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# Convergence and computation of simultaneous rational quadrature formulas

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**Abstract** We discuss the convergence and numerical evaluation of simultaneous quadrature formulas which are exact for rational functions. The problem consists in integrating a single function with respect to different measures using a common set of quadrature nodes. Given a multi-index $\mathbf{n}$, the nodes of the integration rule are the zeros of the multi-orthogonal Hermite–Padé polynomial with respect to $(S, \alpha, \mathbf{n})$, where $S$ is a collection of measures, and $\alpha$ is a polynomial which modifies the measures in $S$. The theory is based on the connection between Gauss-type simultaneous quadrature formulas of rational type and multipoint Hermite–Padé approximation. The numerical treatment relies on the technique of modifying the integrand by means of a change of variable when it has real poles close to the integration interval. The output of some tests show the power of this approach in comparison with other ones in use.

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## 1 Introduction

The simultaneous integration of a given function $f : [a,b] \to \mathbb{R}$, with respect to $m$ distinct measures $\mathrm{d}s_1, \ldots, \mathrm{d}s_m$ on $[a,b]$ is a problem which appears in computer graphics to determine the color of the light emanating from a given point on a surface towards a viewer (see Borges [2], and also Kershaw [18]). Borges introduces a reference index $\Phi = (Ov + 1)/Nu$ (*performance ratio*) to quantify the efficiency of a simultaneous procedure when the $m$ integration rules have the same precision $Ov$, and $Nu$ is the total number of integrand evaluations determined by the procedure. As the author points out, when $m \geq 3$ the use of Gaussian quadrature on each of the $m$ rules yields $\Phi < 1$ (notice that, in general, the nodes to be used will differ from one integral to the next). This indicates a low performance of this method. Instead, he suggests to use $m$ quadrature rules with the same collection of nodes. To attain this, he selects the nodes as the zeros of polynomials which satisfy orthogonality relations equally distributed between all the measures and proves that then $\Phi > 1$. If one does not distribute the orthogonality conditions equally, the precision varies from one rule to the other, but defining $Ov$ as the minimum degree of precision one still has $\Phi \geq 1$.

From a different perspective, a similar problem was studied earlier by Nikishin [21] in connection with simultaneous Hermite–Padé approximation of systems of Markov functions. Having this in mind, in [7] the authors of the present paper consider simultaneous quadrature formulas, which are exact for polynomials, and prove convergence with geometric rate when $f$ is analytic in a neighborhood of $[a,b]$. Despite the theoretical geometric rate of convergence, instability shows up when the numerical method is applied to integrate meromorphic functions with poles close to $[a,b]$.

In the past few years, several methods have been developed to integrate such meromorphic functions. An interesting approach is based on the use of rational Gaussian integration rules which are connected with multipoint Padé approximation of Markov functions (cf. [9,10,12,17]). Let us outline some general features of rational integration rules.

Let $\{\alpha_N\}_{N=1}^{\infty}$ be a sequence of algebraic polynomials with real coefficients, such that $\deg \alpha_N \leq 2N$ and $\alpha_N(x) > 0$, for all $N \in \mathbb{N}, x \in [a,b]$. Let $\sigma$ be a finite positive Borel measure supported on the interval $[a,b]$ and $x_{N,j}, j = 1, \ldots, N$, distinct points on $[a,b]$. Let $\mathrm{P}_N$ denote the finite dimensional space of all polynomials of degree at most $N$. We say that

$$\int_a^b f(x)\,\mathrm{d}\sigma(x) \approx \sum_{j=1}^{N} \lambda_{N,j} f(x_{N,j}), \tag{1}$$

is an interpolatory quadrature formula of rational type with respect to $\alpha_N$ if equality holds in (1) for all $f = P/\alpha_N, P \in \mathrm{P}_{N-1}$. If equality takes place for all $f = P/\alpha_N, P \in \mathrm{P}_{2N-1}$, it is said to be a Gaussian rational quadrature formula (GRQF). When $\alpha_N \equiv 1$ we obtain classical polynomial schemes.

