Aggregator Operators for Dynamic Rationing*

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Abstract

We study dynamic rationing problems. In each period, a fixed group of agents hold claims over an insufficient endowment. The solution to each of these periods' problems might be influenced by the solutions at previous periods. We single out a natural family of aggregator operators, which extend static rules (solving static rationing problems) to construct rules to solve dynamic rationing problems.

Keywords: *axioms, resource allocation, dynamic models, rationing, operators.*

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1 Introduction

How should we divide when there is not enough? This is, allegedly, one of the oldest questions in the history of economic thought and its treatment can indeed be traced back to ancient sources. O'Neill (1982) was the first to introduce a simple model to answer this question. His basic (and extremely influential) model formalized a group of individuals having conflicting claims over an insufficient amount of a perfectly divisible good. Formally, a rationing problem is described by an endowment $E \in \mathbb{R}_+$, which has to be allocated among a group of agents N, each of whom has a claim $c_i \in \mathbb{R}_+$, so that $E \leq \sum_{i \in N} c_i$. The issue was to determine rules that would associate with each of these problems a specific allocation of the endowment. The model generated a sizable literature in the last decades analyzing various aspects of this simple, yet rich, model of rationing. The reader is referred to Thomson (2003, 2015, 2019) for detailed surveys of the literature.

The field of operations research has devoted considerable attention to O'Neill's model (Lahiri, 2001; van den Brink et al., 2013; Giménez-Gómez and Peris, 2014), some of its applications (Casas-Méndez et al., 2011; Gutiérrez et al., 2018), or several of its generalizations (Calleja et al., 2005; Bergantiños and Vidal-Puga, 2004, 2006; Bergantiños and Lorenzo, 2008). Nevertheless, it is somewhat remarkable that no attention has been paid to address the extension of the model to a dynamic setting, which would accommodate an extremely natural aspect of real-life rationing processes. This paper aims to be a first step in that direction.

In general, rationing does not occur in static terms. In refugee camps, for instance, minimum food rations are provided immediately upon identification, to ensure the nutritional status of refugees does not deteriorate. In subsequent months, refugees are provided with food rations composed by a mix of food items (involving cereals, pulses, vegetable oil, and nutrient-enriched flour) and cash, sent through mobile telephones, allowing them to buy food products of their choice from local markets. The extent of these rations depends on the available resources and the amount of refugees (and their needs), among other things.[1]

How should rationing be addressed in a dynamic setting? One trivial answer is to do so by ignoring the dynamic component and solving the problem at each period independently. We believe that is unsatisfactory and aim to proceed differently. More precisely,

[1]For instance, in Dadaab (Kenya, bordering near Somalia), the world's largest refugee camp, the UN World Food Programme was forced to cut food rations by 50 per cent in 2016, due to a lack of funds. See `http://www.un.org/apps/news/story.asp?NewsID=56521`, accessed on January 22, 2020.

imagine we consider a sequence of rationing problems involving the same group of agents, at different periods of time, whose period-wise allocations might not only be determined by the data of the rationing problem at such period, but also by the allocations in previous periods. There is obviously a wide margin to do so and we need to take some stances.

In this paper, we shall concentrate on a plausible way to start approaching this issue, by assuming that, at each period, the corresponding rationing problem is enriched by an index summarizing the amounts each agent obtained in the previous periods. The index could have many forms, ranging from the (arithmetic or geometric) average to some lower or upper bounds, as well as simply the choice of a specific period. In any case, it could be interpreted as a baselines profile, as formalized by Hougaard et al. (2012, 2013a,b).[2]

Formally, let N be a fixed population of n claimants. At each period of time $t = 1, 2, \ldots,$ this population faces a realization of a static rationing problem (c^t, E^t). That is, for each $i \in N$, $c_i^t \in \mathbb{R}_+$ denotes the claim of i at period t, $E^t \in \mathbb{R}_+$ denotes the endowment to be distributed at that period and, to qualify as a rationing problem, they jointly satisfy the condition $\sum_{i \in N} c_i^t \geq E^t$. Suppose we have a static rationing rule R. That is, R is a mapping from the set of problems so defined into the set $\mathbb{R}_+^N$, such that it assigns to each problem (c^t, E^t) a given allocation $x^t \in \mathbb{R}_+^N$, with the proviso that $0 \leq x_i^t \leq c_i^t$, for each $i \in N$, and $\sum_{i \in N} x_i^t = E^t$.

Let x^1 be the solution to the first-period problem (c^1, E^1) that R yields, i.e.,

$$x^1 = R\left(c^1, E^1\right).$$

In the second period, we then consider x^1 as a baseline to solve the problem (c^2, E^2) also via R. More precisely,

$$\left(x^1, c^2, E^2\right) = \left(b^2, c^2, E^2\right) \text{ and } x^2 = R^{b^2}\left(c^2, E^2\right) = b^3,$$

where $R^b(c, E)$ denotes the solution that the *b-baseline extended* rule associated to R yields. That is, R^b is the rule that extends R to solve rationing problems in the presence of baselines b: first allocating b tentatively, and then allocating the resulting deficit of surplus via R and the adjusted claims. In general,

$$x^t = R^{b^t}\left(c^t, E^t\right) = b^{t+1}.$$

Thus, the above protocol allows one to extend a static rule R to solve a sequence of rationing problems. Now, the protocol is using as baselines for a given period the solution

[2]See also Pulido et al. (2002, 2008); Timoner and Izquierdo (2016).

to the problem in the previous period. Alternatively, one could take the average of the solutions in all previous periods. That is,

$$b^t = \frac{1}{n}\sum_{l=1}^{t-1} x^l.$$

Or, more generally,

$$b^t = \rho\left(x^1, \ldots, x^{t-1}\right),$$

where ρ is an *aggregator operator*, to be formally defined next.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we introduce (and characterize) the aggregator operators to solve dynamic problems. In Section 4, we concentrate on a focal family of these operators. We conclude in Section 5.

2 The model

2.1 The benchmark model

We start considering the benchmark (static) model, as initially formalized by O'Neill (1982). There is a finite number of claimants, or **agents**, which are indexed by the set $N = \{1, \ldots, n\}$. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's **claim** and $c := (c_i)_{i\in N}$ the claims profile. Let $0^N \in \mathbb{R}^N_+$ be defined as $0^N_i = 0$ for each $i \in N$. Let $\|c\|_1 := \sum_{i\in N} c_i$ denote the 1-norm (*taxicab* norm) of c. An endowment $E \in \mathbb{R}_+$ is to be allocated among N. Formally, a (rationing) **problem** is a pair (c, E) consisting of a claims profile $c \in \mathbb{R}^N_+$, and an endowment $E \in \mathbb{R}_+$ such that $\|c\|_1 \geq E$. Let $C := \|c\|_1$. Let $\mathbb{P}$ be set of rationing problems.

Given a problem $(c, E) \in \mathbb{P}$, an **allocation** is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$, and (ii) $\|x\|_1 = E$. We refer to (i) as **boundedness**, and (ii) as **budget-balancedness**.

A *static rule* R on $\mathbb{P}$, $R\colon \mathbb{P} \to \mathbb{R}^N$, associates with each problem $(c, E) \in \mathbb{P}$ an allocation $R(c, E) \in \mathbb{R}^N$. Let $\mathcal{R}$ denote set of those static rules. Each static rule $R \in \mathcal{R}$ has a *dual* static rule $R^* \in \mathcal{R}$ defined as $R^*(c, E) = c - R(c, C - E)$, for each $(c, E) \in \mathbb{P}$.[3]

We now consider some classical static rules. The constrained equal losses rule imposes that losses are as equal as possible subject to no one receiving a negative amount. The proportional rule allocates awards proportionally to claims. The constrained equal awards

[3]Note that $(c, C - E)$ is a well-defined problem because $\|c\|_1 = C \geq C - E$.

rule distributes the endowment equally among all agents, subject to no agent receiving more than she claims.

Finally, the Talmud rule behaves like the first or the third rule, depending on whether the endowment exceeds or falls short one half of the aggregate claim, using half-claims instead of claims. Formally,

- The **constrained equal-losses** rule, L, selects for each $(c, E) \in \mathbb{P}$, $L(c, E) = (\max\{0, c_i - \lambda\})_{i\in N}$, where $\lambda \geq 0$ is chosen so that $\sum_{i\in N} \max\{0, c_i - \lambda\} = E$.
- The **proportional** rule, P, selects for each $(c, E) \in \mathbb{P}$ with $C > 0$, $P(c, E) = \left(\frac{E}{C} \cdot c_i\right)_{i\in N}$.[4]
- The **constrained equal-awards** rule, A, selects for each $(c, E) \in \mathbb{P}$, $A(c, E) = (\min\{c_i, \lambda\})_{i\in N}$, where $\lambda \geq 0$ is chosen so that $\sum_{i\in N} \min\{c_i, \lambda\} = E$.
- The **Talmud** rule, T, selects for each $(c, E) \in \mathbb{P}$, $T(c, E) = \left(\min\{\frac{1}{2}c_i, \lambda\}\right)_{i\in N}$ if $E \leq \frac{1}{2}C$ and $T(c, E) = \left(\max\{\frac{1}{2}c_i, c_i - \lambda\}\right)_{i\in N}$ if $E \geq \frac{1}{2}C$, where λ is chosen so that $\sum_{i\in N} T_i(c, E) = E$.

