Journal of Computational and Applied Mathematics 414 (2022) 114403

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The Gibbs–Wilbraham phenomenon in the approximation of $|x|$ by using Lagrange interpolation on the Chebyshev–Lobatto nodal systems ☆

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ARTICLE INFO

Article history:
Received 23 December 2021
Received in revised form 30 April 2022

MSC:
41A05
65D05
42C05

Keywords:
Lagrange interpolation
Chebyshev nodal systems
Chebyshev–Lobatto nodal systems
Absolute value approximation
Rate of convergence
Gibbs–Wilbraham phenomena

ABSTRACT

Along this study we find and deeply analyze a new Gibbs phenomenon. As far as we know, this type of behavior, in different contexts, is connected with functions having jump discontinuities. In our case it is related to the behavior of the Lagrange interpolation polynomials of the continuous absolute value function. Our study is related to the error of the Lagrange polynomial interpolants of the function $|x|$ on $[-1, 1]$ taking as nodal system the $m+2$ nodes of the extended Chebyshev polynomial of the second kind, obtaining that the error behaves like a function of order $\mathcal{O}(1/m)$. A detailed description and approximation of the function is presented.

1. Introduction

The absolute value is not very distant from smooth functions. A singularity on point 0 is the only difference that can be pointed out. However, this singularity has a lot of consequences through the development of mathematics. A very important one is the fact that it is not a good strategy to approximate $|x|$ through polynomials.

In "Sur la meilleure approximation de $|x|$ par des polynômes de degrés donnés" (see [1]), a famous paper of 1914, S. Bernstein established that the best polynomial approximation of degree m of $|x|$, $E_m(|x|, x)$, has an $\mathcal{O}(\frac{1}{m})$ error. Moreover, Bernstein established superior and lower bounds for the asymptotic constant and gave a method to improve these bounds.

The use of polynomial interpolation in approximation is normally considered a bad idea, but this is a prejudged concept. We only have to consider the work of Trefethen and, in particular, [2,3] to claim this. In general, we can consider its use in approximation as a good idea when we deal with smooth functions. Under these assumptions, the numerical methods are reliable and the rate of approximation is fine (see [2]).

☆ This document is the result of the research project partially funded (first two authors) by Ministerio de Ciencia e Innovación of Spain under grant PID2020-116764RB-I00.

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https://doi.org/10.1016/j.cam.2022.114403

So, we can return to the polynomial approximation of $|x|$. The aim of this paper is to study the polynomial approximation of $|x|$ by using Lagrange interpolation on Chebyshev nodes as a step pace to obtain some evidence related to the polynomial approximation of a quasi-smooth function, that is, a function with some local problem; namely, a function with a derivative of any order in any point apart from x_0 where the derivative of order k does not exist.

More precisely, we assume that the nodal points are the zeros of the Chebyshev polynomial of the second kind joined with the end points ± 1, that is, we take $(1-x^2)U_m(x)$ as nodal polynomial. This nodal system is known as Chebyshev–Lobatto points, Fekete points or Chebyshev extrema points. With this nodal system we consider the Lagrange interpolation polynomial related to $f(x)=|x|$, $\ell_{m+1}(|x|,x)$, and our aim is to estimate the difference $|x|-\ell_{m+1}(|x|,x)$. By using Bernstein's result we know that $|x| = E_{m+1}(|x|,x)+\mathcal{O}(\frac{1}{m+1})$. So, by taking into account the Lebesgue constant for this particular interpolation, $\ell_{m+1}(|x|,x)=E_{m+1}(|x|,x)+\mathcal{O}(\frac{\log(m+1)}{m+1})$. Hence, $|x|-\ell_{m+1}(|x|,x)=\mathcal{O}(\frac{\log(m+1)}{m+1})$ and now we try to improve this primary result.

Thus, to solve our problem we transform it by using the Joukowsky transformation $x=\dfrac{z+\frac{1}{z}}{2}$. Although the original problem (on the bounded interval) and the transformed problem (on the unit circle) are similar, we think that its manipulation is more simple if we work in the circle than in the interval. This idea due to Szegő, see [4], has given to us good results in the study of interpolation problems (see [5,6]). The transformed nodal system is constituted by the $2n=2m+2$ roots of the unity and the new function is $F(z)=f(x)=\left|\dfrac{z+\frac{1}{z}}{2}\right|$. We solve the new transformed problem in the space of the Laurent polynomials by obtaining the Lagrange interpolation polynomial on the unit circle, $\mathcal{L}_{-n,n-1}(F,z)$, and by computing the difference $F(z)-\mathcal{L}_{-n,n-1}(F,z)$. In the end we translate the corresponding results to the bounded interval to obtain our main result. Meanwhile, we develop the theory in this particular case and we use techniques that we think that are valid for the different cases.

The more relevant results that we present are

1. The rate of convergence of the approximants $\ell_{m+1}(|x|,x)$. We have seen that the error is $\mathcal{O}(\frac{\log m}{m})$ and we improve this result to $\mathcal{O}(\frac{1}{m})$.
2. A new Gibbs–Wilbraham phenomenon, that we study in detail. Indeed, we establish where and when the phenomenon occurs and give an accuracy approximation. We must point out that Gibbs–Wilbraham phenomena have been widely studied (see [7–14], and [2] among others) but all the studies, in different contexts, refer to functions with jump discontinuities.

Notice that our aim is not to correct the Gibbs phenomenon. Nevertheless several authors have discussed methods to correct this phenomenon in relation with the polynomial interpolation. Among others we highlight these two recent papers [15,16]. A good presentation of Gibbs phenomenon appearing in different applications is given in [17].

The article is structured in seven different sections. After the introduction, the second section is devoted to pose the problem to be studied and to recall the well-known related results. In order to make the paper more readable we include an auxiliary section, titled Technical results, in which we develop trigonometrical formulas that we need for our main results. In Section 4 we present the central results of the article concerning the behavior of the scaled errors for the auxiliary problem on the unit circle and for the original problem on the bounded interval. We remark that we have managed to describe a new Gibbs phenomenon. Section 5 is devoted to the development of the same problem using the Chebyshev system. In Section 6 we present some numerical experiments and finally, the last section summarizes the conclusions.

2. The problem and some background

Among the functions with singularities in a particular point, the usual absolute value $f(x)=|x|$ defined on $[-1,1]$ plays an important role. Since we want to estimate the difference between $|x|$ and its Lagrange polynomial interpolation approximants, we interpolate $|x|$ on $[-1,1]$ by using as nodal system the most frequently selected one, that is, the Chebyshev–Lobatto nodes. Besides, as the singularity is in 0 we choose the nodal polynomial $(1-x^2)U_m(x)$ with m odd, in such a way that 0, the singularity, is a nodal point.

We denote the nodal points as $\{x_j\}_{j=1}^m \cup \{-1,1\}$. Recall that the Lagrange interpolation polynomial related to the function $|x|$ based on the previous nodal system is a polynomial $\ell_{m+1}(|x|,x)\in\mathbb{P}_{m+1}[x]$ characterized by satisfying

$$\ell_{m+1}(|x|,x_j)=|x_j| \text{ for } j=1,\ldots,m \text{ and } \ell_{m+1}(|x|,\pm 1)=1. \tag{1}$$

It is well known that the nodal systems constituted by the zeros of the Chebyshev polynomials of the first kind and the extended nodal systems based upon the zeros of the Chebyshev polynomials of the second, third and fourth kinds are all well related to equispaced nodal systems on the unit circle $\mathbb{T}=\{z\in\mathbb{C}: |z|=1\}$ (see [18] or [19]). Indeed, through the Joukowsky transformation $x=\dfrac{z+\frac{1}{z}}{2}$ our nodal system $\{x_j\}_{j=1}^m\cup\{-1,1\}$ is translated to a nodal system on $\mathbb{T}$ constituted by $2m+2=2n$ roots of the unity, with n even, $\{\alpha_j\}_{j=0}^{2n-1}$ (see Fig. 1).

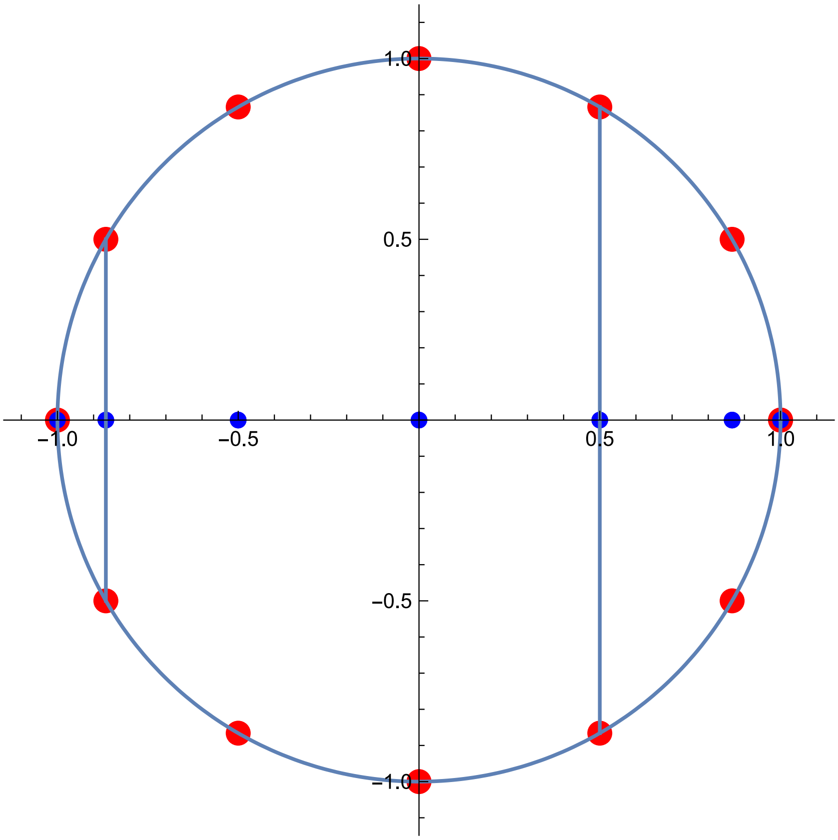

Fig. 1. Roots of $U_5(x)$ plus ± 1 (blue points) and their images (red points) through the Joukowsky transformation (12 points, $2 \times 5 + 2$).

Using the Joukowsky transformation again, $|x|$ can be translated to a function $F(z)$ defined on $\mathbb{T}$ by $F(z) = \left|\frac{z+\frac{1}{z}}{2}\right|$. Also, we can consider $z = e^{i\theta}$ and $F(e^{i\theta})$, which leads us to the function $G(\theta) = F(e^{i\theta})$. With these representations the singularity of $f(x)$ on 0 is translated into singularities on $\pm i$ (being z the variable) and $\pm\frac{\pi}{2}$ (being θ the variable).

To solve our problem our strategy is to study the transformed problem. In order to do this, we have to interpolate $F(z)$ with the nodal system $\{\alpha_j\}_{j=0}^{2n-1}$.

It is well known that polynomial interpolation on the space of algebraic polynomials $\mathbb{P}[z]$ is not a good idea. Notice that $\mathbb{P}[z]$ is not dense on the space of continuous functions on $\mathbb{T}$. The usual solution is that of employing subspaces of the space of Laurent polynomials $\Lambda[z]$ defined as $\mathbb{P}[z] \oplus \mathbb{P}[\frac{1}{z}]$. In particular, when we deal with Lagrange interpolation and nodal systems with $2n$ points, we use the subspace $\Lambda_{-n,n-1}[z] = \mathbb{P}_{n-1}[z] \oplus \mathbb{P}_n[\frac{1}{z}]$.