The characterization of the nodes and coefficients of a rational Gaussian rule can be easily reduced to the polynomial case. The nodes are the zeros of the $N$th orthogonal polynomial $Q_N(z) = \prod_{j=1}^{N}(z - x_{N,j})$ with respect to the (varying) measure $\mathrm{d}\sigma/\alpha_N$ and the generalized Christoffel coefficients are given by

$$\lambda_{N,j} = \alpha_N(x_{N,j}) \int \left( \frac{Q_N(x)}{Q'_N(x_{N,j})(x - x_{N,j})} \right)^2 \frac{\mathrm{d}\sigma(x)}{\alpha_N(x)}.$$

In [11], Gautschi refers to ([8],1993) as the first paper dealing with GRQR. Nevertheless, such quadrature rules were earlier introduced in ([13],1978) in connection with the convergence of multipoint Padé approximants of Markov functions and employed in [15–17] to study the convergence of rational quadrature rules.

Let us assume that $f$ is meromorphic in a neighborhood $V$ of $[a,b]$, and $\{z : f(z) = \infty\} \subset V \backslash [a,b]$. Under these assumptions, the efficiency of any numerical procedure of type (1) heavily depends on the design of the polynomials $\alpha_N$. The presence of poles close to $[a,b]$ causes slow convergence, so we should choose the zeros of $\alpha_N$ appropriately. Making some of the zeros of $\alpha_N$ coincide with those "difficult" poles of $f$ (cf. [9,10,12]) can have a very good effect. The reason for this is explained in the second part of Remark 2 below.

In order to appraise the nature of this approach, consider a triangular array of complex numbers $\mathbf{A} = \{z_{N,j};\, j = 1,\ldots,2N,\ N \in \mathbb{N}\}$, $\mathbf{A} \subset \overline{\mathbb{C}} \backslash [a,b]$, such that all its rows are symmetric with respect to the real axis (counting multiplicities). The polynomials $\alpha_N$ are constructed from table $\mathbf{A}$ by

$$\alpha_N(x) = D_N \prod_{j=1}^{2N} \left( x - z_{N,j} \right), \quad D_N \in \mathbb{R},\ \ N \in \mathbb{N}.$$

By convention, we define $(x - \infty \equiv 1)$. Notice that all these polynomials have real coefficients. For numerical purposes the factor $D_N$ scales $\alpha_N$.

For the moment we assume: $z_{N,\nu} \in \mathbb{R} \backslash [a,b]$, if $\nu = 1,\ldots,N$, and $z_{N,\nu} = \infty$ if $\nu = N+1,\ldots,2N$, $N \in \mathbb{N}$. In addition, $z_{N,\nu} \neq z_{N,\eta}, 1 \leq \nu < \eta \leq N$, and $D_N = 1$. Choose $N$ distinct points $x_{N,1},\ldots,x_{N,N} \subset [a,b]$. In order to obtain an interpolation formula of rational type with respect to $\alpha_N$ it is sufficient to solve the following linear system

$$\int_a^b \frac{\mathrm{d}\sigma(x)}{(x - z_{N,\nu})} = \sum_{j=1}^{N} \frac{\lambda_{N,j}}{(x_{N,j} - z_{N,\nu})}, \quad \nu = 1,\ldots,N, \tag{2}$$

with respect to the unknowns $\lambda_{N,j}$. The determinant of the system is well known to be non zero. Indeed, the polynomials $P_\nu(x) = \prod_{j \neq \nu}(x - z_{N,j})$, $\nu = 1,\ldots,N$, form a basis in $\mathrm{P}_{N-1}$ and $P_\nu(x)/\alpha_N(x) = (x - z_{N,\nu})^{-1}$.

When developing numerical integration schemes based on (2), two questions must be taken into consideration when $x_{N,j} - z_{N,\nu}$ is close to zero: (a) the condition number of the matrix associated with the system (2) becomes large; and (b) numerical instability is detected when the integral in the left side of (2) is calculated.