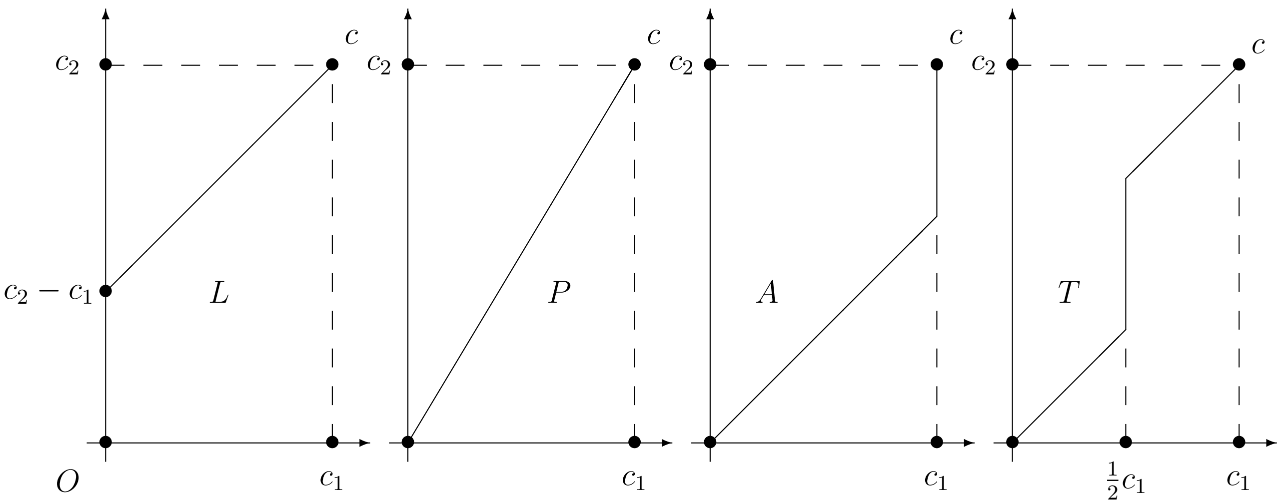

Figure 1: Rules in the two-claimant case. This figure illustrates the "path of awards" of some rules for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of awards for c (the locus of the awards vector chosen by a rule as the endowment E varies from 0 to $c_1 + c_2$) of L follows the vertical axis until the average loss coincides with the lowest claim, i.e., until $E = c_2 - c_1$. After that, it follows the line of slope 1 until it reaches the vector of claims. The path of awards of P follows the segment from the origin to the claims vector. The path of awards of A follows the

[4]The case $C = 0$ implies $(c, E) = (0^N, 0)$. By boundedness and budget-balancedness, $P\left(0^N, 0\right) = 0^N$.

45^o line until it gives the whole claim to the lowest claimant, i.e. until $E = 2c_1$, from where it is vertical until it reaches the vector of claims. Finally, the path of awards of T follows the 45^o line until claimant 1 obtains half of her claim. Then, it is a vertical line until $E = c_2$, from where it follows the line of slope 1 until it reaches the vector of claims.

2.2 The extended model with baselines

A *problem with baselines*, as introduced by Hougaard et al. (2013a), is a triple (b, c, E) consisting of a *baselines* profile $b \in \mathbb{R}^N_+$, a claims profile $c \in \mathbb{R}^N_+$, and an endowment $E \in \mathbb{R}_+$ such that $C \geq E$. Note that baselines can have multiple interpretations: as minimal rights of the agents, as meaningful upper bounds (such as truncated claims), a mixture of both, or just as exogenous references for the agents. We denote by $\mathbb{B}$ the class of rationing problems with baselines. For each problem with baselines $(b, c, E) \in \mathbb{B}$, let $\min_i(b, c) = \min\{b_i, c_i\}$, for each $i \in N$, and $\min(b, c) = \{\min_i(b, c)\}_{i \in N}$ denote the corresponding (baseline-claim) truncated vector. A *baseline rule* S on $\mathbb{B}$, $S\colon \mathbb{B} \to \mathbb{R}^N$, associates with each problem with baselines $(b, c, E) \in \mathbb{B}$ an allocation $x = S\,(b, c, E)$ for the problem, which satisfies the budget-balance and boundedness conditions. Let $\mathcal{B}$ be the set of baseline rules.

An *extension operator* $O\colon \mathcal{R} \to \mathcal{B}$ associates with each static rule a baseline rule.[5] A focal example is the so-called *composition extension operator* (Hougaard et al., 2013b), which is formally defined as follows:

$$O^c(R)\,(b, c, E) = \begin{cases} R(\min(b, c), E) & \text{if } E \leq \|\min(b, c)\|_1 \\ \min(b, c) + R(c - \min(b, c), E - \|\min(b, c)\|_1) & \text{if } E \geq \|\min(b, c)\|_1. \end{cases} \quad (1)$$

Note that if one considers *endogenous* baselines, as in Hougaard et al. (2013b), then the previous family can lead to specific (and well-known) operators within the space of static rules. For instance, if $b_i(c, E) = \max\{0, E - \sum_{j \neq i} c_i\}$, for each $i \in N$, then the corresponding composition extension operator is the so-called *minimal rights operator* (Thomson and Yeh, 2008). Similarly, if $b_i(c, E) = \min\{c_i, E\}$, for each $i \in N$, then the corresponding composition extension operator is the so-called *claims truncation operator* (Thomson and Yeh, 2008).

[5]The concept of operators on the space of static rules (i.e., associating each static rule to a static rule) was originally introduced by Thomson and Yeh (2008).

2.3 The dynamic model

We consider a fixed-population setting for this dynamic environment. Thus, let N be our population of claimants. At each period of time $t = 1, 2, \ldots$, this population faces a realization of a (static) rationing problem $(c^t, E^t) \in \mathbb{P}$.

For convention, we also consider $(c^0, E^0) = (c^1, C^1)$. For simplicity, we write $(c^\bullet, E^\bullet)$ instead of $\left(((c^t, E^t))_{t\in\mathbb{N}}\right)$.

Rather than solving each problem independently, we aim to consider more general rules that might take into account the outcome of previous periods, while solving the problem at a given one.

More precisely, a *dynamic rule* is a function D that assigns to each $(c^\bullet, E^\bullet) \in \mathbb{P}^\mathbb{N}$ with $(c^0, E^0) = (c^1, C^1)$ a history allocation configuration $D\left(c^\bullet, E^\bullet\right) \in \mathbb{N} \times \mathbb{R}^N_+$ such that

$$0 \leq D^t_i\left(c^\bullet, E^\bullet\right) \leq c^t_i$$

for each $i \in N$ and each $t \in \mathbb{N}$, and

$$\sum_{i\in N} D^t_i\left(c^\bullet, E^\bullet\right) = E^t$$

for each $t \in \mathbb{N}$. Notice that this implies $D^0(c^\bullet, E^\bullet) = c^1$. Let $\mathcal{D}$ denote the set of dynamic rules.

3 Aggregator operators

In this section, we concentrate on the rules emerging after converting a history of allocations into a single allocation, by means of an *aggregator function*.[6]

Given a natural number m, let $M = \{1, \ldots, m\}$. An m-aggregator is a mapping $\rho\colon \mathbb{R}^{N\times M}_+ \to \mathbb{R}^N_+$.

Familiar examples of aggregators are the arithmetic and geometric means in each coordinate, which we denote as μ and γ, respectively, the previous aggregator, which we denote as π, and the rank-order aggregators, which we denote as $\rho^{\prec k}$. Formally, for each $i \in N$,

$$\mu_i(a^1, \ldots, a^m) = \frac{1}{m}\sum_{k=1}^m a^k_i,$$

$$\gamma_i(a^1, \ldots, a^m) = \sqrt[m]{\prod_{k=1}^m a^k_i},$$

[6]This function generalizes to n agents the aggregator function defined by de Clippel et al. (2008).

$$\pi_i(a^1, \ldots, a^m) = a_i^m,$$

and, for each $k \in M$,

$$\rho_i^{\prec k}(a^1, \ldots, a^m) = a_i^{\prec k},$$

where $a_i^{\prec k}$ is the k-th largest value within the set $\{a_i^1, \ldots, a_i^m\}$. In other words, for each $i \in N$ and each $a_i = (a_i^1, \ldots, a_i^m) \in \mathbb{R}_+^m$, we construct $a_i^{\prec} \in \mathbb{R}_+^m$, the vector obtained from a_i by rearranging its coordinates increasingly: $a_i^{\prec 1} \le a_i^{\prec m} \le \cdots \le a_i^{\prec m}$.

We shall sometimes refer to $\rho^{\prec 1}$ and $\rho^{\prec m}$ as the min and max aggregators, respectively.