If we consider the equispaced nodal system given by the $2n$-th roots of 1, $\{\alpha_j\}_{j=0}^{2n-1}$, and we also consider the function $F(z)$ defined on $\mathbb{T}$, recall that the Lagrange Laurent interpolation polynomial of $F(z)$ on the nodal system $\{\alpha_j\}_{j=0}^{2n-1}$, $\mathcal{L}_{-n,n-1}(F,z) \in \Lambda_{-n,n-1}[z]$, is characterized by satisfying

$$\mathcal{L}_{-n,n-1}(F,\alpha_j) = F(\alpha_j) \text{ for } j = 0,\ldots,2n-1. \tag{2}$$

The previous problem is a very well-known one and in [9] we have given the expressions for $\mathcal{L}_{-n,n-1}(F,z)$ in a more general situation. We recall them, giving the particularized version for our nodal system of $2n$ points.

(1) The Laurent polynomial $\mathcal{L}_{-n,n-1}(F,z)$ satisfying (2) exists, is unique and has the following expression

$$\mathcal{L}_{-n,n-1}(F,z) = \frac{1}{z^n}\sum_{j=0}^{2n-1}\frac{z^{2n}-1}{2n\,\alpha_j^{2n-1}(z-\alpha_j)}\alpha_j^n F(\alpha_j) = \frac{z^{2n}-1}{2n\,z^n}\sum_{j=0}^{2n-1}\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}F(\alpha_j). \tag{3}$$

(2) To evaluate $\mathcal{L}_{-n,n-1}(F,z)$ we use the so-called barycentric formula of type II, which has the form

$$\mathcal{L}_{-n,n-1}(F,z)=\frac{\sum_{j=0}^{2n-1}\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}F(\alpha_j)}{\sum_{j=0}^{2n-1}\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}}. \tag{4}$$

It is well known that the barycentric formula is easy to use and numerically stable in the sense of [20].

Due to the relationship between the nodal systems on $[-1,1]$ and the nodal systems on $\mathbb{T}$, there exists a close relationship between the interpolation polynomials related to a function defined on $[-1,1]$ and the Laurent interpolation polynomials related to its transformed function through the Joukowsky transformation (see Section 4.2 for details). Hence, we can study the error of the polynomial interpolators of $f(x)=|x|$ by studying the error of the Laurent polynomial interpolators of $F(z)=\left|\frac{z+\frac{1}{z}}{2}\right|$.

2.1. The transformed problem

Since i is a node of the complex nodal system, for simplicity we consider that the nodal system is clockwise ordered and beginning at $i=\alpha_0$ (see Fig. 2).

Any point z on the right-hand arc of $\mathbb{T}$ ($\Re(z)\geq 0$ case) can be described as $z=ie^{-i\frac{2\pi d}{2n}}=ie^{-i\frac{\pi d}{n}}$, where $\frac{\pi d}{n}$ is the shortest arc between i and z. Thus, the nodes can be written as $\alpha_\ell=ie^{-i\frac{2\pi\ell}{2n}}=ie^{-i\frac{\pi\ell}{n}}$ and $\alpha_{2n-\ell}=ie^{i\frac{2\pi\ell}{2n}}=ie^{i\frac{\pi\ell}{n}}$. So, our aim is to obtain a detailed description of the error $\mathcal{E}(F,z)=F(z)-\mathcal{L}_{-n,n-1}(F,z)$ when $\Re(z)\geq 0$. The description will also cover, as we will prove, the case $\Re(z)<0$.

Thinking that Bernstein's result can be translated to the problem on the unit circle and that in consequence $\mathcal{E}(F,z)$ uniformly converges to 0 at least as an $\mathcal{O}\left(\frac{\log n}{n}\right)$ on $\mathbb{T}$, we propose to study $2n\mathcal{E}(F,z)$.

It is clear that $F(z)=|\frac{z+\frac{1}{z}}{2}|$ can be defined on $\mathbb{T}$ in terms of two Laurent polynomials as

$$F(z)=\begin{cases} F_1(z)=\frac{z+\frac{1}{z}}{2} & \text{if} \quad z=e^{i\theta} \text{ with } -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}, \\ F_2(z)=-\frac{z+\frac{1}{z}}{2} & \text{if} \quad z=e^{i\theta} \text{ with } \frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}. \end{cases}$$

As $F_1(z)$ and $F_2(z)$ are Laurent polynomials of low degree on $\mathbb{T}$, we have that $\mathcal{L}_{-n,n-1}(F_1,z)=F_1(z)$ on $\mathbb{T}$ and $\mathcal{L}_{-n,n-1}(F_2,z)=F_2(z)$ on $\mathbb{T}$.

So, if we consider A the arc of $\mathbb{T}$ with $\Re(z)\geq 0$ we have, by using (3),

$$\mathcal{E}(F,z)=F(z)-\sum_{j=0}^{2n-1}F(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}=$$

$$F_1(z)-\sum_{\alpha_j\in A}F_1(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}-\sum_{\alpha_j\in\bar{A}}F_2(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}=$$

$$F_1(z)-\sum_{\alpha_j\in\mathbb{T}}F_1(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}+$$

$$\sum_{\alpha_j\in\bar{A}}F_1(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}-\sum_{\alpha_j\in\bar{A}}F_2(\alpha_j)\frac{1}{z^n}\frac{z^{2n}-1}{2n\alpha_j^{n-1}(z-\alpha_j)}=$$

$$-2\sum_{j=n+1}^{2n-1}F_2(\alpha_j)\frac{z^{2n}-1}{z^n 2n}\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}=-2\sum_{j=n+1}^{2n-1}F(\alpha_j)\frac{z^{2n}-1}{z^n 2n}\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}.$$

Hence, we have

$$2n\mathcal{E}(F,z)=-4n\frac{z^{2n}-1}{2n\,z^n}\sum_{j=n+1}^{2n-1}F(\alpha_j)\frac{1}{\alpha_j^{n-1}(z-\alpha_j)}=$$

$$-2\frac{z^{2n}-1}{i^n z^n}\sum_{j=n+1}^{2n-1}F(\alpha_j)\frac{i^n}{\alpha_j^{n-1}(z-\alpha_j)}=\underbrace{-2\frac{z^{2n}-1}{i^n z^n}}_{*}\sum_{\ell=1}^{n-1}\underbrace{F(\alpha_{2n-\ell})\frac{i^n}{\alpha_{2n-\ell}^{n-1}(z-\alpha_{2n-\ell})}}_{**}.$$

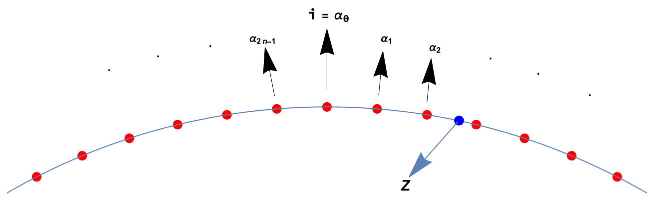

Fig. 2. $2n$ equispaced node system on $\mathbb{T}$ including i and an arbitrary point $z\in\mathbb{T}$.

It is easy to obtain a nice expression for $*$ with the representation $z=ie^{-\frac{i\pi d}{n}}$ and by taking into account that n is even, we get

$$*=-2\frac{z^{2n}-1}{i^nz^n}=-2\left(\frac{z^n}{i^n}-\frac{i^n}{z^n}\right)=$$

$$-2\left(\frac{(ie^{-\frac{i\pi d}{n}})^n}{i^n}-\frac{i^n}{(ie^{-\frac{i\pi d}{n}})^n}\right)=-2\left(e^{-i\pi d}-\frac{1}{e^{-i\pi d}}\right)=4i\sin d\pi\,.$$

In a similar way, we can obtain a convenient representation for expression $**$

$$F(\alpha_{2n-\ell})=-\frac{\alpha_{2n-\ell}+\frac{1}{\alpha_{2n-\ell}}}{2}=-\frac{ie^{i\frac{\pi\ell}{n}}+\frac{1}{ie^{i\frac{\pi\ell}{n}}}}{2}=-\Re(ie^{i\frac{\pi\ell}{n}})=\sin\frac{\ell\pi}{n},$$

and

$$\frac{i^n}{\alpha_{2n-\ell}^{n-1}(z-\alpha_{2n-\ell})}=\frac{i^n}{\alpha_{2n-\ell}^n}\frac{\alpha_{2n-\ell}}{z-\alpha_{2n-\ell}}=\frac{i^n}{i^n\left(e^{i\frac{\pi\ell}{n}}\right)^n}\frac{\alpha_{2n-\ell}}{z-\alpha_{2n-\ell}}=\frac{1}{(e^{i\pi})^\ell}\frac{\alpha_{2n-\ell}}{z-\alpha_{2n-\ell}}=$$

$$(-1)^\ell\frac{\alpha_{2n-\ell}}{z-\alpha_{2n-\ell}}=\frac{(-1)^\ell}{\frac{z}{\alpha_{2n-\ell}}-1}=(-1)^\ell\frac{1}{\frac{ie^{-i\frac{\pi d}{n}}}{ie^{i\frac{\pi\ell}{n}}}-1}=(-1)^\ell\frac{1}{e^{-i\frac{\pi(d+\ell)}{n}}-1}.$$

Hence, we have

$$2n\mathcal{E}(F,z)=4i\sin d\pi\sum_{\ell=1}^{n-1}(-1)^\ell\frac{1}{e^{-i\frac{\pi(d+\ell)}{n}}-1}\sin\frac{\ell\pi}{n}. \tag{5}$$

Proposition 1. *For $z=ie^{-\frac{i\pi d}{n}}$, the imaginary and real parts of $2n\mathcal{E}(F,z)$ are given by*

$$\Im\left(2n\mathcal{E}(F,z)\right)=-2\sin d\pi\sum_{\ell=1}^{n-1}(-1)^\ell\sin\frac{\ell\pi}{n}, \tag{6}$$

$$\Re\left(2n\mathcal{E}(F,z)\right)=-4\sin d\pi\sum_{\ell=1}^{n-1}\frac{(-1)^\ell\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}\sin\frac{\ell\pi}{n}. \tag{7}$$

Proof. By taking into consideration (5), we only have to compute $\frac{1}{e^{-i\frac{\pi(d+\ell)}{n}}-1}$. The expression $\frac{1}{e^{-ix}-1}$ can be transformed as

$$\frac{1}{e^{-ix}-1}=\frac{1}{\cos x-i\sin x-1}=\frac{\cos x-1+i\sin x}{(\cos x-1)^2+\sin^2x}=$$

$$\frac{\cos x-1+i\sin x}{2(1-\cos x)}=-\frac{1}{2}+i\frac{\sin x}{4\sin^2\frac{x}{2}}=-\frac{1}{2}+i\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{4\sin^2\frac{x}{2}}=-\frac{1}{2}+i\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Thus,

$$\frac{1}{e^{-i\frac{\pi(d+\ell)}{n}}-1}=-\frac{1}{2}+i\frac{\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}} \tag{8}$$

and (6) and (7) are straightforward consequences from (5) and (8). □

Other interesting properties of $\mathcal{L}_{-n,n-1}(F,z)$ are gathered in the next proposition.

Proposition 2. *In our conditions for $z \in \mathbb{T}$ it holds*

(i) $\mathcal{L}_{-(n-1),n}(F,z) = \mathcal{L}_{-n,n-1}(F,\frac{1}{z}) = \overline{\mathcal{L}_{-n,n-1}(F,z)}$.
(ii) $\mathcal{L}_{-n,n-1}(F,-z) = \mathcal{L}_{-n,n-1}(F,z)$.