In the case of Gaussian rules the nodes and coefficients should not be calculated from the system (2) (which is now non linear and underdetermined) but using the corresponding Jacobi matrix (cf. [9,10,17,24]). The numerical method to be applied is based on recursion formulas which in turn require a discretization procedure to evaluate accurately integrals of the form

$$\int_a^b \frac{P(x)}{\alpha_N(x)}\,\mathrm{d}\sigma(x), \tag{3}$$

where $P$ is a polynomial and some zeros of $\alpha_N$ may be poles of a given integrand.

Suppose we have a collection of measures $S = (s_1, \ldots, s_m)$ supported on $[a, b]$ and we wish to integrate $f$ simultaneously with respect to all of them. Since the function is the same, it is natural to take the same $\alpha_N$ in the quadrature formulas to be used. Certainly, one can use independent quadrature procedures for each integral. There are several inconveniences. If we choose a fixed set of $N$ nodes for all the procedures, in order to reduce the number of evaluations of the function, we can only expect to have precision of order $N$ on each process. On the contrary, if we select its nodes to be the zeros of the $N$-th orthogonal polynomial with respect to $\mathrm{d}s_k/\alpha_N, k = 1, \ldots, m$, to evaluate the $k$-th integral, so as to increase to $2N$ the order of precision in each formula, we obtain different nodes for each integral and we will be forced to produce in general $mN$ evaluations of $f$. We can also single out one of the measures (say $s_1$), choose as nodes the zeros of the $N$-th orthogonal polynomial with respect to $\mathrm{d}s_1/\alpha_N$ and use those nodes in all the procedures, but then we only improve the precision in one of the rules. Borges suggests to make all the measures intervene in the selection of the nodes. One way of doing so is to distribute the orthogonality conditions between all the measures.

Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m, \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and set $|\mathbf{n}| = n_1 + \cdots + n_m$. Fix a polynomial with real coefficients $\alpha_{\mathbf{n}}, \alpha_{\mathbf{n}}(x) \neq 0, x \in [a, b]$. Let $Q_{\mathbf{n}}$ be the monic polynomial of least degree ($\leq |\mathbf{n}|$) satisfying

$$\int_a^b x^\nu Q_{\mathbf{n}}(x) \frac{\mathrm{d}s_k(x)}{\alpha_{\mathbf{n}}(x)} = 0, \quad \nu = 0, \ldots, n_k - 1, \;\; k = 1, \ldots, m. \tag{4}$$

The polynomial $Q_{\mathbf{n}}$ is uniquely determined. In general, its degree may be $< |\mathbf{n}|$, the zeros could lie outside the interval $[a, b]$ or have multiplicity greater than 1. These facts are important drawbacks if we plan to use these zeros as nodes of quadrature rules on $[a, b]$.

A multi-index is said to be *normal* if $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. If, additionally, its zeros are simple and lie on $[a,b]$ the multi-index is said to be *strongly normal*. Let us make some restrictions on the collection of measures to guarantee strong normality. Assume that $\mathrm{d}s_k(x) = w_k(x)\mathrm{d}\sigma(x), x \in [a,b], k = 1,\ldots,m$, where $\sigma$ is a finite positive Borel measure on $[a,b]$ whose support contains infinitely many points.

**Definition 1** We say that the system of functions $(w_1,\ldots,w_m)$ is an AT-system for the multi-index $\mathbf{n} = (n_1,\ldots,n_m)$ on the interval $[a,b]$ if for every non-trivial collection of polynomials $P_1,\ldots,P_m$ with deg $P_j \le n_j - 1$, the function

$$\mathrm{H}(x) = \mathrm{H}(P_1,\ldots,P_m;x) = P_1(x)w_1(x) + \cdots + P_m(x)w_m(x),$$

has at most $|\mathbf{n}| - 1$ zeros on $[a,b]$.