Combining the aggregator concept with the baselines operators introduced above, we can define a family of operators from the domain of static rules to the domain of dynamic rules. Formally, for each $t \in \mathbb{N}$, let ρ^t be a t-aggregator, i.e., $\rho^t \colon \mathbb{R}_+^{N \times T} \to \mathbb{R}_+^N$, where $T = \{1, \ldots, t\}$. Let $\rho = (\rho^t)_{t \in \mathbb{N}}$. A *$\rho$-extension operator* $O^\rho \colon \mathcal{R} \to \mathcal{D}$ is an operator assigning to each static rule $R \in \mathcal{R}$ a dynamic rule $O^\rho(R) \in \mathcal{D}$ arising from inductively applying an extension operator $O \colon \mathcal{R} \to \mathcal{B}$ to R as follows:

$$O^\rho(R)^1 \left(c^\bullet, E^\bullet\right) = R(c^1, E^1)$$

and

$$O^\rho(R)^t \left(c^\bullet, E^\bullet\right) = O(R) \left(\rho^{t-1} \left(O^\rho(R)^1 \left(c^\bullet, E^\bullet\right), \ldots, O^\rho(R)^{t-1} \left(c^\bullet, E^\bullet\right)\right), c^t, E^t\right).$$

We shall refer to the operators so defined as *aggregator operators.*

The family of aggregator operators is derived from two basic axioms of dynamic rules reflecting the principle of impartiality, a principle with a long tradition in the theory of justice (Moreno-Ternero and Roemer, 2006). The two axioms formalize alternative versions of *Equal Treatment of Equals.* The first one states that sequences that are identical up to a given period yield the same solutions (up to that period). The second one states that sequences with identical solutions up to a given period yield the same solution for the next period.

Equal Treatment of Equal Problem Histories: For each pair $\left(c^\bullet, E^\bullet\right), (\overline{c}^\bullet, \overline{E}^\bullet) \in \mathbb{P}^{\mathbb{N}}$, and each $\hat{t} \in \mathbb{N}$ such that $(c^t, E^t) = (\overline{c}^t, \overline{E}^t)$ for each $t \le \hat{t}$, then

$$D^t \left(c^\bullet, E^\bullet\right) = D^t(\overline{c}^\bullet, \overline{E}^\bullet)$$

for each $t \le \hat{t}$.

Equal Treatment of Equal Solution Histories: For each $\left(c^\bullet, E^\bullet\right), (\overline{c}^\bullet, \overline{E}^\bullet) \in \mathbb{P}^{\mathbb{N}}$ with $D^t(c^\bullet, E^\bullet) = D^t(\overline{c}^\bullet, \overline{E}^\bullet)$ for each $t \le \hat{t}$ and $(c^{\hat{t}}, E^{\hat{t}}) = (\overline{c}^{\hat{t}}, \overline{E}^{\hat{t}})$ for some $\hat{t} \in \mathbb{N}$,

$$D^{\hat{t}} \left(c^\bullet, E^\bullet\right) = D^{\hat{t}}(\overline{c}^\bullet, \overline{E}^\bullet).$$

Theorem 3.1 *A dynamic rule satisfies Equal Treatment of Equal Problem Histories and Equal Treatment of Equal Solution Histories if and only if it is the image of a static rule via a ρ-extension operator for some aggregator ρ.*

Proof. It is straightforward to see that, for any (static) rule R and any aggregator ρ, $O^{\rho}(R)$ satisfies both axioms. Thus, we focus on the converse implication. Let D be a dynamic rule satisfying both axioms. Under Equal Treatment of Equal Problem Histories, given $t \in \mathbb{N}$, $D^t(c^\bullet, E^\bullet)$ does not depend on (c^k, E^k) for each $k > t$ and, under Equal Treatment of Equal Solution Histories, $D^t(c^\bullet, E^\bullet)$ only depends on $D^k(c^\bullet, E^\bullet) = a^k$ for $k < t$ and (c^t, E^t). Hence, $D^t(c^\bullet, E^\bullet)$ can be rewritten as $D^t((a^1, \ldots, a^{t-1}), c^t, E^t)$. Analogously, $O^{\rho}(R)^t(c^\bullet, E^\bullet)$ can be rewritten as $O^{\rho}(R)^t((a^1, \ldots, a^{t-1}), c^t, E^t)$. Thus, we need to prove that

$$D^t\left(\left(a^1, \ldots, a^{t-1}\right), c^t, E^t\right) = O^{\rho}(R)^t\left((a^1, \ldots, a^{t-1}), c^t, E^t\right)$$

for each $t \in \mathbb{N}$ and some appropriate ρ, O, and R. For each $t \in \mathbb{N}$, $t > 0$, let $\psi^t \colon \mathbb{R}^N_+ \setminus \left\{0^N\right\} \to \left(\mathbb{R}^N_+\right)^{t-1}$ be a bijective function, and $\psi^{-t}\colon \left(\mathbb{R}^N_+\right)^{t-1} \to \mathbb{R}^N_+ \setminus \left\{0^N\right\}$ its inverse.

- Given $a = (a^1, \ldots, a^{t-1}) \in \left(\mathbb{R}^N_+\right)^{t-1}$, we define

  $$\rho^{t-1}(a) = \left(\frac{t}{\|\psi^{-t}(a)\|_1} - \frac{1}{\|\psi^{-t}(a)\|_1 + 1}\right) \cdot \psi^{-t}(a)$$

  and let $\rho = (\rho^t)_{t \in \mathbb{N}}$.

- We define O as

  $$O(R')(b, c, E) = D^{\lceil \|b\|_1 \rceil}\left(\psi^{\lceil \|b\|_1 \rceil}\left(\frac{\lceil \|b\|_1 \rceil - \|b\|_1}{(1 - \lceil \|b\|_1 \rceil + \|b\|_1) \cdot \|b\|_1} \cdot b\right), c, E\right)$$

  for each $R' \in \mathcal{R}$ and (b, c, E) problem with baselines with $b \neq 0^N$, and $O(R')(0^N, c, E) = R'(c, E)$.

- We take $R \in \mathcal{R}$ given by $R(c, E) = D^1(c, E)$ for each $(c, E) \in \mathbb{P}$.

We proceed by induction. For $t = 1$, $O^{\rho}(R)^1(c^1, E^1) = O(R)(c^1, E^1) = R(c^1, E^1) = D(c^1, E^1)$. Assume now the result holds for m with $m < t$ and let $a^m = D^m(c^\bullet, E^\bullet)$ for each $m < t$. By induction hypothesis, $a^m = O^{\rho}(R)(\rho(a^1, \ldots, a^{m-1}), c^m, E^m)$ for each $m < t$. Hence, $a^m = D(a^1, \ldots, a^{m-1}, c^m, E^m)$ for each $m < t$. Let $a = (a^1, \ldots, a^{t-1})$. Then,

$$\begin{aligned} O^{\rho}(R)^t\left(\rho(a), c^t, E^t\right) &= O(R)\left(\rho^{t-1}(a), c^t, E^t\right) \\ &= O(R)\left(\left(\frac{t}{\|\psi^{-t}(a)\|_1} - \frac{1}{\|\psi^{-t}(a)\|_1 + 1}\right) \cdot \psi^{-t}(a), c^t, E^t\right). \quad (2) \end{aligned}$$

Let $\chi = \|\psi^{-t}(a)\|_1$, so that (2) can be rewritten as

$$O(R)\left(\left(\frac{t}{\chi}-\frac{1}{\chi+1}\right)\cdot\psi^{-t}(a),c^t,E^t\right)=O(R)\left(\frac{(\chi+1)\cdot t-\chi}{(\chi+1)\cdot\chi}\cdot\psi^{-t}(a),c^t,E^t\right). \quad (3)$$

Let

$$b=\frac{(\chi+1)\cdot t-\chi}{(\chi+1)\cdot\chi}\cdot\psi^{-t}(a).$$

It is not difficult to check that $\|b\|_1 = t - \frac{\chi}{\chi+1}$ and hence $\lceil \|b\|_1 \rceil = t$. By definition of O, (3) equals

$$\begin{aligned}D^t\left(\psi^t\left(\frac{t-\|b\|_1}{(1-t+\|b\|_1)\cdot\|b\|_1}\cdot b\right),c^t,E^t\right)&=D^t\left(\psi^t\left(\frac{\frac{\chi}{\chi+1}}{\left(1-\frac{\chi}{\chi+1}\right)\cdot\left(t-\frac{\chi}{\chi+1}\right)}\cdot b\right),c^t,E^t\right)\\&=D^t\left(\psi^t\left(\frac{(\chi+1)\cdot\chi}{(\chi+1)\cdot t-\chi}\cdot b\right),c^t,E^t\right)\\&=D^t\left(\psi^t\left(\psi^{-t}(a)\right),c^t,E^t\right)=D^t\left(a,c^t,E^t\right).\end{aligned}$$

■

Notice that the aggregator, the operator, and the rule defined in the "only if" part in the proof of Theorem 3.1 are not unique. In fact, the aggregator is technical and not what one would expect for a reasonable element to define any dynamic rule. In particular, in the proof, the operator uses the aggregator as a codex to deduce the complete family of awards in the history, including the number of previous rounds. Yet, this aggregator conveys the idea of a baseline which summarizes previous awards, and the static rule is a way to deal with a problem in the absence of baselines.