Proof. (i) If we consider that the last two polynomials in (i) are elements of $\Lambda_{-(n-1),n}[z]$ and that they satisfy the same interpolation conditions as $\mathcal{L}_{-(n-1),n}(F,z)$, that is,

$$\mathcal{L}_{-n,n-1}\left(F,\frac{1}{z}\right)\Big|_{z=\alpha_j} = \mathcal{L}_{-n,n-1}\left(F,z\right)\big|_{z=\frac{1}{\alpha_j}} = \left|\frac{\alpha_j+\frac{1}{\alpha_j}}{2}\right|,$$
$$\overline{\mathcal{L}_{-n,n-1}(F,z)}\big|_{z=\alpha_j} = |\frac{\alpha_j+\frac{1}{\alpha_j}}{2}|,$$

we conclude that they are equal to $\mathcal{L}_{-(n-1),n}(F,z)$. As a consequence, we have that the coefficients of these three polynomials are real.

(ii) It can be obtained by using the same arguments as in (i). □

Next, our aim is to study the asymptotic behavior of (6) and (7) for which we need to present some auxiliary results. Taking into account Proposition 2 we only need to study the case $z = ie^{-\frac{i\pi d}{n}}$ with $0 \le d \le \frac{n}{2}$, that is, z with $\Re(z) \ge 0$ and $\Im(z) \ge 0$.

We gather the auxiliary results in the next section.

3. Technical results

This section is devoted to obtain some useful formulas in order to get suitable expressions for (6) and (7). In general, they are based on well-known trigonometric equalities. We are conscious that this section can be tedious, although the proofs are not difficult and we have tried to simplify them. In Lemma 1 we obtain the expression (9) that allows us to simplify the expression (6). In Lemmas 2 and 3 we obtain expressions (10), (11), (12), (13), (15) and (16) that allow us to rewrite (7) in a more manageable way. In the next lemmas (Lemmas 4, 5, 6, 7 and 8) we present some intermediary results to approximate the expression (12). Finally, in the last Lemma 9 we present the properties of a special function that we are going to use in the next section.

Lemma 1. *If n is even, then it holds that*

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{\ell\pi}{n} = -\tan\frac{\pi}{2n}. \tag{9}$$

Proof. Taking into account that $\sin\frac{\ell\pi}{n}$ has the representation $\sin\frac{\ell\pi}{n} = \frac{1}{2i}\left(e^{\frac{i\pi\ell}{n}} - e^{-\frac{i\pi\ell}{n}}\right)$, we can write

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{\ell\pi}{n} = \frac{1}{2i}\left(\sum_{\ell=1}^{n-1}(-1)^{\ell}e^{\frac{i\pi\ell}{n}}\right) - \frac{1}{2i}\left(\sum_{\ell=1}^{n-1}(-1)^{\ell}e^{\frac{-i\pi\ell}{n}}\right),$$

that is, the sum of two partial sums of different geometric progressions. Moreover, these sums are conjugates each other. After obtaining the sums and after simplifying the corresponding expressions, we get

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{\ell\pi}{n} = \frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\frac{(-1+2n^2)\pi}{2n} + \sin\frac{(1-2n)\pi}{2n}\right).$$

We have also checked the last expression with a symbolic calculator. Using the fact that n is even, we obtain

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{\ell\pi}{n} = \frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\left(\frac{-\pi}{2n}+n\pi\right) + \sin\left(\frac{\pi}{2n}-\pi\right)\right) =$$
$$\frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\frac{-\pi}{2n} - \sin\frac{\pi}{2n}\right) = \frac{1}{2}\sec\frac{\pi}{2n}(-2\sin\frac{\pi}{2n}) = -\tan\frac{\pi}{2n}.$$

Therefore, (9) is proved. □

Lemma 2. *It holds that*

$$\sum_{\ell=1}^{n-1}(-1)^\ell \frac{\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}\sin\frac{\ell\pi}{n} = P_{1,n}(d)+P_{2,n}(d)+P_{3,n}(d), \tag{10}$$

with

$$P_{1,n}(d) = \cos\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^\ell\cos^2\frac{(d+\ell)\pi}{2n}, \tag{11}$$

$$P_{2,n}(d) = -\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}\frac{(-1)^\ell\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}} = -\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^\ell\cot\frac{(d+\ell)\pi}{2n}, \tag{12}$$

$$P_{3,n}(d) = \frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^\ell\sin\frac{(d+\ell)\pi}{n}. \tag{13}$$

Proof. First, we prove

$$\sin\frac{\ell\pi}{n} = 2\sin\frac{(d+\ell)\pi}{2n}\cos\frac{(d+\ell)\pi}{2n}\cos\frac{d\pi}{n} - \sin\frac{d\pi}{n} + 2\sin^2\frac{(d+\ell)\pi}{2n}\sin\frac{d\pi}{n}. \tag{14}$$

Indeed

$$\sin\frac{\ell\pi}{n} = \sin\left(\frac{(d+\ell)\pi}{n}-\frac{d\pi}{n}\right) = \sin\frac{(d+\ell)\pi}{n}\cos\frac{-d\pi}{n} + \cos\frac{(d+\ell)\pi}{n}\sin\frac{-d\pi}{n} =$$
$$2\sin\frac{(d+\ell)\pi}{2n}\cos\frac{(d+\ell)\pi}{2n}\cos\frac{d\pi}{n} - \left(\cos^2\frac{(d+\ell)\pi}{2n} - \sin^2\frac{(d+\ell)\pi}{2n}\right)\sin\frac{d\pi}{n} =$$
$$2\sin\frac{(d+\ell)\pi}{2n}\cos\frac{(d+\ell)\pi}{2n}\cos\frac{d\pi}{n} - \left(1-2\sin^2\frac{(d+\ell)\pi}{2n}\right)\sin\frac{d\pi}{n}.$$

Hence, we can obtain the decomposition (10) taking into account (14). Then

$$P_{1,n}(d) = \sum_{\ell=1}^{n-1}\frac{(-1)^\ell\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}2\sin\frac{(d+\ell)\pi}{2n}\cos\frac{(d+\ell)\pi}{2n}\cos\frac{d\pi}{n} =$$
$$\cos\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^\ell\cos^2\frac{(d+\ell)\pi}{2n},$$

$$P_{2,n}(d) = -\left(\sum_{\ell=1}^{n-1}\frac{(-1)^\ell\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}\right)\sin\frac{d\pi}{n},$$

and

$$P_{3,n}(d) = \sum_{\ell=1}^{n-1}\frac{(-1)^\ell\cos\frac{(d+\ell)\pi}{2n}2\sin^2\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}\sin\frac{d\pi}{n} = \frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^\ell\sin\frac{(d+\ell)\pi}{n}. \quad \square$$

In the next lemma we obtain expressions that enable us to simplify the formulas (11) and (13).

Lemma 3. *If n is even, then it holds*

$$\sum_{\ell=1}^{n-1}(-1)^\ell\cos^2\frac{(d+\ell)\pi}{2n} = -\frac{1}{2}+\frac{1}{2}\sec\frac{\pi}{2n}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}, \tag{15}$$

and

$$\sum_{\ell=1}^{n-1}(-1)^\ell\sin\frac{(d+\ell)\pi}{n} = -\sec\frac{\pi}{2n}\cos\frac{d\pi}{n}\sin\frac{\pi}{2n}. \tag{16}$$

Proof. To obtain these sums we recall that $\cos\frac{(d+\ell)\pi}{2n} = \frac{1}{2}(e^{i\frac{(d+l)\pi}{2n}} + e^{-i\frac{(d+l)\pi}{2n}})$ and, in consequence,

$$(-1)^\ell\cos^2\frac{(d+\ell)\pi}{2n} = (-1)^\ell\frac{1}{4}\left(e^{i\frac{(d+l)\pi}{n}} + 2 + e^{-i\frac{(d+l)\pi}{n}}\right) =$$
$$\frac{1}{4}\left((-1)^\ell e^{i\frac{(d+l)\pi}{n}} + (-1)^\ell 2 + (-1)^\ell e^{-i\frac{(d+l)\pi}{n}}\right).$$

So, the first expression can be computed as the sum of three partial sums of three different geometric progressions. Using these facts and after some simplifications (we omit the intermediate steps for both expressions but we have checked the results with Mathematica®), we have for the first expression (remember that n is even):

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\cos^2\frac{(d+\ell)\pi}{2n}=$$

$$\frac{1}{4}\left(-1+(-1)^{n+1}-\sec\frac{\pi}{2n}\left(\cos\frac{(1+2d)\pi}{2n}+\cos\frac{(-1+2d+2n+2n^2)\pi}{2n}\right)\right)$$

$$\frac{1}{4}\left(-1-1-\sec\frac{\pi}{2n}\left(\cos\frac{(1+2d)\pi}{2n}+\cos\left(\frac{(-1+2d)\pi}{2n}+\pi+n\pi\right)\right)\right)=$$

$$\frac{1}{4}\left(-2-\sec\frac{\pi}{2n}\left(\cos\frac{(1+2d)\pi}{2n}-\cos\left(\frac{(-1+2d)\pi}{2n}\right)\right)\right)=$$

$$\frac{1}{4}\left(-2-\sec\frac{\pi}{2n}\left(-2\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\right)\right)=-\frac{1}{2}+\frac{1}{2}\sec\frac{\pi}{2n}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}.$$

On the other hand, $\sin\frac{(d+\ell)\pi}{n}=\frac{1}{2i}(e^{i\frac{(d+l)\pi}{n}}-e^{-i\frac{(d+l)\pi}{n}})$. Hence, the second expression can be computed as the sum of two partial sums of two different geometric progressions (actually they are conjugates). With an analogous technique we have for this expression

$$\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{(d+\ell)\pi}{n}=\frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\frac{(-1+2d+2n^2)\pi}{2n}+\sin\frac{(1+2d-2n)\pi}{2n}\right)=$$

$$\frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\left(\frac{(-1+2d)\pi}{2n}+n\pi\right)+\sin\left(\frac{(1+2d)\pi}{2n}-\pi\right)\right)=$$

$$=\frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\frac{(-1+2d)\pi}{2n}-\sin\frac{(1+2d)\pi}{2n}\right)=$$

$$\frac{1}{2}\sec\frac{\pi}{2n}2\cos\frac{d\pi}{n}\sin\frac{-\pi}{2n}=-\sec\frac{\pi}{2n}\cos\frac{d\pi}{n}\sin\frac{\pi}{2n}.$$

Then, (15) and (16) are proved. □

Next, we present some intermediary results to approximate the expression of $P_{2,n}(d)$ given in (12).

Lemma 4. *We have*

(i)

$$-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+2)\pi}{2n}=-\frac{\sin\frac{\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}}. \tag{17}$$

(ii)

$$-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+3)\pi}{2n}=-\frac{\sin\frac{2\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+3)\pi}{2n}}. \tag{18}$$

(iii) Moreover, if $\sqrt{n}\le d\le\frac{n}{2}$ then

$$\frac{-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+2)\pi}{2n}}{-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+3)\pi}{2n}}=\frac{1}{2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right). \tag{19}$$

Proof. (i) In order to obtain (17), we proceed with the sum and we have

$$-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+2)\pi}{2n}=-\frac{\cos\frac{(d+\ell+1)\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}}+\frac{\cos\frac{(d+\ell+2)\pi}{2n}}{\sin\frac{(d+\ell+2)\pi}{2n}}=$$

$$\frac{-\cos\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}+\cos\frac{(d+\ell+2)\pi}{2n}\sin\frac{(d+\ell+1)\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}}=$$

$$\frac{\sin\left(\frac{(d+\ell+1)\pi}{2n}-\frac{(d+\ell+2)\pi}{2n}\right)}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}}=\frac{-\sin\frac{\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}}.$$

(ii) Clearly (18) can be obtained as (17).