The concept of AT system was introduced by Nikishin [21]. It means that the collection of functions

$$(w_1(x),\ldots,x^{n_1-1}w_1(x),\ldots,w_m(x),\ldots,x^{n_m-1}w_m(x))$$

forms an integrable Markov system on $[a,b]$. For example (see [14], pp. 37–39), $(e^{\alpha_1},\ldots,e^{\alpha_m})$, where $\alpha_1 < \cdots < \alpha_m$, is an AT system with respect to any multi-index on any interval of the real line. It is easy to prove that if $(w_1,\ldots,w_m)$ is an AT-system for the multi-index $\mathbf{n} = (n_1,\ldots,n_m)$ on the interval $[a,b]$ then $Q_{\mathbf{n}}$ has exactly $|\mathbf{n}|$ simple zeros in $(a,b)$.

We investigate the convergence of Gauss-type simultaneous quadrature formulas, exact for rational functions. The paper is organized as follows.

The definition of simultaneous rational quadrature formula (SRQF) and some aspects of simultaneous multipoint Hermite–Padé approximation are laid out in Sect. 2. Using known techniques of rational interpolation, we prove convergence of SRQF for certain general classes of functions. These results are obtained assuming strong or weak stability of the quadrature rule and strong normality of the multi-indices. The problem of obtaining sufficient conditions so that these assumptions are fulfilled is discussed in Sects. 3 and 4.

In Sect. 3, we study the case of sequences of multi-indices of the form $\mathbf{n} = (0,\ldots,0,N,0,\ldots,0), N \in \mathbb{Z}_+$, which reduces to the problem of evaluating two integrals simultaneously, one of them by means of a rational Gaussian rule, and the other by an interpolation quadrature formula.

Section 4 is dedicated to the so called Nikishin systems of measures which were introduced in [21]. Here, we extend some results on simultaneous integration rules exact for polynomials, obtained earlier by the authors in [7], to the rational version.

Section 5 presents a numerical method which allows to evaluate simultaneously three different integrals of a single function with real poles close to the integration interval. The efficiency of the method is also based on the use of smoothing transformations to compute (3) in contrast with Gautschi's approach which basically relies on iterative algorithms. This section also describes correction techniques to make all integrators participate equally and to reduce instability. The numerical results obtained in two examples are compared with those given in earlier papers where Gaussian rules of rational type are applied.

## 2 Simultaneous rational quadrature formulas

In the previous section, we pointed out that rational quadrature formulas of Gauss type arise from the theory of multipoint Padé approximation of Markov functions. Similarly, simultaneous rational quadrature formulas are intimitely connected with multipoint Hermite–Padé approximation of Markov systems of functions.

Let $m \in \mathbb{N}$ be fixed and $S = (s_1, \dots, s_m)$ be a system of $m$ finite measures with constant sign on $[a, b]$. From $S$ we construct $\widehat{S} = (\widehat{s}_1, \dots, \widehat{s}_m)$, the corresponding system of Markov functions; namely,

$$\widehat{s}_k(z) = \int_a^b \frac{\mathrm{d}s_k(x)}{z - x}, \quad k = 1, \dots, m.$$

Notice that all the functions $\widehat{s}_k$ are holomorphic in $D = \overline{\mathbb{C}} \setminus [a, b]$. We restrict our attention to the case $\mathrm{d}s_k(x) = w_k(x)\mathrm{d}\sigma(x)$, where $w_k$, $k = 1, \dots, m$, are weight functions and $\sigma$ is a finite positive Borel measure on $[a, b]$ whose support contains infinitely many points. By a weight we mean a continuous function with constant sign on $(a, b)$, with possibly integrable singularities at the end points $x = a, b$.

Fix a sequence of distinct multi-indices $\Lambda \subset \mathbb{Z}_+^m$. For $k = 1, \dots, m$, let

$$\mathbf{A}_k = \{z_{k,\mathbf{n},j} : \mathbf{n} \in \Lambda, j = 1, \dots, |\mathbf{n}| + n_k = K_{\mathbf{n},k}\},$$

where $\mathbf{A}_k \subset \overline{\mathbb{C}} \setminus [a, b]$, is stable by complex conjugation. We will consider $m$ quadrature formulas of rational type, so we need $m$ polynomial sequences as those given below $(x - \infty \equiv 1)$

$$\alpha_{\mathbf{n},k}(x) = \prod_{j=1}^{K_{\mathbf{n},k}} (x - z_{k,\mathbf{n},j}). \tag{5}$$

Most frequently, we can consider only one table $\mathbf{A}$ and $\mathbf{A}_k = \mathbf{A}, k = 1, \dots, m$, and by $\alpha_{\mathbf{n}}$ we denote the associated polynomials. Nevertheless, it is convenient to allow the polynomials to depend on $k$.