4 Composition aggregator operators

A natural family of aggregator operators arises from combining an aggregator with the composition extension operator defined at (1). Formally, for each aggregator ρ, the ρ-*composition operator* $O^{\rho c}\colon \mathcal{R}\to\mathcal{D}$, assigns to each rule $R\in\mathcal{R}$ the dynamic rule arising from applying $O^c(R)$ to each problem with baselines $(b^{\rho c,t-1},c^t,E^t)\in\mathbb{B}$, where $b^{\rho c,0}=c^1$ and

$$b^{\rho c,t-1}=\rho^{t-1}\left(O^c(R)\left(b^{\rho c,0},c^1,E^1\right),\ldots,O^c(R)\left(b^{\rho c,t-2},c^{t-1},E^{t-1}\right)\right)$$

for each $t>1$.

4.1 A numerical example

In order to illustrate the family just singled out, we consider two claimants $N = \{1, 2\}$ who face four rationing problems (at four consecutive points in time) with the following data:

$$(c^2, E^2) = ((20, 20), 20) \qquad (c^4, E^4) = ((15, 15), 5)$$

$$(c^1, E^1) = ((10, 20), 10) \qquad (c^3, E^3) = ((20, 30), 10)$$

We consider four possible aggregators; namely, the arithmetic mean aggregator μ, the previous allocation aggregator π, the min aggregator $\rho^{\prec 1}$ and the max aggregator $\rho^{\prec m}$.

4.1.1 The arithmetic mean aggregator

We start with the arithmetic mean aggregator μ. If the static rule is the proportional rule, then the allocation for the first period will be

$$O^c(P)(b^{\mu c,0}, c^1, E^1) = P(c^1, E^1) = \left(\frac{10}{3}, \frac{20}{3}\right) = b^{\mu c,1}.$$

As $E^2 = 20 \geq \min\{\frac{10}{3}, 20\} + \min\{\frac{20}{3}, 20\} = \min\{b_1^{\mu c,1}, c_1^2\} + \min\{b_2^{\mu c,1}, c_2^2\}$, it follows that

$$O^c(P)(b^{\mu c,1}, c^2, E^2) = \left(\frac{10}{3}, \frac{20}{3}\right) + P\left(\left(\frac{50}{3}, \frac{40}{3}\right), 10\right) = \left(\frac{80}{9}, \frac{100}{9}\right).$$

As $b^{\mu c,2} = \mu\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right)\right) = \left(\frac{55}{9}, \frac{80}{9}\right)$, and $E^3 = 10 \leq \min\{20, \frac{55}{9}\} + \min\{30, \frac{80}{9}\} = \min\{b_1^{\mu c,2}, c_1^3\} + \min\{b_2^{\mu c,2}, c_2^3\}$, it follows that

$$O^c(P)\left(b^{\mu c,2}, c^3, E^3\right) = O^c(P)\left(\left(\frac{55}{9}, \frac{80}{9}\right), c^3, E^3\right) = P\left(\left(\frac{55}{9}, \frac{80}{9}\right), 10\right) = \left(\frac{110}{27}, \frac{160}{27}\right).$$

Thus, $b^{\mu c,3} = \mu\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right), \left(\frac{110}{27}, \frac{160}{27}\right)\right) = \left(\frac{440}{81}, \frac{640}{81}\right)$. As $E^4 = 5 \leq \min\{15, \frac{440}{81}\} + \min\{15, \frac{640}{81}\} = \min\{b_1^{\mu c,3}, c_1^4\} + \min\{b_2^{\mu c,3}, c_2^4\}$, it follows that

$$O^c(P)\left(b^{\mu c,3}, c^4, E^4\right) = O^c(P)\left(\left(\frac{440}{81}, \frac{640}{81}\right), c^4, E^4\right) = P\left(\left(\frac{440}{81}, \frac{640}{81}\right), 5\right) = \left(\frac{55}{27}, \frac{80}{27}\right).$$

If we now replicate the exercise for the constrained equal-awards rule A, the allocation for the first period will be

$$O^c(A)(b^{\mu c,0}, c^1, E^1) = A(c^1, E^1) = (5, 5).$$

As $E^2 = 20 \geq \min\{5, 20\} + \min\{5, 20\} = \min\{b_1^{\mu c,1}, c_1^2\} + \min\{b_2^{\mu c,1}, c_2^2\}$, it follows that

$$O^c(A)(b^{\mu c,1}, c^2, E^2) = (5,5) + A\left((15,15), 10\right) = (10,10).$$

As $b^{\mu c,2} = \mu\left((5,5), (10,10)\right) = \left(\frac{15}{2}, \frac{15}{2}\right)$, and $E^3 = 10 \leq \min\{20, \frac{55}{9}\} + \min\{30, \frac{80}{9}\} = \min\{b_1^{\mu c,2}, c_1^3\} + \min\{b_2^{\mu c,2}, c_2^3\}$, it follows that

$$O^c(A)\left(b^{\mu c,2}, c^3, E^3\right) = O^c(A)\left(\left(\frac{15}{2}, \frac{15}{2}\right), c^3, E^3\right) = A\left(\left(\frac{15}{2}, \frac{15}{2}\right), 10\right) = (5,5).$$

Thus, $b^{\mu c,3} = \mu\left((5,5), (10,10), (5,5)\right) = \left(\frac{20}{3}, \frac{20}{3}\right)$. As $E^4 = 5 \leq \min\{15, \frac{20}{3}\} + \min\{15, \frac{20}{3}\} = \min\{b_1^{\mu c,3}, c_1^4\} + \min\{b_2^{\mu c,3}, c_2^4\}$, it follows that

$$O^c(A)\left(b^{\mu c,3}, c^4, E^4\right) = O^c(A)\left(\left(\frac{20}{3}, \frac{20}{3}\right), c^4, E^4\right) = A\left(\left(\frac{20}{3}, \frac{20}{3}\right), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right).$$

We conclude replicating the exercise for the constrained equal-losses rule, L. The allocation for the first period will be

$$O^c(L)(b^{\mu c,0}, c^1, E^1) = L(c^1, E^1) = (0, 10).$$

As $E^2 = 20 \geq \min\{0, 20\} + \min\{10, 20\} = \min\{b_1^{\mu c,1}, c_1^2\} + \min\{b_2^{\mu c,1}, c_2^2\}$, it follows that

$$O^c(L)(b^{\mu c,1}, c^2, E^2) = (0, 10) + L\left((20, 10), 10\right) = (10, 10).$$

As $b^{\mu c,2} = \mu\left((0,10), (10,10)\right) = (5, 10)$, and $E^3 = 10 \leq \min\{20, 5\} + \min\{30, 10\} = \min\{b_1^{\mu c,2}, c_1^3\} + \min\{b_2^{\mu c,2}, c_2^3\}$, it follows that

$$O^c(L)\left(b^{\mu c,2}, c^3, E^3\right) = O^c(L)\left((5,10), c^3, E^3\right) = L\left((5,10), 10\right) = \left(\frac{5}{2}, \frac{15}{2}\right).$$

Thus, $b^{\mu c,3} = \mu\left((0,10), (10,10), \left(\frac{5}{2}, \frac{15}{2}\right)\right) = \left(\frac{25}{6}, \frac{55}{6}\right)$. As $E^4 = 5 \leq \min\{15, \frac{25}{6}\} + \min\{15, \frac{55}{6}\} = \min\{b_1^{\mu c,3}, c_1^4\} + \min\{b_2^{\mu c,3}, c_2^4\}$, it follows that

$$O^c(L)\left(b^{\mu c,3}, c^4, E^4\right) = O^c(L)\left(\left(\frac{25}{6}, \frac{55}{6}\right), c^4, E^4\right) = L\left(\left(\frac{25}{6}, \frac{55}{6}\right), 5\right) = (0,5).$$

In summary,

μ	(c^1, E^1)	(c^2, E^2)	(c^3, E^3)	(c^4, E^4)
P	$\left(\frac{10}{3}, \frac{20}{3}\right)$	$\left(\frac{80}{9}, \frac{100}{9}\right)$	$\left(\frac{110}{27}, \frac{160}{27}\right)$	$\left(\frac{55}{27}, \frac{80}{27}\right)$
A	$(5,5)$	$(10,10)$	$(5,5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$
L	$(0,10)$	$(10,10)$	$\left(\frac{5}{2}, \frac{15}{2}\right)$	$(0,5)$

4.1.2 The previous aggregator

We now consider the previous aggregator π. If the static rule is the proportional rule, P, then the allocation for the first period will be

$$O^c(P)(b^{\pi c,0}, c^1, E^1) = P(c^1, E^1) = \left(\frac{10}{3}, \frac{20}{3}\right) = b^{\pi c,1}.$$