(iii) Finally, using (17) and (18) we can write

$$\frac{-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+2)\pi}{2n}}{-\cot\frac{(d+\ell+1)\pi}{2n}+\cot\frac{(d+\ell+3)\pi}{2n}}=$$

$$\frac{-\frac{\sin\frac{\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+2)\pi}{2n}}}{-\frac{\sin\frac{2\pi}{2n}}{\sin\frac{(d+\ell+1)\pi}{2n}\sin\frac{(d+\ell+3)\pi}{2n}}}=\frac{\sin\frac{\pi}{2n}}{\sin\frac{2\pi}{2n}}\frac{\sin\frac{(d+\ell+3)\pi}{2n}}{\sin\frac{(d+\ell+2)\pi}{2n}}=\frac{1}{2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right). \tag{20}$$

For the last step of (20) we have taken into account that

$$\frac{\sin\frac{\pi}{2n}}{\sin\frac{2\pi}{2n}}=\frac{\sin\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}=\frac{1}{2}\frac{1}{\cos\frac{\pi}{2n}}=\frac{1}{2}\frac{1}{1+\mathcal{O}\left(\frac{1}{n^2}\right)}=\frac{1}{2}\left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right),$$

and, applying the Mean Value Theorem (MVT), we have for some ξ that it holds

$$\frac{\sin\frac{(d+\ell+3)\pi}{2n}}{\sin\frac{(d+\ell+2)\pi}{2n}}=\frac{\sin\frac{(d+\ell+2)\pi}{2n}+\cos\xi\ \frac{\pi}{2n}}{\sin\frac{(d+\ell+2)\pi}{2n}}=1+\cos\xi\ \frac{\frac{\pi}{2n}}{\sin\frac{(d+\ell+2)\pi}{2n}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

For the last equality we take into account that $\frac{(d+\ell+2)\pi}{2n}$ satisfies

$$0\le\frac{\sqrt{n}\,\pi}{2n}\le\frac{(d+\ell+2)\pi}{2n}\le\frac{(\frac{n}{2}+n+1)\pi}{2n}$$

and, as a consequence of the concavity of $\sin x$ in $[0,\frac{3\pi}{4}+\frac{\pi}{2n}]$, we can say that $\sin\frac{(d+\ell+2)\pi}{2n}\ge A\frac{\sqrt{n}\,\pi}{2n}$ ($A=\sin\frac{5\pi}{6}=\frac{1}{2}$ for example), and the proof is complete. □

Lemma 5. *If $\sqrt{n}\le d\le\frac{n}{2}$, then*

(i)

$$\cot\frac{(d+n-2)\pi}{2n}=\cot\frac{(d+n-1)\pi}{2n}+\mathcal{O}\left(\frac{1}{n}\right). \tag{21}$$

(ii) Besides, if n is even, then

$$\frac{1}{2}\sum_{\ell=1}^{n-1}(-1)^{\ell}\cot\frac{(d+\ell)\pi}{2n} =$$
$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n}-\cot\frac{(d+n-1)\pi}{2n}\right)+\frac{1}{4}\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\left(-\cot\frac{(d+1)\pi}{2n}+\cot\frac{(d+n-1)\pi}{2n}\right). \tag{22}$$

(iii)

$$\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\left(-\cot\frac{(d+1)\pi}{2n}+\cot\frac{(d+n-1)\pi}{2n}\right)\sin\frac{d\pi}{n}=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{23}$$

Proof. (i) In our conditions, and applying the MVT we have (for n large enough)

$$\cot\frac{(d+n-1)\pi}{2n}-\cot\frac{(d+n-2)\pi}{2n}=\frac{-1}{\sin^2\xi}\frac{\pi}{2n},\ \text{with } \xi\in\left[\frac{(d+n-2)\pi}{2n},\frac{(d+n-1)\pi}{2n}\right]\subset\left[\frac{\pi}{2},\frac{3\pi}{4}\right].$$

So, $\frac{1}{\sin^2\xi}\le 2$ and we obtain the result.

(ii) Since

$$\frac{1}{2}\sum_{\ell=1}^{n-1}(-1)^{\ell}\cot\frac{(d+\ell)\pi}{2n}=\sum_{\ell=1,\ \ell\text{ odd}}^{n-3}\frac{1}{2}\left(-\cot\frac{(d+\ell)\pi}{2n}+\cot\frac{(d+\ell+1)\pi}{2n}\right)-\frac{1}{2}\cot\frac{(d+n-1)\pi}{2n},$$

if we apply Lemma 4(iii), the last expression is equal to

$$\frac{1}{4}\sum_{\ell=1,\ \ell\text{ odd}}^{n-3}\left(-\cot\frac{(d+\ell)\pi}{2n}+\cot\frac{(d+\ell+2)\pi}{2n}\right)\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)-\frac{1}{2}\cot\frac{(d+n-1)\pi}{2n}=$$
$$\frac{1}{4}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)\left(-\cot\frac{(d+1)\pi}{2n}+\cot\frac{(d+n-1)\pi}{2n}\right)-\frac{1}{2}\cot\frac{(d+n-1)\pi}{2n}=$$
$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n}-\cot\frac{(d+n-1)\pi}{2n}\right)+\frac{1}{4}\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\left(-\cot\frac{(d+1)\pi}{2n}+\cot\frac{(d+n-1)\pi}{2n}\right).$$

(iii) In our conditions it is easy to see that $\cot\frac{(d+n-1)\pi}{2n}=-\tan\frac{(d-1)\pi}{2n}$ and since $0\le\frac{(d-1)\pi}{2n}\le\frac{\pi}{4}$, we obtain that $\cot\frac{(d+n-1)\pi}{2n}=\mathcal{O}(1)$ and $\cot\frac{(d+1)\pi}{2n}\sin\frac{d\pi}{n}=\mathcal{O}(1)$.

The result is a straightforward consequence of these facts. □

Lemma 6. *If $\sqrt{n}\le d\le\frac{n}{2}$, then*

$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n}-\cot\frac{(d+n-1)\pi}{2n}\right)\sin\frac{d\pi}{n}=$$
$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{24}$$

Proof. Applying the MVT, we can say

$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n}-\cot\frac{(d+n-1)\pi}{2n}\right)\sin\frac{d\pi}{n}=$$
$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}+\frac{1}{\sin^2\xi_1}\frac{\pi}{2n}-\frac{1}{\sin^2\xi_2}\frac{\pi}{2n}\right)\sin\frac{d\pi}{n}=$$
$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n}+\frac{1}{4}\left(\frac{1}{\sin^2\xi_1}-\frac{1}{\sin^2\xi_2}\right)\frac{\pi}{2n}\sin\frac{d\pi}{n},$$

with $\xi_1\in[\frac{d\pi}{2n},\frac{(d+1)\pi}{2n}]$ and $\xi_2\in[\frac{(d+n-1)\pi}{2n},\frac{(d+n)\pi}{2n}]$.

In our conditions $\xi_2\in[\frac{\pi}{2},\frac{3\pi}{4}]$, in the way that $\frac{1}{\sin^2\xi_2}=\mathcal{O}(1)$ and in consequence the corresponding term $\frac{1}{\sin^2\xi_2}\frac{\pi}{2n}\sin\frac{d\pi}{n}=\mathcal{O}\left(\frac{1}{n}\right)$.

For the other term we can see that, in the interval of ξ_1, $\sin\frac{d\pi}{n}\le\frac{d\pi}{n}$ and $\sin\xi_1\ge\frac{1}{2}\xi_1$. Consequently $\frac{\sin\frac{d\pi}{n}}{\sin\xi_1}=\mathcal{O}(1)$, and $\frac{1}{\sin^2\xi_1}\frac{\pi}{2n}\sin\frac{d\pi}{n}=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. □

Lemma 7. *It holds*

$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n}=-\frac{1}{2}\cos\frac{d\pi}{n}. \tag{25}$$

Proof. We can write

$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n}=\frac{1}{4}\left(-\cot\frac{d\pi}{2n}+\tan\frac{d\pi}{2n}\right)\sin\frac{d\pi}{n}=$$

$$\frac{1}{4}\left(-\frac{\cos\frac{d\pi}{2n}}{\sin\frac{d\pi}{2n}}+\frac{\sin\frac{d\pi}{2n}}{\cos\frac{d\pi}{2n}}\right)\sin\frac{d\pi}{n}=\frac{1}{4}\frac{-\cos^2\frac{d\pi}{2n}+\sin^2\frac{d\pi}{2n}}{\sin\frac{d\pi}{2n}\cos\frac{d\pi}{2n}}\sin\frac{d\pi}{n}=-\frac{1}{2}\frac{\cos\frac{d\pi}{n}}{\sin\frac{d\pi}{n}}\sin\frac{d\pi}{n}=-\frac{1}{2}\cos\frac{d\pi}{n}. \quad \square$$

The next results are devoted to a further manipulation of $P_{2,n}(d)$.

Lemma 8. *It holds*

(i) If $0\le d\le\sqrt{n}$, then

$$\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}=\frac{\pi^2 d}{2n^2}+\mathcal{O}\left(\frac{1}{n^2\sqrt{n}}\right). \tag{26}$$

(ii) For all x, such that $0\le x\le\frac{2\pi}{3}$ we have

$$\frac{1}{\sin x}=\frac{1}{x}+A_x x \text{ with } 0\le A_x\le 2. \tag{27}$$

(iii) If $0\le d\le\sqrt{n}$, $0\le\ell$ and $2\ell+2\le n$, then

$$\frac{1}{\sin\frac{(d+2\ell+1)\pi}{2n}}\frac{1}{\sin\frac{(d+2\ell+2)\pi}{2n}}=\frac{4n^2}{(d+2\ell+1)(d+2\ell+2)\pi^2}+\mathcal{O}(1). \tag{28}$$

Proof.

(i) When $0\le d\le\sqrt{n}$ we can apply that $\sin x=x+\mathcal{O}(x^3)$, taking $x=\frac{d\pi}{n}=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and $x=\frac{\pi}{2n}=\mathcal{O}\left(\frac{1}{n}\right)$. So, we have

$$\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}=\left(\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{n\sqrt{n}}\right)\right)\left(\frac{\pi}{2n}+\mathcal{O}\left(\frac{1}{n^3}\right)\right)=\frac{d\pi^2}{2n^2}+\mathcal{O}\left(\frac{1}{n^2\sqrt{n}}\right)$$

and (26) is proved.

(ii) We know that $\sin x=x-\frac{x^3}{6}+\frac{\cos\xi_x}{120}x^5$ for some ξ_x. So, we can write

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-(x-\frac{x^3}{6}+\frac{\cos\xi_x}{120}x^5)}{x\sin x}=\frac{\frac{x^3}{6}-\frac{\cos\xi_x}{120}x^5}{x^2-\frac{x^4}{6}+\frac{\cos\xi_x}{120}x^6}=\frac{\frac{x}{6}-\frac{\cos\xi_x}{120}x^3}{1-\frac{x^2}{6}+\frac{\cos\xi_x}{120}x^4}=$$

$$x\frac{\frac{1}{6}-\frac{\cos\xi_x}{120}x^2}{1-\frac{x^2}{6}+\frac{\cos\xi_x}{120}x^4}=A_x x.$$

But, in our case, $0\le\frac{1}{6}-\frac{\cos\xi_x}{120}x^2\le\frac{1}{6}+\frac{(\frac{2\pi}{3})^2}{120}\approx 0.203221$. On the other hand, we have that $0.1<0.108574\approx 1-\frac{(\frac{2\pi}{3})^2}{6}-\frac{(\frac{2\pi}{3})^4}{120}\le 1-\frac{x^2}{6}+\frac{\cos\xi_x}{120}x^4$.