Given the polynomials $\alpha_{\mathbf{n},k}$, $k = 1, \dots, m$, there exists a unique vector of rational functions $R_{\mathbf{n}} = (R_{\mathbf{n},1}, \dots, R_{\mathbf{n},m}), R_{\mathbf{n},k} = P_{\mathbf{n},k}/Q_{\mathbf{n}}$, $k = 1, \dots, m$, with a monic common denominator $Q_{\mathbf{n}}$ of least possible degree such that:

$$\deg P_{\mathbf{n},k} \le |\mathbf{n}| - 1, \quad \deg Q_{\mathbf{n}} \le |\mathbf{n}|, \;\; Q_{\mathbf{n}} \not\equiv 0, \tag{6}$$

$$\frac{Q_{\mathbf{n}}(z)\hat{s}_k(z) - P_{\mathbf{n},k}(z)}{\alpha_{\mathbf{n},k}(z)} = \mathrm{O}\left(\frac{1}{z^{n_k+1}}\right), \quad k = 1, 2, \ldots, m. \tag{7}$$

In fact, existence is easily guaranteed solving the homogeneous linear system of equations determined by (6)–(7) which has one more unknown than equations. Assume that there are two distinct solutions with monic common denominators of least degree given by $(Q_{\mathbf{n}}; P_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m}), (\widetilde{Q}_{\mathbf{n}}; \widetilde{P}_{\mathbf{n},1}, \ldots, \widetilde{P}_{\mathbf{n},m})$. If $Q_{\mathbf{n}} \equiv \widetilde{Q}_{\mathbf{n}}$, from (7) it follows that

$$\frac{\widetilde{P}_{\mathbf{n},k}(z) - P_{\mathbf{n},k}(z)}{\alpha_{\mathbf{n},k}(z)} = \mathrm{O}\left(\frac{1}{z^{n_k+1}}\right).$$

This implies that $\widetilde{P}_{\mathbf{n},k} \equiv P_{\mathbf{n},k}$, in contradiction with our assumption that the solutions were distinct. So, the monic common denominators of the two distinct solutions must be different and at the same time have the same (minimal) degree. Then, it is easy to verify that $(Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}}; P_{\mathbf{n},1} - \widetilde{P}_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m} - \widetilde{P}_{\mathbf{n},m})$ also satisfies (6), (7) and the common denominator $Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}}$ is not identically equal to zero. Let $c_{\mathbf{n}}$ be an appropriate constant so that $c_{\mathbf{n}}(Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}})$ is monic. Then $(c_{\mathbf{n}}(Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}}); c_{\mathbf{n}}(P_{\mathbf{n},1} - \widetilde{P}_{\mathbf{n},1}), \ldots, c_{\mathbf{n}}(P_{\mathbf{n},m} - \widetilde{P}_{\mathbf{n},m}))$ is also a solution of (6), (7) and the common denominator has smaller degree against the assumption that our initial solutions were of least possible degree. With this contradiction, we conclude the proof of uniqueness. We wish to underline that without the assumption that the common denominator have least possible degree, in general, there is not a unique vector of rational functions satisfying (6),(7).

$Q_{\mathbf{n}}$ is called the *Hermite–Padé polynomial* associated to $(S, (\alpha_{\mathbf{n},k})_{k=1}^{m})$, and $R_{\mathbf{n}}$ is the *simultaneous multipoint Hermite–Padé approximant*.

