circ \mathbf{v})$. ■

Observe that inclusion (15) holds also when $\mathcal{C}(\mathbf{v}) = \emptyset$. Inclusion $C(\hat{\alpha} \circ \mathbf{v}) \subseteq \hat{\alpha} \circ \mathcal{C}(\mathbf{v})$ is not always true, as in the following example.

Example 6.1 *Consider $n = 2$ and the interval cooperative game $\mathbf{v}$ given by $\mathbf{v}(\{1\}) = \mathbf{v}(\{2\}) = [1,3]$ and $\mathbf{v}(\{1,2\}) = [2,4]$. For this game, $\mathcal{C}(\mathbf{v}) = \emptyset$, while if we consider $\alpha_1 = \alpha_2 = 0$, the game $\hat{\alpha} \circ \mathbf{v}$ has non emptycore: $0 \circ \mathbf{v}(\{1\}) = 1$, $0 \circ \mathbf{v}(\{2\}) = 1$ and $0 \circ \mathbf{v}(\{1,2\}) = 2$, so $C(\hat{\alpha} \circ \mathbf{v}) = \{(1,1)\}$.*

Then we can deduce the next result.

Proposition 6.2 *The interval core is not uniform Hurwicz-compatible in $\mathcal{IG}$.*

Remark 6.1 *If we consider in the previous example the grand coalition certainty, in particular if we assume $\mathbf{v}(\{1,2\}) = [2,2]$, we have still that the inclusion $C(\hat{\alpha} \circ \mathbf{v}) \subseteq \hat{\alpha} \circ \mathcal{C}(\mathbf{v})$ does not hold.*

6.2 The interval square core

Given an interval cooperative game $\mathbf{v} \in \mathcal{IG}$, we call *border games* the two (classical) cooperative games defined as $\underline{v}$ and $\overline{v}$ where for any $S \subseteq N$, $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$. In Alparslan Gök et al. (2009c) the *square interval core* $C(\underline{v}) \square C(\overline{v})$ has been defined as:

$$C(\underline{v}) \square C(\overline{v}) := \left\{ \mathbf{a} \in \mathbb{I}^N : \sum_{i \in N} \mathbf{a}_i = \mathbf{v}(N), \underline{a} \in C(\underline{v}), \overline{a} \in C(\overline{v}) \right\}$$

and it has been proved that if $\mathcal{C}(\mathbf{v}) \neq \emptyset$ then $C(\underline{v}) \square C(\overline{v}) = \mathcal{C}(\mathbf{v})$. Then, the same considerations of the interval core on the Hurwicz compatibility can be done.

7 Concluding remarks

In this paper we study interval cooperative games. These are games where the worth of coalitions are uncertain. Both a lower and an upper bound of the possible outcome is assigned to each coalition. For these games, several solution concepts provide interval allocations to the players and leave uncertainty on the exit. To mitigate this uncertainty, assuming some degree of optimism (or pessimism) of the players (given by real numbers between 0 and 1), we introduce a TU cooperative game applying the Hurwicz criterion. This procedure allows having a standard solution concept once the degree of optimism is fixed.

Another possibility of approaching the uncertainty is the following: consider any interval solution concept in the original interval game and then apply the Hurwicz criterion to the interval allocation. The question posed in the paper is if the two approaches lead to the same result. We give the idea of Hurwicz compatibility and investigate under which conditions it holds. In the case of a uniform degree of optimism/pessimism or of the grand coalition certainty, we prove that the only compatible solutions are the proportional ones, or the egalitarian in case symmetry is required. Some considerations on the Shapley value and the core solution are also discussed.

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