(iii) By applying (27) we get

$$\frac{1}{\sin\frac{(d+2\ell+1)\pi}{2n}}\frac{1}{\sin\frac{(d+2\ell+2)\pi}{2n}}=$$

$$\left(\frac{2n}{(d+2\ell+1)\pi}+A_{x_1}\frac{(d+2\ell+1)\pi}{2n}\right)\left(\frac{2n}{(d+2\ell+2)\pi}+A_{x_2}\frac{(d+2\ell+2)\pi}{2n}\right)=$$

$$\frac{4n^2}{(d+2\ell+1)(d+2\ell+2)\pi^2}+A_{x_2}\frac{d+2\ell+2}{d+2\ell+1}+A_{x_1}\frac{d+2\ell+1}{d+2\ell+2}+\mathcal{O}(1).$$

Furthermore, in our conditions, it is easy to see that $\frac{d+2\ell+2}{d+2\ell+1}$ is a decreasing function on ℓ, always less than or equal to 2 and always greater than 1. In a similar way $\frac{d+2\ell+1}{d+2\ell+2}$ is an increasing function on ℓ, with minimum $\frac{1}{2}$ and always less than or equal to 1. Hence, (28) is obtained. $\square$

In the next Lemma we use the special function Phi of Hurwitz–Lerch with -1 as first argument, that is, $HurwitzLerchPhi[-1, s, d]$ defined by

$$HurwitzLerchPhi[-1, s, a] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s}.$$

In our case we always use $s = 1$. So, we use $HurwitzLerchPhi[-1, 1, a]$, which we denote as $\eta(a)$ (see [21] for the details).

Lemma 9. *It holds*

(i)

$$\sum_{\ell=0}^{\frac{n}{2}-1} \frac{1}{(d+2\ell+1)(d+2\ell+2)} = \eta(d+1) - \eta(d+n+1). \tag{29}$$

(ii)

$$\sum_{\ell=0}^{\frac{n}{2}-1} \frac{1}{(d+2\ell+1)(d+2\ell+2)} = \eta(d+1) + \mathcal{O}\left(\frac{1}{n}\right). \tag{30}$$

Proof. We have

$$\eta(d+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+d+1} = -\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{d+\ell} =$$

$$-\sum_{\ell=0}^{\infty} \left(\frac{-1}{d+2\ell+1} + \frac{1}{d+2\ell+2}\right) = \sum_{\ell=0}^{\infty} \frac{1}{(d+2\ell+1)(d+2\ell+2)}.$$

In a similar way, we get

$$\eta(d+n+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+d+n+1} = -\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{d+n+\ell} =$$

$$-\sum_{\ell=0}^{\infty} \left(\frac{-1}{d+n+2\ell+1} + \frac{1}{d+n+2\ell+2}\right) = \sum_{\ell=\frac{n}{2}}^{\infty} \frac{1}{(d+2\ell+1)(d+2\ell+2)}.$$

Consequently, we obtain (29).

(ii) Since

$$\sum_{\ell=0}^{\infty} \frac{1}{(d+n+2\ell+1)(d+n+2\ell+2)} \leq \sum_{\ell=\frac{n}{2}}^{\infty} \frac{1}{\ell^2} \leq \frac{4}{n-2},$$

it is immediate to get (30). □

4. Main results

The first part of this section is devoted to present the central results of the article related to the unit circle, that is the behavior of $2n\mathcal{E}(F, z)$ along the unit circle. In the second one, they are presented for the original problem.

4.1. The unit circle

Proposition 3. *The imaginary part of $2n\mathcal{E}(F, z)$ uniformly converges to 0 as an $\mathcal{O}\left(\frac{1}{n}\right)$.*

Proof. From (6) and taking into account (9), we set for $z = ie^{-i\frac{\pi d}{n}}$

$$\Im\left(2n\mathcal{E}(F, z)\right) = -2\sin d\pi \sum_{\ell=1}^{n-1} (-1)^\ell \sin\frac{\ell\pi}{n} = 2\sin d\pi \tan\frac{\pi}{2n},$$

which is equivalent, when n is large enough, to $\frac{\pi}{n}\sin d\pi$. □

Next, we are going to study the behavior of the real part of $2n\mathcal{E}(F,z)$. Thus, we recall that as a consequence of (10) we can write (7) as

$$\Re(2n\mathcal{E}(F,z)) = -4\sin d\pi \sum_{\ell=1}^{n-1}(-1)^{\ell}\frac{\cos\frac{(d+\ell)\pi}{2n}}{2\sin\frac{(d+\ell)\pi}{2n}}\sin\frac{\ell\pi}{n} = \\ -4\sin d\pi(P_{1,n}(d)+P_{2,n}(d)+P_{3,n}(d)).$$

Taking into account that the behavior depends on d, we study in the next propositions $P_{1,n}(d)+P_{3,n}(d)$ and $P_{2,n}(d)$ in terms of d.

Proposition 4. *It holds*

(i) The function $P_{1,n}(d)+P_{3,n}(d)+\frac{1}{2}\cos\frac{d\pi}{n}$ uniformly converges to 0 as an $\mathcal{O}\left(\frac{1}{n}\right)$. In other words, we can consider that $P_{1,n}(d)+P_{3,n}(d)=-\frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{n}\right)$.

(ii) Moreover, if $0\le d\le\sqrt{n}$, then

$$P_{1,n}(d)+P_{3,n}(d) = -\frac{1}{2}+\mathcal{O}\left(\frac{1}{n}\right). \tag{31}$$

Proof. (i) Taking into account (11) and (15), we can write

$$P_{1,n}(d)=\cos\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^{\ell}\cos^2\frac{(d+\ell)\pi}{2n} = \\ \cos\frac{d\pi}{n}\left(-\frac{1}{2}+\frac{1}{2}\sec\frac{\pi}{2n}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\right) = -\frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{n}\right).$$

On the other hand, using (13) and (16),

$$P_{3,n}(d)=\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}(-1)^{\ell}\sin\frac{(d+\ell)\pi}{n} = -\frac{1}{2}\sin\frac{d\pi}{n}sec\frac{\pi}{2n}\cos\frac{d\pi}{n}\sin\frac{\pi}{2n} = \mathcal{O}\left(\frac{1}{n}\right).$$

Now, the result is a straightforward consequence.

(ii) We apply the general result and consider that $0\le d\le\sqrt{n}$. □

To study the behavior of $P_{2,n}(d)$ we have to distinguish two situations depending on d.

Proposition 5. *If $\sqrt{n}\le d\le\frac{n}{2}$, then*

$$P_{2,n}(d)=\frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{32}$$

Proof. Recalling (12) and (22)

$$P_{2,n}(d) = -\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n}-\cot\frac{(d+n-1)\pi}{2n}\right)\sin\frac{d\pi}{n} - \\ \frac{1}{4}\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\left(-\cot\frac{(d+1)\pi}{2n}+\cot\frac{(d+n-1)\pi}{2n}\right)\sin\frac{d\pi}{n}. \tag{33}$$

The last term of (33) is an $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ as we have proved in (23). Thus, by using (24) and (25), we can write

$$P_{2,n}(d) = -\frac{1}{4}\left(-\cot\frac{d\pi}{2n}-\cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad \square$$

Next, we study the behavior of $P_{2,n}(d)$ when $0\le d\le\sqrt{n}$.

Proposition 6. *If $0\le d\le\sqrt{n}$, then*

$$P_{2,n}(d) = d\,\eta(d+1)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{34}$$

Proof. Using (12) we have

$$P_{2,n}(d)=-\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n-1}\frac{(-1)^{\ell}\cos\frac{(d+\ell)\pi}{2n}}{\sin\frac{(d+\ell)\pi}{2n}}=\underbrace{-\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n}\frac{(-1)^{\ell}\cos\frac{(d+\ell)\pi}{2n}}{\sin\frac{(d+\ell)\pi}{2n}}}_{*}-\underbrace{\frac{1}{2}\sin\frac{d\pi}{n}\frac{\cos\frac{(d+n)\pi}{2n}}{\sin\frac{(d+n)\pi}{2n}}}_{**}.$$

Since $**=-\frac{1}{2}\sin\frac{d\pi}{n}\tan\frac{d\pi}{2n}=\mathcal{O}\left(\frac{1}{n}\right)$, we can neglect it.
Applying (17) and (28), we get

$$*=-\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=1}^{n}\frac{(-1)^{\ell}\cos\frac{(d+\ell)\pi}{2n}}{\sin\frac{(d+\ell)\pi}{2n}}=-\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=0}^{\frac{n}{2}-1}\left(-\frac{\cos\frac{(d+2\ell+1)\pi}{2n}}{\sin\frac{(d+2\ell+1)\pi}{2n}}+\frac{\cos\frac{(d+2\ell+2)\pi}{2n}}{\sin\frac{(d+2\ell+2)\pi}{2n}}\right)=$$

$$\frac{1}{2}\sin\frac{d\pi}{n}\sum_{\ell=0}^{\frac{n}{2}-1}\frac{\sin\frac{\pi}{2n}}{\sin\frac{(d+2\ell+1)\pi}{2n}\sin\frac{(d+2\ell+2)\pi}{2n}}=$$

$$\frac{1}{2}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\sum_{\ell=0}^{\frac{n}{2}-1}\left(\frac{4n^2}{(d+2\ell+1)(d+2\ell+2)\pi^2}+\mathcal{O}(1)\right)=$$

$$\frac{1}{2}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\sum_{\ell=0}^{\frac{n}{2}-1}\frac{4n^2}{(d+2\ell+1)(d+2\ell+2)\pi^2}+\frac{1}{2}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\sum_{\ell=0}^{\frac{n}{2}-1}\mathcal{O}(1)=$$

$$\underbrace{\frac{1}{2}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\frac{4n^2}{\pi^2}}_{***}\underbrace{\sum_{\ell=0}^{\frac{n}{2}-1}\frac{1}{(d+2\ell+1)(d+2\ell+2)}}_{****}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

In order to obtain a good approximation of $***$, we use (26). By doing this we have

$$***=\frac{1}{2}\sin\frac{d\pi}{n}\sin\frac{\pi}{2n}\frac{4n^2}{\pi^2}=\frac{1}{2}\frac{4n^2}{\pi^2}\left(\frac{\pi^2 d}{2n^2}+\mathcal{O}\left(\frac{1}{n^2\sqrt{n}}\right)\right)=d+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

On the other hand, we know by (30) that $****=\eta(d+1)+\mathcal{O}\left(\frac{1}{n}\right)$. Hence, we conclude (34). □

Theorem 1. *In our conditions and for $z=ie^{-i\frac{\pi d}{n}}$ we have*

(i) If $\sqrt{n}\le d\le\frac{n}{2}$, then

$$2n\mathcal{E}(F,z)=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

(ii) If $0\le d\le\sqrt{n}$, then

$$2n\mathcal{E}(F,z)=-4\sin d\pi\left(d\,\eta(d+1)-\frac{1}{2}\right)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. We know from Proposition 3 that $2n\Im(\mathcal{E}(F,z))=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$.

(i) From Propositions 4 and 5 we have

$$2n\Re(\mathcal{E}(F,z))=-4\sin d\pi(P_{1,n}(d)+P_{2,n}(d)+P_{3,n}(d))=$$
$$-4\sin d\pi\left(-\frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{n}\right)+\frac{1}{2}\cos\frac{d\pi}{n}+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

(ii) We proceed as in (i) using Propositions 4 and 6 and we obtain it. □

Remark 1. We want to point out the following considerations.

1. As a consequence of Proposition 2 we can extend the results to all the unit circle.
2. The detailed behavior of $2n\mathcal{E}(F,z)$ presents a Gibbs–Wilbraham phenomenon.

   (1) When z is close to $\pm i$, then $2n\mathcal{E}(F,z)$ has a strongly oscillatory behavior decaying as z goes far away from $\pm i$.

(2) When z is far away $\pm i$, then $2n\mathcal{E}(F,z)$ can be considered as 0.