As $E^2 = 20 \geq \min\{\frac{10}{3}, 20\} + \min\{\frac{20}{3}, 20\} = \min\{b_1^{\pi c,1}, c_1^2\} + \min\{b_2^{\pi c,1}, c_2^2\}$, it follows that

$$O^c(P)(b^{\pi c,1}, c^2, E^2) = \left(\frac{10}{3}, \frac{20}{3}\right) + P\left(\left(\frac{50}{3}, \frac{40}{3}\right), 10\right) = \left(\frac{80}{9}, \frac{100}{9}\right) = b^{\pi c,2}.$$

As $b^{\pi c,2} = \left(\frac{80}{9}, \frac{100}{9}\right)$, and $E^3 = 10 \leq \min\{20, \frac{80}{9}\} + \min\{30, \frac{100}{9}\} = \min\{b_1^{\pi c,2}, c_1^3\} + \min\{b_2^{\pi c,2}, c_2^3\}$, it follows that

$$O^c(P)\left(b^{\pi c,2}, c^3, E^3\right) = O^c(P)\left(\left(\frac{80}{9}, \frac{100}{9}\right), c^3, E^3\right) = P\left(\left(\frac{80}{9}, \frac{100}{9}\right), 10\right) = \left(\frac{40}{9}, \frac{50}{9}\right).$$

Thus, $b^{\pi c,3} = \left(\frac{40}{9}, \frac{50}{9}\right)$. As $E^4 = 5 \leq \min\{15, \frac{40}{9}\} + \min\{15, \frac{50}{9}\} = \min\{b_1^{\pi c,3}, c_1^4\} + \min\{b_2^{\pi c,3}, c_2^4\}$, it follows that

$$O^c(P)\left(b^{\pi c,3}, c^4, E^4\right) = O^c(P)\left(\left(\frac{40}{9}, \frac{50}{9}\right), c^4, E^4\right) = P\left(\left(\frac{40}{9}, \frac{50}{9}\right), 5\right) = \left(\frac{20}{9}, \frac{25}{9}\right).$$

If we now replicate the exercise for the constrained equal-awards rule A, the allocation for the first period will be

$$O^c(A)(b^{\pi c0}, c^1, E^1) = A(c^1, E^1) = (5, 5).$$

As $E^2 = 20 \geq \min\{5, 20\} + \min\{5, 20\} = \min\{b_1^{\pi c,1}, c_1^2\} + \min\{b_2^{\pi c,1}, c_2^2\}$, it follows that

$$O^c(A)(b^{\pi c,1}, c^2, E^2) = (5, 5) + A\left((15, 15), 10\right) = (10, 10) = b^{\pi c,2}.$$

As $E^3 = 10 \leq \min\{20, 10\} + \min\{30, 10\} = \min\{b_1^{\pi c,2}, c_1^3\} + \min\{b_2^{\pi c,2}, c_2^3\}$, it follows that

$$O^c(A)\left(b^{\pi c,2}, c^3, E^3\right) = O^c(A)\left((10, 10), c^3, E^3\right) = A\left((10, 10), 10\right) = (5, 5) = b^{\pi c,3}.$$

As $E^4 = 5 \leq \min\{15, 5\} + \min\{15, 5\} = \min\{b_1^{\pi c,3}, c_1^4\} + \min\{b_2^{\pi c,3}, c_2^4\}$, it follows that

$$O^c(A)\left(b^{\pi c,3}, c^4, E^4\right) = O^c(A)\left((5, 5), c^4, E^4\right) = A\left((5, 5), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right).$$

We conclude replicating the exercise for the constrained equal-losses rule, L. The allocation for the first period will be

$$O^c(L)(b^{\pi c0}, c^1, E^1) = L(c^1, E^1) = (0, 10).$$

As $E^2 = 20 \geq \min\{0, 20\} + \min\{10, 20\} = \min\{b_1^{\pi c,1}, c_1^2\} + \min\{b_2^{\pi c,1}, c_2^2\}$, it follows that

$$O^c(L)(b^{\pi c,1}, c^2, E^2) = (0, 10) + L\left((20, 10), 10\right) = (10, 10) = b^{\pi c,2}.$$

As $E^3 = 10 \leq \min\{20, 10\} + \min\{30, 10\} = \min\{b_1^{\pi c,2}, c_1^3\} + \min\{b_2^{\pi c,2}, c_2^3\}$, it follows that

$$O^c(L)\left(b^{\pi c,2}, c^3, E^3\right) = O^c(L)\left((10, 10), c^3, E^3\right) = L\left((10, 10), 10\right) = (5, 5) = b^{\pi c,3}.$$

As $E^4 = 5 \leq \min\{15, 5\} + \min\{15, 5\} = \min\{b_1^{\pi c,3}, c_1^4\} + \min\{b_2^{\pi c,3}, c_2^4\}$, it follows that

$$O^c(L)\left(b^{\pi c,3}, c^4, E^4\right) = O^c(L)\left((5, 5), c^4, E^4\right) = L\left((5, 5), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right).$$

In summary,

π	(c^1, E^1)	(c^2, E^2)	(c^3, E^3)	(c^4, E^4)
P	$\left(\frac{10}{3}, \frac{20}{3}\right)$	$\left(\frac{80}{9}, \frac{100}{9}\right)$	$\left(\frac{40}{9}, \frac{50}{9}\right)$	$\left(\frac{20}{9}, \frac{25}{9}\right)$
A	$(5, 5)$	$(10, 10)$	$(5, 5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$
L	$(0, 10)$	$(10, 10)$	$(5, 5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$

4.1.3 The min aggregator

We now consider the min aggregator $\rho^{\prec 1}$. If the static rule is the proportional rule, then the allocation for the first period will be

$$O^c(P)(b^{\rho^{\prec 1}c,0}, c^1, E^1) = P(c^1, E^1) = \left(\frac{10}{3}, \frac{20}{3}\right) = b^{\rho^{\prec 1}c,1}.$$

As $E^2 = 20 \geq \min\{\frac{10}{3}, 20\} + \min\{\frac{20}{3}, 20\} = \min\{b_1^{\rho^{\prec 1}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec 1}c,1}, c_2^2\}$, it follows that

$$O^c(P)(b^{\rho^{\prec 1}c,1}, c^2, E^2) = \left(\frac{10}{3}, \frac{20}{3}\right) + P\left(\left(\frac{50}{3}, \frac{40}{3}\right), 10\right) = \left(\frac{80}{9}, \frac{100}{9}\right).$$

As $b^{\rho^{\prec 1}c,2} = \rho^{\prec 1}\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right)\right) = \left(\frac{10}{3}, \frac{20}{3}\right)$, and $E^3 = 10 = \min\{20, \frac{10}{3}\} + \min\{30, \frac{20}{3}\} = \min\{b_1^{\rho^{\prec 1}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec 1}c,2}, c_2^3\}$, it follows that

$$O^c(P)\left(b^{\rho^{\prec 1}c,2}, c^3, E^3\right) = O^c(P)\left(\left(\frac{10}{3}, \frac{20}{3}\right), c^3, E^3\right) = P\left(\left(\frac{10}{3}, \frac{20}{3}\right), 10\right) = \left(\frac{10}{3}, \frac{20}{3}\right).$$

Thus, $b^{\rho^{\prec 1}c,3} = \rho^{\prec 1}\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right), \left(\frac{10}{3}, \frac{20}{3}\right)\right) = \left(\frac{10}{3}, \frac{20}{3}\right)$. As $E^4 = 5 \leq \min\{15, \frac{10}{3}\} + \min\{15, \frac{20}{3}\} = \min\{b_1^{\rho^{\prec 1}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec 1}c,3}, c_2^4\}$, it follows that

$$O^c(P)\left(b^{\rho^{\prec 1}c,3}, c^4, E^4\right) = O^c(P)\left(\left(\frac{10}{3}, \frac{20}{3}\right), c^4, E^4\right) = P\left(\left(\frac{10}{3}, \frac{20}{3}\right), 5\right) = \left(\frac{5}{3}, \frac{10}{3}\right).$$

If we now replicate the exercise for the constrained equal-awards rule A, the allocation for the first period will be

$$O^c(A)(b^{\rho^{\prec 1}c,0}, c^1, E^1) = A(c^1, E^1) = (5, 5)\,.$$

As $E^2 = 20 \geq \min\{5, 20\} + \min\{5, 20\} = \min\{b_1^{\rho^{\prec 1}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec 1}c,1}, c_2^2\}$, it follows that

$$O^c(A)(b^{\rho^{\prec 1}c,1}, c^2, E^2) = (5, 5) + A\left((15, 15), 10\right) = (10, 10)\,.$$

As $b^{\rho^{\prec 1}c,2} = \rho^{\prec 1}\left((5, 5), (10, 10)\right) = (5, 5)$, and $E^3 = 10 = \min\{20, 5\} + \min\{30, 5\} = \min\{b_1^{\rho^{\prec 1}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec 1}c,2}, c_2^3\}$, it follows that

$$O^c(A)\left(b^{\rho^{\prec 1}c,2}, c^3, E^3\right) = O^c(A)\left((5, 5), c^3, E^3\right) = A\left((5, 5), 10\right) = (5, 5)\,.$$