3. We can approximate asymptotically $2n\mathcal{E}(F,z)$ near $\pm i$ as in the previous theorem and obtain the extrema of $-4\sin d\pi\left(d\,\eta(d+1)-\frac{1}{2}\right)$. Below, we describe them in terms of d.

4.2. The real case

The interpolation polynomial $\mathcal{L}_{-n,n-1}(F,z)\in\Lambda_{-n,n-1}[z]$ satisfying (2) has been studied along the article. Then it is clear that

$$\mathcal{L}_{-n,n-1}(F,\alpha_j)=\left|\frac{\alpha_j+\frac{1}{\alpha_j}}{2}\right|$$

for all $j=0,\dots,2n-1$. In a similar way, the interpolation polynomial $\mathcal{L}_{-n,n-1}(F,\frac{1}{z})\in\Lambda_{-(n-1),n}[z]$ satisfies the interpolation conditions

$$\mathcal{L}_{-n,n-1}(F,\frac{1}{\alpha_j})=\left|\frac{\alpha_j+\frac{1}{\alpha_j}}{2}\right|$$

for all $j=0,\dots,2n-1$. So, if we consider the polynomial

$$\frac{\mathcal{L}_{-n,n-1}(F,z)+\mathcal{L}_{-n,n-1}(F,\frac{1}{z})}{2}$$

we have that it is an element of $\mathbb{P}_n[x]=\mathbb{P}_{m+1}[x]$ which satisfies the interpolation conditions (1). The uniqueness of the interpolation polynomial ensures us that these two polynomials are the same.

We summarize these facts saying that

$$\ell_{m+1}(|x|,x)=\frac{\mathcal{L}_{-n,n-1}(F,z)+\mathcal{L}_{-n,n-1}(F,\frac{1}{z})}{2}=$$
$$\frac{\mathcal{L}_{-n,n-1}(F,z)+\overline{\mathcal{L}_{-n,n-1}(F,z)}}{2}=\Re(\mathcal{L}_{-n,n-1}(F,z)). \tag{35}$$

Theorem 2. *For $x=\sin\frac{d\pi}{m+1}$ it holds*

(i) If $\sqrt{m+1}\le d\le\dfrac{m+1}{2}$, then

$$2(m+1)(|x|-\ell_{m+1}(|x|,x))=\mathcal{O}\left(\frac{1}{\sqrt{m+1}}\right). \tag{36}$$

(ii) If $0\le d\le\sqrt{m+1}$, then

$$2(m+1)(|x|-\ell_{m+1}(|x|,x))=-4\sin d\pi\,(d\,\eta(d+1)-\frac{1}{2})+\mathcal{O}\left(\frac{1}{\sqrt{m+1}}\right). \tag{37}$$

Proof. Taking into account (35) to obtain

$$2(m+1)(|x|-\ell_{m+1}(|x|,x))=2nF(z)-\Re(\mathcal{L}_{-n,n-1}(F,z))=2n\Re(\mathcal{E}(F,z))$$

and using Theorem 1 lead to (40) and (41). □

Remark 2. With the previous result we conclude that

1. The polynomial approximation process of $|x|$ in $[-1,1]$, by using $\ell_{m+1}(|x|,x)$, presents a Gibbs–Wilbraham phenomenon.
2. Near 0, $\ell_{m+1}(|x|,x)$ presents a decaying oscillatory phenomenon around $|x|$ with a maximal amplitude of order $\frac{1}{m}$.
3. Far from 0, $\ell_{m+1}(|x|,x)$ presents a convergence to $|x|$ with an error of order $\frac{1}{m\sqrt{m}}$.

5. Using simply Chebyshev points

This section is devoted to present the results concerning to the Lagrange interpolation process of $f(x)=|x|$ when we use as nodal system the roots of T_m, with m odd. In this case, 0 is root of T_m. The relevant changes along this section are given below.

When we translate the problem using Joukowsky–Szegő transformation we obtain new equispaced nodal system on $\mathbb{T}$ of $2n = 2m$ points and i and $-i$ are nodes. Now n is odd and the nodal polynomial is $z^{2n} + 1$. We introduce T in the notation to distinguish the objects of this case, related to the Chebyshev first kind polynomial.

Eq. (3) becomes to:

$$\mathcal{L}_{-n,n-1}(F, z, T) = \frac{z^{2n}+1}{2n\, z^n} \sum_{j=0}^{2n-1} \frac{1}{\alpha_j^{n-1}(z-\alpha_j)} F(\alpha_j) \tag{38}$$

and therefore the barycentric expression (4) has no changes.

Following the same steps as in Section 2.1, we have for this new problem

$$2n\mathcal{E}(F, z, T) = \underbrace{-2\frac{z^{2n}+1}{i^n\, z^n}}_{*} \sum_{\ell=1}^{n-1} \underbrace{F(\alpha_{2n-\ell}) \frac{i^n}{\alpha_{2n-\ell}^{n-1}(z-\alpha_{2n-\ell})}}_{**}.$$

With the representation $z = ie^{-\frac{i\pi d}{n}}$, $**$ can be treated as in the original problem. To simplify $*$, we use now that n is odd and therefore $i^{2n} = -1$; doing the same we obtain $* = -2\frac{z^{2n}+1}{i^n z^n} = 4i \sin d\pi$. This is the same result as in the original case. Hence, we have the same expression for Eq. (5), and consequently Proposition 1 has no changes. Also, Proposition 2 has no changes. To summarize, we must change some proofs but the expression obtained for $2n\mathcal{E}(F, z, T)$ is still the same.

Next we present the changes in Section 3.

Lemma 1 is treated in the same way, but now n is odd and we get

$$\sum_{\ell=1}^{n-1} (-1)^\ell \sin\frac{\ell\pi}{n} = \frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\left(\frac{-\pi}{2n} + n\pi\right) + \sin\left(\frac{\pi}{2n} - \pi\right)\right) = \frac{1}{2}\sec\frac{\pi}{2n}\left(\sin\frac{\pi}{2n} - \sin\frac{\pi}{2n}\right) = 0.$$

Lemma 2 has no changes, meanwhile Lemma 3 has some changes. Using the same ideas, we have for n odd

$$\sum_{\ell=1}^{n-1} (-1)^\ell \cos^2\frac{(d+\ell)\pi}{2n} = -\frac{1}{2}\cos\frac{d\pi}{n} \text{ and}$$

$$\sum_{\ell=1}^{n-1} (-1)^\ell \sin\frac{(d+\ell)\pi}{n} = -\sin\frac{d\pi}{n}. \tag{39}$$

Lemma 4 has no changes, as well as Lemma 5(i) and (iii). However, there is an important change in Lemma 5(ii). Indeed, if n is odd we must write

$$\frac{1}{2}\sum_{\ell=1}^{n-1} (-1)^\ell \cot\frac{(d+\ell)\pi}{2n} = \sum_{\ell=1,\ \ell \text{ odd}}^{n-2} \frac{1}{2}\left(-\cot\frac{(d+\ell)\pi}{2n} + \cot\frac{(d+\ell+1)\pi}{2n}\right).$$

Then, if we apply Lemma 4(iii), the last expression is equal to

$$\frac{1}{4}\sum_{\ell=1,\ \ell \text{ odd}}^{n-2}\left(-\cot\frac{(d+\ell)\pi}{2n} + \cot\frac{(d+\ell+2)\pi}{2n}\right)\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right) =$$
$$\frac{1}{4}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)\left(-\cot\frac{(d+1)\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right) =$$
$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right) + \frac{1}{4}\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\left(-\cot\frac{(d+1)\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right).$$

Lemmas 6 and 7 (valid for all n) must be rewritten to describe the new situation. It is easy to obtain their corresponding expressions:

$$\frac{1}{4}\left(-\cot\frac{(d+1)\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n} = \frac{1}{4}\left(-\cot\frac{d\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

and

$$\frac{1}{4}\left(-\cot\frac{d\pi}{2n} + \cot\frac{(d+n)\pi}{2n}\right)\sin\frac{d\pi}{n} = \frac{1}{4}\left(-\cot\frac{d\pi}{2n} - \tan\frac{d\pi}{2n}\right)\sin\frac{d\pi}{n} = -\frac{1}{2}.$$

No other changes are needed to adapt the section Technical results.

We pass to establish the central results for this case.

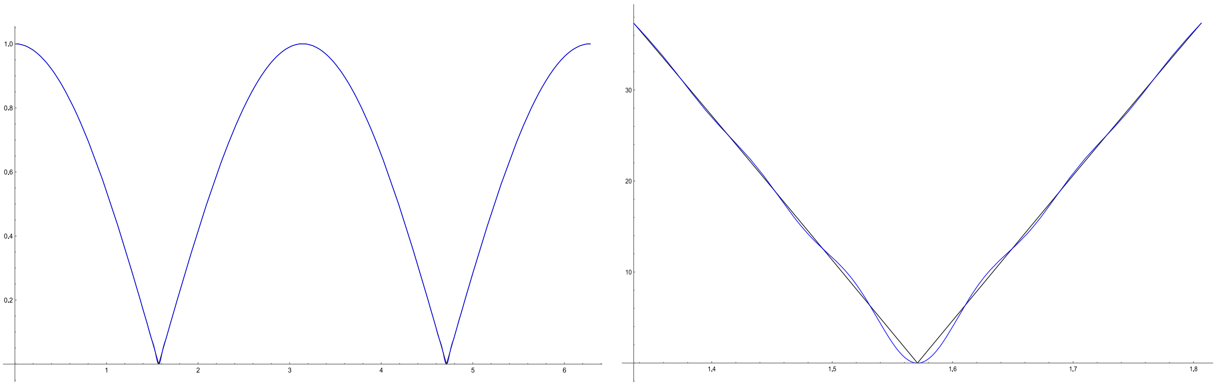

Fig. 3. General view of the interpolation process. $F(z)$ in black and $\Re(\mathcal{L}_{-80,79}(F,z))$ in blue on the left. $160F(z)$ in black and $160\Re(\mathcal{L}_{-80,79}(F,z))$ in blue on the right.

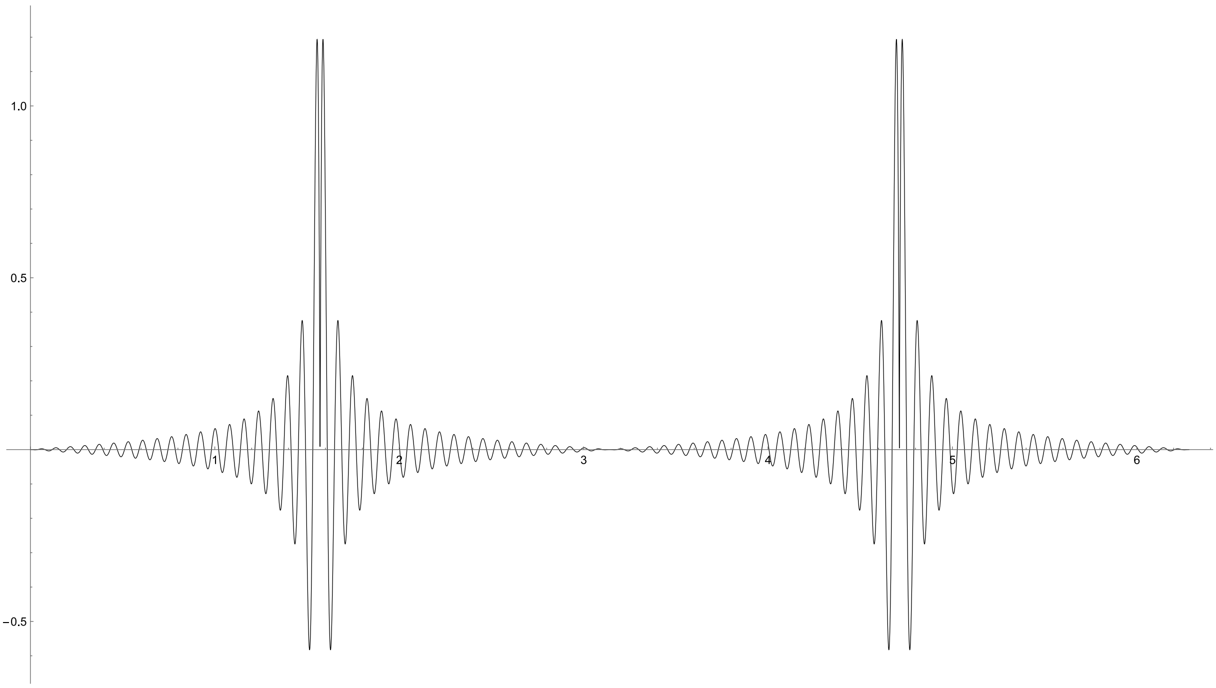

Fig. 4. $\Re(160\,\mathcal{E}(F, e^{i\theta}))$ along the interval $[0, 2\pi]$.

Proposition 7. *The imaginary part of* $2n\mathcal{E}(F, z, T)$ *is* 0.

We recall that as a consequence of (10) we can write (7) as

$$\Re(2n\mathcal{E}(F, z, T)) = -4\sin d\pi(P_{1,n}(d) + P_{2,n}(d) + P_{3,n}(d)).$$

Taking into account that the behavior depends on d, next we establish the expressions of $P_{1,n}(d) + P_{3,n}(d)$ and $P_{2,n}(d)$ in terms of d. In particular and taking into account (39) we obtain:

(i) $P_{1,n}(d) + P_{3,n}(d) = -\frac{1}{2}$.
(ii) If $\sqrt{n} \le d \le \frac{n}{2}$, then $P_{2,n}(d) = \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$.
(iii) If $0 \le d \le \sqrt{n}$, then $P_{2,n}(d) = d\,\eta(d+1) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$.

Proceeding like in Section 4, next we obtain the main results of this section.

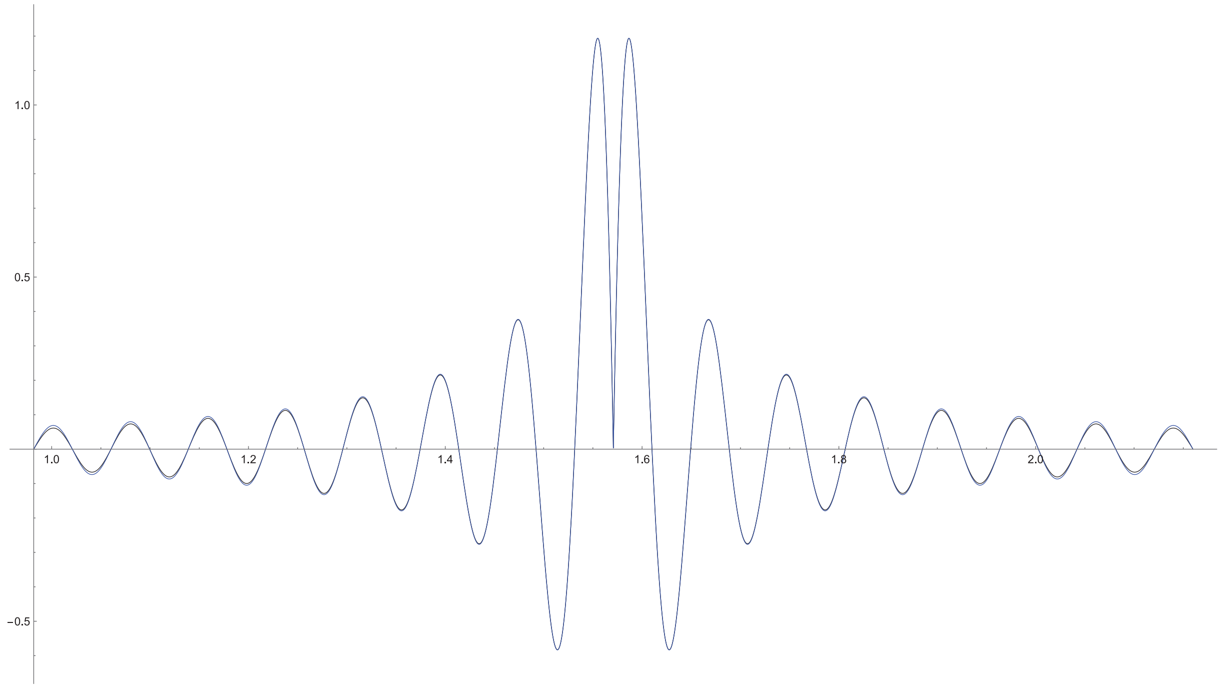

Fig. 5. $\Re(160\,\mathcal{E}(F, e^{i\theta}))$ near $\frac{\pi}{2}$ together with the approximation given by Theorem 1.

Theorem 3. *In our conditions and for* $z = ie^{-i\frac{\pi d}{n}}$ *we have*

(i) If $\sqrt{n} \le d \le \frac{n}{2}$*, then*

$$2n\mathcal{E}(F, z, T) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

(ii) If $0 \le d \le \sqrt{n}$*, then*

$$2n\mathcal{E}(F, z, T) = -4\sin d\pi\left(d\,\eta(d+1) - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Theorem 4. *For* $x = \sin\frac{d\pi}{m}$ *it holds*

(i) If $\sqrt{m} \le d \le \dfrac{m}{2}$*, then*

$$2m(|x| - \ell_{m-1}(|x|, x, T)) = \mathcal{O}\left(\frac{1}{\sqrt{m}}\right). \tag{40}$$

(ii) If $0 \le d \le \sqrt{m}$*, then*

$$2m(|x| - \ell_{m-1}(|x|, x, T)) = -4\sin d\pi\,(d\,\eta(d+1) - \frac{1}{2}) + \mathcal{O}\left(\frac{1}{\sqrt{m}}\right). \tag{41}$$

All the proofs of these results are straightforward consequences of the previous results. Notice that there are no relevant changes in the behavior of $2n\mathcal{E}(F, z)$ and $2n\mathcal{E}(F, z, T)$, as well as in the real problem.

6. Numerical experiments and graphics

All the numerical experiments that we present in the sequel have been obtained for the function $F(z) = \frac{|z+\frac{1}{z}|}{2}$, the nodal system constituted by the roots of order $n = 80$ of the unity, and the interpolation polynomial $\mathcal{L}_{-80,79}(F, z)$. As in the former sections, $2n\mathcal{E}(F, z)$ represents the error between $F(z)$ and $\mathcal{L}_{-80,79}(F, z)$ multiplied by $2n$. For simplicity, we use for the plots the variable θ with $z = e^{i\theta}$. The translation to the bounded interval is immediate.

Fig. 3 gives us a general view of the interpolation process. On the left-hand side we have a joint representation of $F(z)$ and $\Re(\mathcal{L}_{-80,79}(F, z))$ (we use $z = e^{i\theta}$) for a variable $\theta \in [0, 2\pi]$. As we have remarked from the beginning they

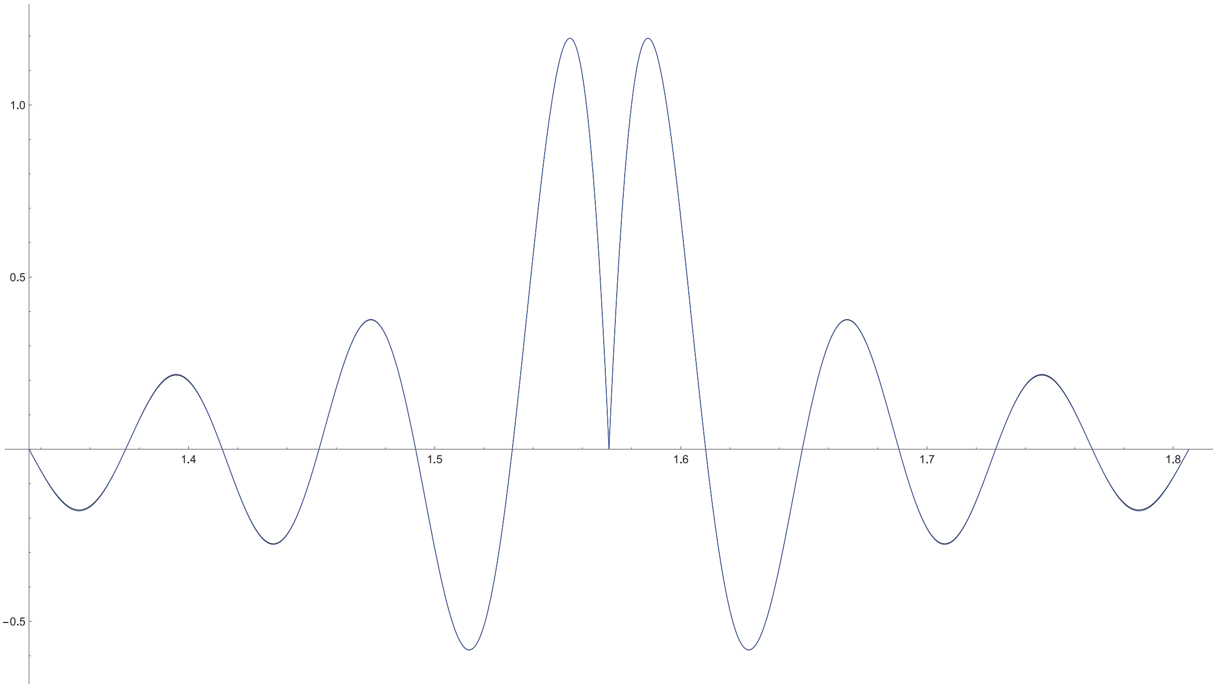

Fig. 6. $\Re(160\,\mathcal{E}(F, e^{i\theta}))$ very close to $\frac{\pi}{2}$ along with the approximation given by Theorem 1.

Table 1
Extrema of $-4\sin d\pi\left(d\,\eta(d+1)-\frac{1}{2}\right)$.

In the interval	The extremum is attained at (d value)	Being the extremum
[0, 1]	0.403614	1.19386
[1, 2]	1.44636	−0.583363
[2, 3]	2.4637	0.377061
[3, 4]	3.47283	−0.27652
[4, 5]	4.47838	0.217691
[5, 6]	5.48209	−0.179272

are indistinguishable. The situation changes when we represent $160F(z)$ and $160\Re(\mathcal{L}_{-80,79}(F, z))$ near i (or, if we prefer, $\theta = \frac{\pi}{2}$). This can be seen on the right-hand side of the figure and we can observe neatly the difference between both functions.

Fig. 4 represents $2n\Re(\mathcal{E}(F, e^{i\theta}))$ with θ varying in the interval $[0, 2\pi]$ and $2n = 160$. As we can see, the function is great near the points $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Clearly when θ is far from these points the function is close to zero and the approximation is better.

Fig. 5 represents $2n\Re(\mathcal{E}(F, e^{i\theta}))$ and the approximation given by Theorem 1 with θ varying in an interval close to $\frac{\pi}{2}$. The more relevant fact is that they are indistinguishable. Indeed, the difference between the functions is less than or equal to $3 * 10^{-3}$. We see in this figure the oscillating nature of the error, with greater amplitudes in the proximities of $\frac{\pi}{2}$.

Finally Fig. 6 represents $160\Re(\mathcal{E}(F, e^{i\theta}))$ and the approximation given by Theorem 1 with θ varying in an interval close to $\frac{\pi}{2}$. Our comments are similar to those related to Fig. 5; but we must add that the difference between the functions is smaller and the amplitudes of the oscillation are clearly related to the extrema in Table 1.