Thus, $b^{\rho^{\prec 1}c,3} = \rho^{\prec 1}\left((5, 5), (10, 10), (5, 5)\right) = (5, 5)$. As $E^4 = 5 \leq \min\{15, 5\} + \min\{15, 5\} = \min\{b_1^{\rho^{\prec 1}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec 1}c,3}, c_2^4\}$, it follows that

$$O^c(A)\left(b^{\rho^{\prec 1}c,3}, c^4, E^4\right) = O^c(A)\left((5, 5), c^4, E^4\right) = A\left((5, 5), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right)\,.$$

We conclude replicating the exercise for the constrained equal-losses rule, L. The allocation for the first period will be

$$O^c(L)(b^{\rho^{\prec 1}c,0}, c^1, E^1) = L(c^1, E^1) = (0, 10)\,.$$

As $E^2 = 20 \geq \min\{0, 20\} + \min\{10, 20\} = \min\{b_1^{\rho^{\prec 1}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec 1}c,1}, c_2^2\}$, it follows that

$$O^c(L)(b^{\rho^{\prec 1}c,1}, c^2, E^2) = (0, 10) + L\left((20, 10), 10\right) = (10, 10)\,.$$

As $b^{\rho^{\prec 1}c,2} = \rho^{\prec 1}\left((0, 10), (10, 10)\right) = (0, 10)$, and $E^3 = 10 = \min\{20, 0\} + \min\{30, 10\} = \min\{b_1^{\rho^{\prec 1}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec 1}c,2}, c_2^3\}$, it follows that

$$O^c(L)\left(b^{\rho^{\prec 1}c,2}, c^3, E^3\right) = O^c(L)\left((0, 10), c^3, E^3\right) = L\left((0, 10), 10\right) = (0, 10)\,.$$

Thus, $b^{\rho^{\prec 1}c,3} = \rho^{\prec 1}\left((0, 10), (10, 10), (0, 10)\right) = (0, 10)$. As $E^4 = 5 \leq \min\{15, 0\} + \min\{15, 10\} = \min\{b_1^{\rho^{\prec 1}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec 1}c,3}, c_2^4\}$, it follows that

$$O^c(L)\left(b^{\rho^{\prec 1}c,3}, c^4, E^4\right) = O^c(L)\left((0, 10), c^4, E^4\right) = L\left((0, 10), 5\right) = (0, 5)\,.$$

In summary,

$\rho^{\prec 1}$	(c^1, E^1)	(c^2, E^2)	(c^3, E^3)	(c^4, E^4)
P	$\left(\frac{10}{3}, \frac{20}{3}\right)$	$\left(\frac{80}{9}, \frac{100}{9}\right)$	$\left(\frac{10}{3}, \frac{20}{3}\right)$	$\left(\frac{5}{3}, \frac{10}{3}\right)$
A	$(5, 5)$	$(10, 10)$	$(5, 5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$
L	$(0, 10)$	$(10, 10)$	$(0, 10)$	$(0, 5)$

4.1.4 The max aggregator

We conclude considering the max aggregator $\rho^{\prec m}$. If the static rule is the proportional rule, then the allocation for the first period will be

$$O^c(P)(b^{\rho^{\prec m}c,0}, c^1, E^1) = P(c^1, E^1) = \left(\frac{10}{3}, \frac{20}{3}\right) = b^{\rho^{\prec m}c,1}.$$

As $E^2 = 20 \geq \min\{\frac{10}{3}, 20\} + \min\{\frac{20}{3}, 20\} = \min\{b_1^{\rho^{\prec m}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec m}c,1}, c_2^2\}$, it follows that

$$O^c(P)(b^{\rho^{\prec m}c,1}, c^2, E^2) = \left(\frac{10}{3}, \frac{20}{3}\right) + P\left(\left(\frac{50}{3}, \frac{40}{3}\right), 10\right) = \left(\frac{80}{9}, \frac{100}{9}\right).$$

As $b^{\rho^{\prec m}c,2} = \rho^{\prec m}\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right)\right) = \left(\frac{80}{9}, \frac{100}{9}\right)$, and $E^3 = 10 \leq \min\{20, \frac{80}{9}\} + \min\{30, \frac{100}{9}\} = \min\{b_1^{\rho^{\prec m}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec m}c,2}, c_2^3\}$, it follows that

$$O^c(P)\left(b^{\rho^{\prec m}c,2}, c^3, E^3\right) = O^c(P)\left(\left(\frac{80}{9}, \frac{100}{9}\right), c^3, E^3\right) = P\left(\left(\frac{80}{9}, \frac{100}{9}\right), 10\right) = \left(\frac{40}{9}, \frac{50}{9}\right).$$

Thus, $b^{\rho^{\prec m}c,3} = \rho^{\prec m}\left(\left(\frac{10}{3}, \frac{20}{3}\right), \left(\frac{80}{9}, \frac{100}{9}\right), \left(\frac{40}{9}, \frac{50}{9}\right)\right) = \left(\frac{80}{9}, \frac{100}{9}\right)$. As $E^4 = 5 \leq \min\{15, \frac{80}{9}\} + \min\{15, \frac{100}{9}\} = \min\{b_1^{\rho^{\prec m}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec m}c,3}, c_2^4\}$, it follows that

$$O^c(P)\left(b^{\rho^{\prec m}c,3}, c^4, E^4\right) = O^c(P)\left(\left(\frac{80}{9}, \frac{100}{9}\right), c^4, E^4\right) = P\left(\left(\frac{80}{9}, \frac{100}{9}\right), 5\right) = \left(\frac{20}{9}, \frac{25}{9}\right).$$

If we now replicate the exercise for the constrained equal-awards rule A, the allocation for the first period will be

$$O^c(A)(b^{\rho^{\prec m}c,0}, c^1, E^1) = A(c^1, E^1) = (5, 5).$$

As $E^2 = 20 \geq \min\{5, 20\} + \min\{5, 20\} = \min\{b_1^{\rho^{\prec m}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec m}c,1}, c_2^2\}$, it follows that

$$O^c(A)(b^{\rho^{\prec m}c,1}, c^2, E^2) = (5, 5) + A\left((15, 15), 10\right) = (10, 10).$$

As $b^{\rho^{\prec m}c,2} = \rho^{\prec m}\left((5, 5), (10, 10)\right) = (10, 10)$, and $E^3 = 10 \leq \min\{20, 10\} + \min\{30, 10\} = \min\{b_1^{\rho^{\prec m}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec m}c,2}, c_2^3\}$, it follows that

$$O^c(A)\left(b^{\rho^{\prec m}c,2}, c^3, E^3\right) = O^c(A)\left((10, 10), c^3, E^3\right) = A\left((10, 10), 10\right) = (5, 5).$$

Thus, $b^{\rho^{\prec m}c,3} = \rho^{\prec m}\left((5, 5), (10, 10), (5, 5)\right) = (10, 10)$. As $E^4 = 5 \leq \min\{15, 10\} + \min\{15, 10\} = \min\{b_1^{\rho^{\prec m}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec m}c,3}, c_2^4\}$, it follows that

$$O^c(A)\left(b^{\rho^{\prec m}c,3}, c^4, E^4\right) = O^c(A)\left((10, 10), c^4, E^4\right) = A\left((10, 10), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right).$$

We conclude replicating the exercise for the constrained equal-losses rule, L. The allocation for the first period will be

$$O^c(L)(b^{\rho^{\prec m}c,0}, c^1, E^1) = L(c^1, E^1) = (0, 10)\,.$$

As $E^2 = 20 \geq \min\{0, 20\} + \min\{10, 20\} = \min\{b_1^{\rho^{\prec m}c,1}, c_1^2\} + \min\{b_2^{\rho^{\prec m}c,1}, c_2^2\}$, it follows that

$$O^c(L)(b^{\rho^{\prec m}c,1}, c^2, E^2) = (0, 10) + L\left((20, 10), 10\right) = (10, 10)\,.$$

As $b^{\rho^{\prec m}c,2} = \rho^{\prec m}\left((0, 10), (10, 10)\right) = (10, 10)$, and $E^3 = 10 \leq \min\{20, 10\} + \min\{30, 10\} = \min\{b_1^{\rho^{\prec m}c,2}, c_1^3\} + \min\{b_2^{\rho^{\prec m}c,2}, c_2^3\}$, it follows that

$$O^c(L)\left(b^{\rho^{\prec m}c,2}, c^3, E^3\right) = O^c(L)\left((10, 10), c^3, E^3\right) = L\left((10, 10), 10\right) = (5, 5)\,.$$

Thus, $b^{\rho^{\prec m}c,3} = \rho^{\prec m}\left((0, 10), (10, 10), (0, 10)\right) = (10, 10)$. As $E^4 = 5 \leq \min\{15, 10\} + \min\{15, 10\} = \min\{b_1^{\rho^{\prec m}c,3}, c_1^4\} + \min\{b_2^{\rho^{\prec m}c,3}, c_2^4\}$, it follows that

$$O^c(L)\left(b^{\rho^{\prec m}c,3}, c^4, E^4\right) = O^c(L)\left((10, 10), c^4, E^4\right) = L\left((10, 10), 5\right) = \left(\frac{5}{2}, \frac{5}{2}\right).$$