Next we present the same graphics (Figs. 7, 8, 9 and 10) related to the case of Chebyshev points studied in Section 5, taking $n = 81$. We must point out, as we have proved, that the situation is the same.

The codes used in order to produce the figures were done with Mathematica® version 12.2. In the link https://github.com/eberriochoa we share the codes as well as the graphics.

7. Conclusions

As we have proved along the paper, the Lagrange interpolation polynomial related to the usual absolute value and using Chebyshev nodal systems of second kind has an interesting behavior that we can summarize as:

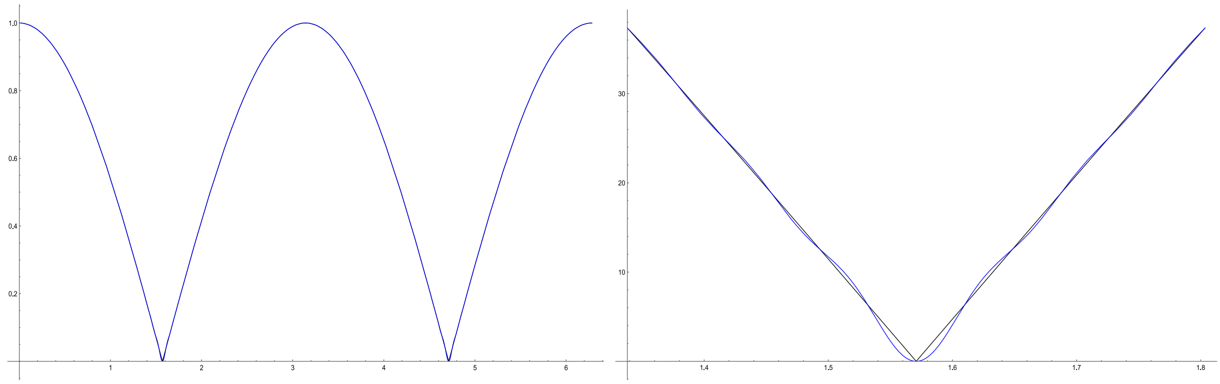

Fig. 7. General view of the interpolation process. $F(z)$ in black and $\Re(\mathcal{L}_{-81,80}(F,z,T))$ in blue on the left. $162F(z)$ in black and $162\Re(\mathcal{L}_{-81,80}(F,z,T))$ in blue on the right.

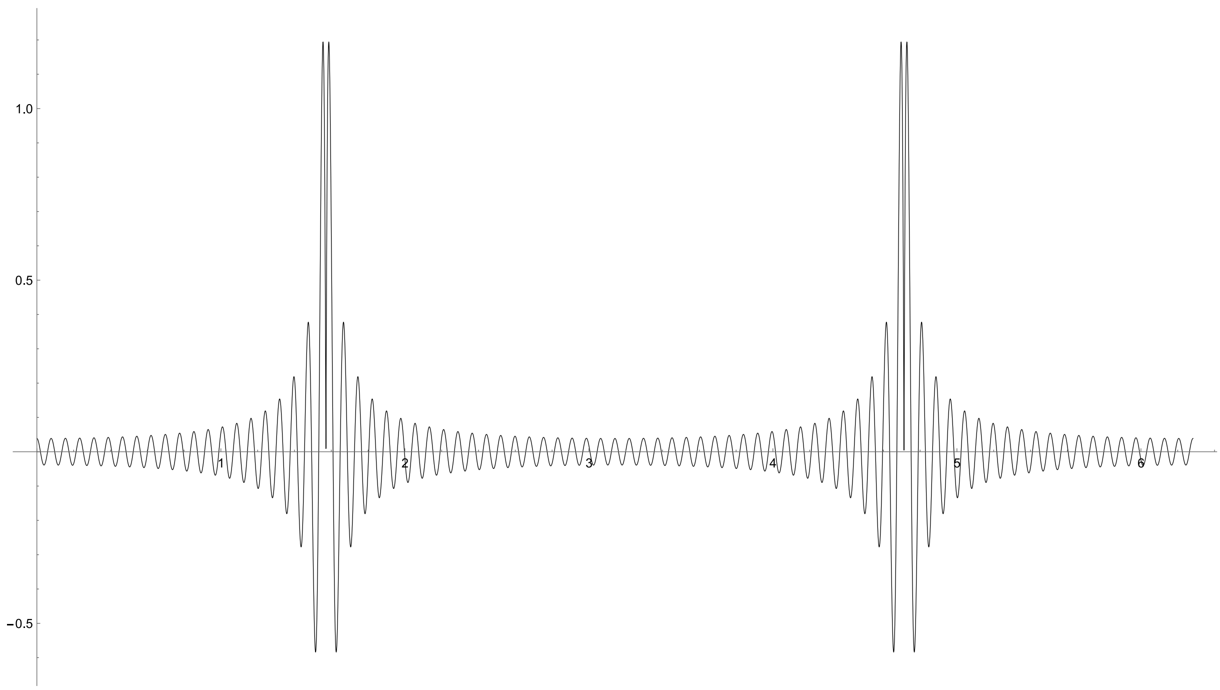

Fig. 8. $\Re(162\,\mathcal{E}(F,e^{i\theta},T))$ along the interval $[0,2\pi]$.

1. In a general sense this approximation is not far from the polynomial of best approximation.
2. The approximation presents a Gibbs–Wilbraham phenomenon.
3. We describe the corresponding Gibbs–Wilbraham phenomenon in detail.

The results could be considered as a stepace to study the behavior of Lagrange interpolation based on Chebyshev nodal systems of second and first kind for functions having similar singularities like the absolute value. Notice that a good understanding of the Gibbs–Wilbraham phenomenon could be of interest to correct it.

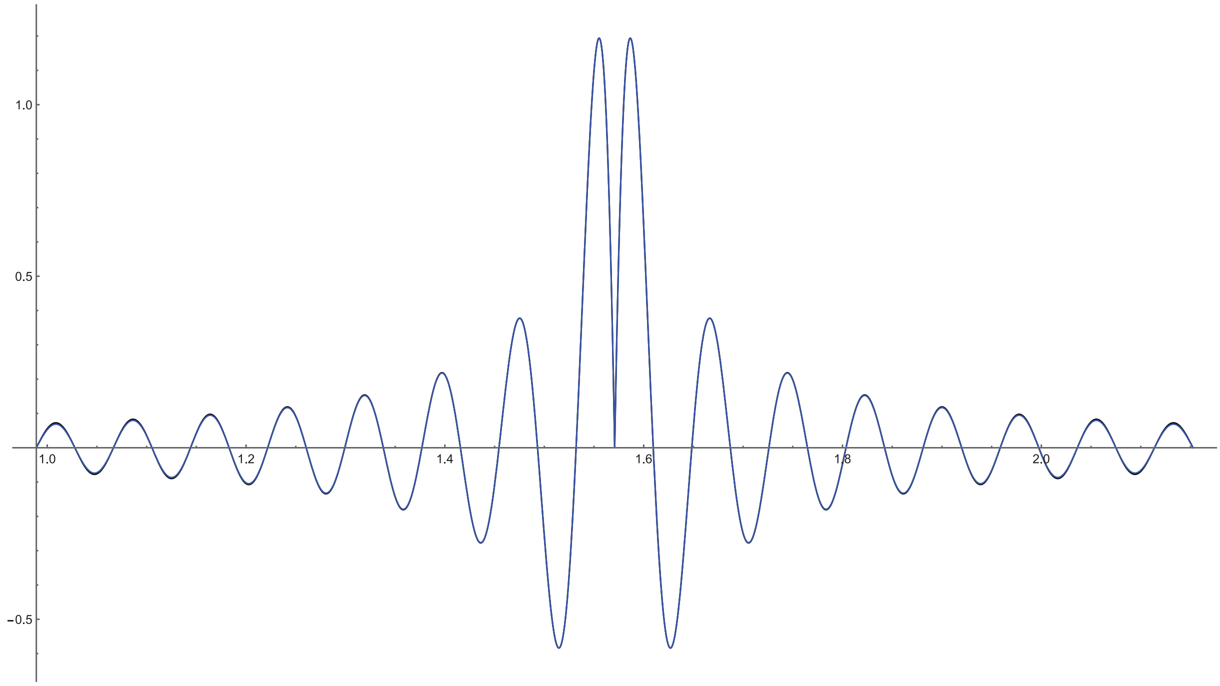

Fig. 9. $\Re(162\,\mathcal{E}(F, e^{i\theta}, T))$ near $\frac{\pi}{2}$ together with the approximation given by Theorem 3.

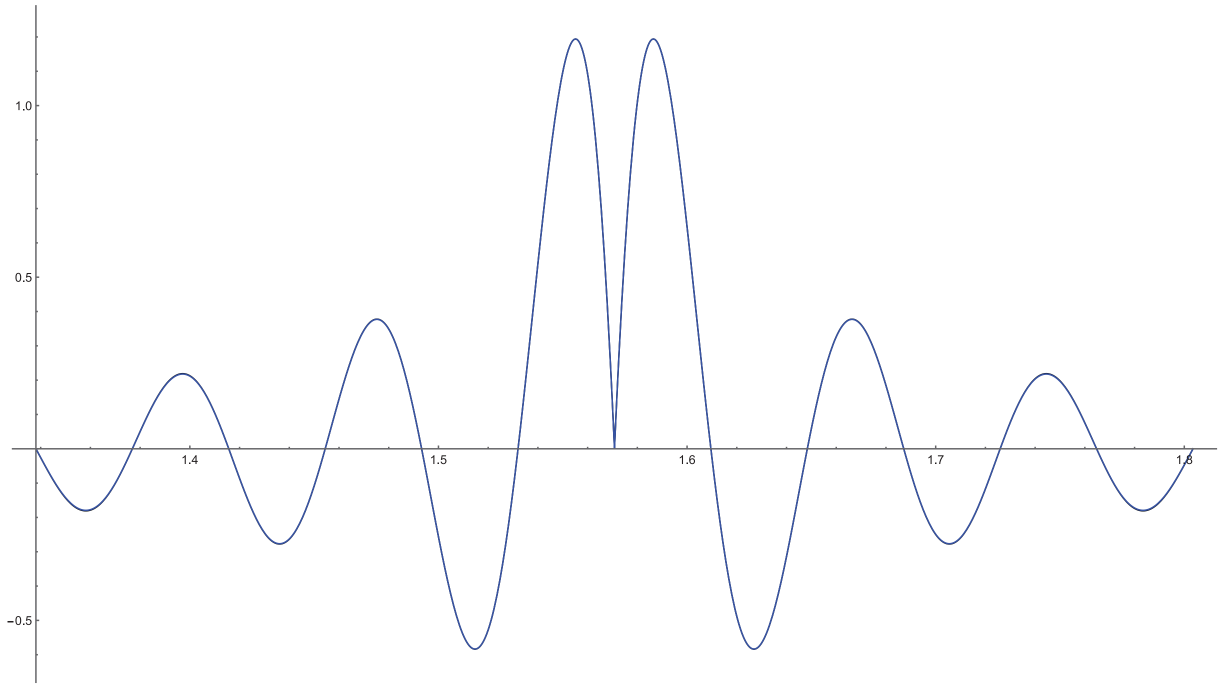

Fig. 10. $\Re(162\,\mathcal{E}(F, e^{i\theta}, T))$ very close to $\frac{\pi}{2}$ along with the approximation given by Theorem 3.

Acknowledgments

We thank to the anonymous referees for the careful read of the paper and for the interesting comments that have significantly improved the paper. Indeed Section 5 is due to a suggestion of one of the referees.

Funding for open access charge: Universidade de Vigo/CISUG.

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