In summary,

$\rho^{\prec m}$	(c^1, E^1)	(c^2, E^2)	(c^3, E^3)	(c^4, E^4)
P	$\left(\frac{10}{3}, \frac{20}{3}\right)$	$\left(\frac{80}{9}, \frac{100}{9}\right)$	$\left(\frac{40}{9}, \frac{50}{9}\right)$	$\left(\frac{20}{9}, \frac{25}{9}\right)$
A	$(5, 5)$	$(10, 10)$	$(5, 5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$
L	$(0, 10)$	$(10, 10)$	$(5, 5)$	$\left(\frac{5}{2}, \frac{5}{2}\right)$

4.2 A result

The numbers from the previous example suggest a pattern. If endowments are not increasing, endowments are below n times the minimum claim in the previous period, the static rule is the constrained equal-awards rule, and the aggregator is either the previous, or the min, then equal division of the endowment is persistent in the long run (for the composition extension operator). Formally,

Proposition 4.1 *Let $(c^\bullet, E^\bullet)$ be a sequence of problems such that*

$$E^t \leq \min\{E^{t-1}, n \cdot \min_{i \in N} c_i^{t-1}\},$$

for each $t > 1$. Then, for each $\rho \in \{\pi, \rho^{\prec 1}\}$,

$$O^{\rho c}(A)_i\left(b^{\rho c,t-1}, c^t, E^t\right) = \frac{E^t}{n}$$

for each $i \in N$ and each $t > 1$.

Proof. Let $\rho \in \{\pi, \min\}$. Let $\{(c^t, E^t)\}_{t=1,2,\ldots}$ be a sequence of rationing problems such that

$$E^t \leq E^{t-1} \qquad \text{for each } t > 1 \tag{4}$$

$$E^1 \leq n \cdot \min_{i \in N} c_i^1 \tag{5}$$

$$E^t \leq n \cdot \min_{i \in N} c_i^{t-1} \qquad \text{for each } t > 2. \tag{6}$$

By (5) and (6), $b^{\rho c,1} = A\left(c^1, E^1\right) = \left(\frac{E^1}{n}, \ldots, \frac{E^1}{n}\right) \leq c^2$. By (4), $E^2 \leq E^1 = \sum_{i \in N} \min\{b_i^1, c_i^2\}$ and, thus, the solution in the second period is

$$O^c(A)\left(b^{\rho c,1}, c^2, E^2\right) = A(b^{\rho c,1}, E^2) = b^{\rho c,2} = \left(\frac{E^2}{n}, \ldots, \frac{E^2}{n}\right) \leq c^3.$$

The solution in the third period is

$$O^c(A)\left(\rho(b^{\rho c,1}, b^{\rho c,2}), c^3, E^3\right).$$

It is straightforward to see that $\rho_i(b^{\rho c,1}, b^{\rho c,2}) = \rho_j(b^{\rho c,1}, b^{\rho c,2})$ for each $i, j \in N$. By (4) and (6), it follows that, for each $i \in N$, $\min\{b_i^{\rho c,1}, b_i^{\rho c,2}, c_i^3\} = \min\left\{\frac{E^1}{n}, \frac{E^2}{n}, c_i^3\right\} = \frac{E^2}{n}$. Thus,

$$O^c(A)\left(\rho(b^{\rho c,1}, b^{\rho c,2}), c^3, E^3\right) = A\left(b^{\rho c,2}, E^3\right) = b^{\rho c,3} = \left(\frac{E^3}{n}, \ldots, \frac{E^3}{n}\right) \leq c^4.$$

The proof follows the same process from here. ■

4.3 Inheritance of some basic properties

Suppose a static rule R satisfies a given property (in the benchmark model). Is it the case that $O^\rho(R)$ satisfies the corresponding property in the dynamic model? This question is quite broad. Thus, we provide only some partial answers. To do so, we consider the composition aggregator operators described above, and concentrate first on some standard properties in the static case, introduced next, reflecting ethical or operational principles.

We start with *Equal Treatment of Equals*, a basic requirement of *impartiality*, which requires allotting equal amounts to those agents with equal claims. Formally, a rule R satisfies equal treatment of equals if, for each $(c, E) \in \mathbb{P}$, and each pair $i, j \in N$, we have $R_i(c, E) = R_j(c, E)$, whenever $c_i = c_j$. A strengthening is *Order Preservation in Gains*, which says that agents with larger claims receive larger awards. That is, $c_i \geq c_j$ implies that $R_i(c, E) \geq R_j(c, E)$, for each $(c, E) \in \mathbb{P}$, and each pair $i, j \in N$. Finally, we say that a rule R satisfies *Scale Invariance* when if claims and endowment are multiplied by the same positive number, then so should all awards. Formally, for each $(c, E) \in \mathbb{P}$ and $\lambda \in \mathbb{R}_+$, $R(\lambda c, \lambda E) = \lambda R(c, E)$.

We now need to define the *alter ego* properties in the dynamic setting. We say that a dynamic rule satisfies *Equal Treatment of Equals* if for each sequence of rationing problems $(c^\bullet, E^\bullet) \in \mathbb{P}^\mathbb{N}$, each period $\hat{t}$, and each pair of agents $i, j \in N$ such that $c_i^t = c_j^t$ for each $t \le \hat{t}$, we have $D_i^{\hat{t}}(c^\bullet, E^\bullet) = D_j^{\hat{t}}(c^\bullet, E^\bullet)$.

We say that a dynamic rule satisfies *Order Preservation in Gains* if for each sequence of rationing problems $(c^\bullet, E^\bullet) \in \mathbb{P}^\mathbb{N}$, each period $\hat{t}$, and each pair of agents $i, j \in N$ such that $c_i^t \le c_j^t$ for each $t \le \hat{t}$, we have $D_i^{\hat{t}}(c^\bullet, E^\bullet) \le D_j^{\hat{t}}(c^\bullet, E^\bullet)$.

We say that a dynamic rule satisfies *Scale Invariance* if for each sequence of rationing problems $(c^\bullet, E^\bullet) \in \mathbb{P}^\mathbb{N}$, and each $\lambda > 0$,

$$D^t(\lambda c^\bullet, \lambda E^\bullet) = \lambda D^t(c^\bullet, E^\bullet),$$

for each period t, where $(\lambda c^\bullet, \lambda E^\bullet) := \left(((\lambda c^t, \lambda E^t))_{t \in \mathbb{N}}\right)$.

It is not difficult to show that the previous properties are not preserved in general. One could simply resort to the numerical example at the previous section, but using different aggregators. To avoid those (somewhat pathological) cases, we follow Hougaard et al. (2012) exploring *consequent preservation* instead. More precisely, we say that a property P is *consequently preserved* if, when a rule R satisfies a property P, and the aggregator satisfies the *corresponding* property too, then $O^{\rho c}(R)$ satisfies the *alter ego* property in the dynamic setting.

We need to define the *corresponding* properties for aggregators. Formally, for each $m \in \mathbb{N} \setminus \{0\}$, let $M = \{1, \ldots, m\}$. We say that an aggregator ρ satisfies *Equal Treatment of Equals* if, for each m and each $a = (a^1, \ldots, a^m) \in \mathbb{R}_+^{N \times M}$, and for each pair $i, j \in N$, such that $a_i^k = a_j^k$, for each $k \in M$,

$$\rho_i(a) = \rho_j(a).$$

Similarly, we say that an aggregator ρ satisfies *Order Preservation in Gains* if, for each m and for each $a = (a^1, \ldots, a^m) \in \mathbb{R}_+^{N \times M}$, and for each pair $i, j \in N$, such that $a_i^k \le a_j^k$, for each $k \in M$,

$$\rho_i(a) \le \rho_j(a).$$

Finally, we say that an aggregator ρ satisfies *Scale Invariance* if, for each m and for each $a \in \mathbb{R}_+^{N \times M}$, and for each $\lambda > 0$,

$$\rho(\lambda a) = \lambda \rho(a).$$

In what follows, we restrict ourselves to operators satisfying the following basic requirement. We say that an aggregator ρ is *proper* if

$$\rho(x,\ldots,x)=x$$

for each $x\in\mathbb{R}^N_+$. Note that all of the aggregators presented in Section 3 satisfy this property.

We then have the following result.

Proposition 4.2 *Given a proper aggregator, Equal Treatment of Equals, Order Preservation in Gains, and Scale Invariance are consequently preserved.*

Proof. Let us start with Equal Treatment of Equals. Let R and ρ be a rule and a proper aggregator, respectively, satisfying Equal Treatment of Equals. Let D denote the dynamic rule arising after submitting R to the aggregator operator $O^{\rho c}$, i.e., $D\equiv O^{\rho c}(R)$. Let $(c^\bullet,E^\bullet)\in\mathbb{P}^{\mathbb{N}}$ be given, $i,j\in N$, and $\hat{t}\in\mathbb{N}$ such that $c_i^t=c_j^t$, for each $t\leq\hat{t}$. We prove, by induction, that $D_i^{\hat{t}}(c^\bullet,E^\bullet)=D_j^{\hat{t}}(c^\bullet,E^\bullet)$.

Case $\hat{t}=2$. In this base case, $x^1=D^1(c^\bullet,E^\bullet)=R(c^1,E^1)$, and $x^2=D^2(c^\bullet,E^\bullet)=O^c(R)(b,c^2,E^2)$, where $b=\rho^1(x^1)=x^1$ (the last equality holds because ρ is a proper aggregator). Let $i,j\in N$ be such that $c_i^1=c_j^1$ and $c_i^2=c_j^2$. As R satisfies Equal Treatment of Equals, $b_i=x_i^1=x_j^1=b_j$. Then, by Proposition 2 in Hougaard et al. (2012), $O^c(R)_i(b,c^2,E^2)=O^c(R)_j(b,c^2,E^2)$.

Case $\hat{t}=k$. Suppose, as induction hypothesis, that, for each pair $i,j\in N$, such that $c_i^t=c_j^t$, for each $t\leq k$, then $D_i^k(c^\bullet,E^\bullet)=D_j^k(c^\bullet,E^\bullet)$.

Case $\hat{t}=k+1$. Let $i,j\in N$ be such that $c_i^t=c_j^t$, for each $t\leq\hat{t}$. Then, for $l=i,j$,

$$D_l^{\hat{t}}(c^\bullet,E^\bullet)=O^c(R)_l\left(b^{\hat{t}-1},c^{\hat{t}},E^{\hat{t}}\right),$$

where $b^{\hat{t}-1}=\rho(x^1,\ldots,x^{\hat{t}-1})$, and $x^t=D^t(c^\bullet,E^\bullet)$, for each $t=1,\ldots,\hat{t}-1$. By the induction hypothesis, $x_i^t=x_j^t$, for each $t=1,\ldots\hat{t}-1$. As ρ satisfies Equal Treatment of Equals, $b_i^{\hat{t}-1}=b_j^{\hat{t}-1}$. Finally, by Proposition 2 in Hougaard et al. (2012), as R satisfies Equal Treatment of Equals, $O^c(R)_i(b^{\hat{t}-1},c^{\hat{t}},E^{\hat{t}})=O^c(R)_j(b^{\hat{t}-1},c^{\hat{t}},E^{\hat{t}})$, which concludes the proof.

We now move to Order Preservation in Gains. Let R and ρ be an order-preserving in gains rule and a proper aggregator, respectively. Let D denote the dynamic rule arising after submitting R to the aggregator operator $O^{\rho c}$, i.e., $D\equiv O^{\rho c}(R)$. Let $(c^\bullet,E^\bullet)\in\mathbb{P}^{\mathbb{N}}$ be given, $i,j\in N$, and $\hat{t}\in\mathbb{N}$ such that $c_i^t\leq c_j^t$, for each $t\leq\hat{t}$. We prove, by induction, that $D_i^{\hat{t}}(c^\bullet,E^\bullet)\leq D_j^{\hat{t}}(c^\bullet,E^\bullet)$.

Case $\hat{t} = 2$. In this base case, $x^1 = D^1(c^\bullet, E^\bullet) = R(c^1, E^1)$, and $x^2 = D^2(c^\bullet, E^\bullet) = O^c(R)(b, c^2, E^2)$, where $b = \rho(x^1) = x^1$ (the last equality holds because ρ is a proper aggregator). Let $i, j \in N$ be such that $c_i^1 \leq c_j^1$ and $c_i^2 \leq c_j^2$. As R satisfies Order Preservation in Gains, $b_i = x_i^1 \leq x_j^1 = b_j$. Then, by Proposition 2 in Hougaard et al. (2012), $O^c(R)_i(b, c^2, E^2) \leq O^c(R)_j(b, c^2, E^2)$.

Case $\hat{t} = k$. Suppose, as induction hypothesis, that, for each pair $i, j \in N$, such that $c_i^t \leq c_j^t$ for each $t < \hat{t}$, then $D_i^{\hat{t}}(c^\bullet, E^\bullet) \leq D_j^{\hat{t}}(c^\bullet, E^\bullet)$.

Case $\hat{t} = k+1$. Let $i, j \in N$ be such that $c_i^t \leq c_j^t$, for each $t < \hat{t}$. Then, for for $l = i, j$,

$$D_l^{\hat{t}}(c^\bullet, E^\bullet) = O^c(R)_l\left(b^{\hat{t}-1}, c^{\hat{t}}, E^{\hat{t}}\right),$$

where $b^{\hat{t}-1} = \rho(x^1, \ldots, x^{\hat{t}-1})$, and $x^t = D^t(c^\bullet, E^\bullet)$, for each $t = 1, \ldots, \hat{t}-1$. By the induction hypothesis, $x_i^t \leq x_j^t$, for each $t = 1, \ldots, \hat{t}-1$. As ρ satisfies Order Preservation in Gains, $b_i^{\hat{t}-1} \leq b_j^{\hat{t}-1}$. Then, by Proposition 2 in Hougaard et al. (2012), $O^c(R)_i\left(b^{\hat{t}}, c^{\hat{t}}, E^{\hat{t}}\right) \leq O^c(R)_j(b^{\hat{t}}, c^{\hat{t}}, E^{\hat{t}})$.

Finally, we move to Scale Invariance. Let R and ρ be a scale-invariant rule and a proper aggregator, respectively. Let D denote the dynamic rule arising after submitting R to the aggregator operator $O^{\rho c}$, i.e., $D \equiv O^{\rho c}(R)$. Let $(c^\bullet, E^\bullet) \in \mathbb{P}^{\mathbb{N}}$ be given and let $\lambda > 0$ be given too. Then, for each period $\hat{t}$,

$$D^{\hat{t}}(\lambda c^\bullet, \lambda E^\bullet) = O^c(R)\left(b_\lambda^{\hat{t}-1}, \lambda c^{\hat{t}}, \lambda E^{\hat{t}}\right),$$

where $b_\lambda^{\hat{t}-1} = \rho(x_\lambda^1, \ldots, x_\lambda^{\hat{t}-1})$, and $x_\lambda^t = D^t(\lambda c^\bullet, \lambda E^\bullet)$, for each $t = 1, \ldots, \hat{t}-1$. As ρ satisfies scale invariance, $b_\lambda^{\hat{t}-1} = \lambda b^{\hat{t}-1} = \lambda \rho(x^1, \ldots, x^{\hat{t}-1})$. Thus, by Proposition 2 in Hougaard et al. (2012), as R satisfies Scale Invariance, $O^c(R)(b_\lambda^{\hat{t}}, \lambda c^{\hat{t}}, \lambda E^{\hat{t}}) = \lambda O^c(R)(b^{\hat{t}}, c^{\hat{t}}, E^{\hat{t}})$. ■

The previous proposition does not seem to be generally extended to many other properties. The reason being that most of the standard properties for static rules cannot be adjusted for operators. For instance, a point in case is the property of *Order Preservation in Losses*, which says that agents with larger claims face larger losses. That is, $c_i \geq c_j$ implies that $c_i - R_i(c, E) \geq c_j - R_j(c, E)$, for each $(c, E) \in \mathbb{P}$, and each pair $i, j \in N$.[7] This property, which could indeed be adapted for dynamic rules, relates claims and awards and that is an aspect that could not be transferred to operators. More precisely, the corresponding property of *Order Preservation in Losses* for operators would require to introduce claims as an additional input in the definition of operators. Alternatively, it

[7]The combination of *Order Preservation in Gains* and *Order Preservation in Losses* is usually referred to as *Order Preservation*.

could be defined with respect to a specific claims vector. But each period in a dynamic setting might have a different claims vector. Similar issues would arise, for instance, with many other standard properties such as *Claims Monotonicity* (Thomson, 2003), *Additivity* (Bergantiños and Vidal-Puga, 2004), *Securement* (Moreno-Ternero and Villar, 2004), or *Exemption* (van den Brink et al., 2013), to name just a few.

5 Discussion

We have analyzed in this paper dynamic rationing problems. We have introduced a natural family of operators, which extend rules in the (static) benchmark model into rules able to solve dynamic problems. Each operator is associated to an *aggregator* indicating how to aggregate the solutions from past periods into a *baseline* to be used in order to solve the problem in a given period. We have studied the basic properties of these operators.

We conclude stressing that our model is able to accommodate a variety of realistic situations that cannot be fully addressed with the benchmark (static) model. We referred in the introduction to the case of food rationing in refugee camps. Another instance is the allocation of public resources (collected via taxes by a central government) among the regional governments of a country, with a certain degree of decentralization, when the government approves the budget for the upcoming fiscal year (Chambers and Moreno-Ternero, 2019). University budgeting procedures (Pulido et al., 2002, 2008), some resource allocation procedures in the public health care sector (Daniels, 2016), protocols for the reduction of greenhouse gas emissions (Ju et al., 2019), or the allocation of revenues collected from selling broadcasting rights of (typically, yearly) sports leagues (Bergantiños and Moreno-Ternero, 2019) can also fit this general setting. Further research within the field of operations research will help shed light on some of these problems.

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