From (7), it follows that

$$\frac{z^{\nu}(Q_{\mathbf{n}}\hat{s}_k - P_{\mathbf{n},k})(z)}{\alpha_{\mathbf{n},k}(z)} = \mathrm{O}\left(\frac{1}{z^2}\right), \quad k = 1, 2, \ldots, m, \;\; \nu = 0, \ldots, n_k - 1,$$

and the function on the left is analytic in the complement of $[a, b]$. Take a smooth integration path $\Gamma$ with winding number 1 with respect to all its interior points which surrounds $[a, b]$ in such a way that the zeros of $\alpha_{\mathbf{n},k}$ lie outside $\Gamma$. Integrating over $\Gamma$, from Cauchy's and Fubini's theorems it follows that

$$0 = \int_{\Gamma} \frac{z^{\nu}(Q_{\mathbf{n}}\hat{s}_k - P_{\mathbf{n},k})(z)}{\alpha_{\mathbf{n},k}(z)}\,\mathrm{d}z = \int_a^b \int_{\Gamma} \frac{z^{\nu} Q_{\mathbf{n}}(z)}{\alpha_{\mathbf{n},k}(z)} \frac{\mathrm{d}z}{z - x}\,\mathrm{d}s_k(x)$$

$$= 2\pi i \int_a^b x^{\nu} Q_{\mathbf{n}}(x) \frac{\mathrm{d}s_k(x)}{\alpha_{\mathbf{n},k}(x)}, \quad \nu = 0, \ldots, n_k - 1,$$

which amounts to (4).

The Hermite–Padé polynomial $Q_{\mathbf{n}}$ can also be defined as the monic polynomial of least degree which satisfies (6) and (4). $Q_{\mathbf{n}}$ is also called the $\mathbf{n}$-th multi-orthogonal polynomial associated to the pair $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$.

**Definition 2** A multi-index $\mathbf{n}$ is said to be normal for $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$ if any non trivial solution $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. If $Q_{\mathbf{n}}$ has exactly $|\mathbf{n}|$ simple zeros and all of them lie in the interior of $[a,b]$ the multi-index is called strongly normal for $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$.

An elementary example of strongly normal multi-index $\mathbf{n}$ is that of the form $n_k = N$, and $n_i = 0$ for $i \neq k$ (see Sect. 3). In the sequel, we will be interested only in subsequences $\Lambda \subset \mathbb{Z}_+^m$ of strongly normal multi-indices. Set $W_{\mathbf{n},k} = w_k/\alpha_{\mathbf{n},k}$, $k = 1, \ldots, m$.

**Definition 3** We say that the system of functions $(W_{\mathbf{n},1}, \ldots, W_{\mathbf{n},m})$ is an AT-system for the multi-index $\mathbf{n} = (n_1, \ldots, n_m)$ on $[a,b]$ if for every non-trivial collection of polynomials $P_1,\ldots,P_m$ with $\deg P_j \leq n_j - 1$, the function

$$\mathrm{H}(x) = \mathrm{H}(P_1, \ldots, P_m, x) = P_1(x)W_{\mathbf{n},1}(x) + \cdots + P_m(x)W_{\mathbf{n},m}(x),$$

has at most $|\mathbf{n}| - 1$ zeros in $[a,b]$.

Notice that if $\alpha_{\mathbf{n},k} = \alpha_{\mathbf{n}}, k = 1, \ldots, m$, we can factor out $\alpha_{\mathbf{n}}$ in the previous sum and we conclude that $(W_{\mathbf{n},1}, \ldots, W_{\mathbf{n},m})$ is an AT-system for the multi-index $\mathbf{n}$ if and only if $(w_1, \ldots, w_m)$ is an AT-system for $\mathbf{n}$ (see Definition 1).

**Proposition 1** *If* $(W_{\mathbf{n},1}, \ldots, W_{\mathbf{n},m})$ *is an AT-system for the multi-index* $\mathbf{n} = (n_1, \ldots, n_m)$ *on* $[a,b]$ *then* $\mathbf{n}$ *is strongly normal for* $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$.

*Proof* Indeed, assume that $Q_{\mathbf{n}}$ has less than $|\mathbf{n}|$ sign changes on $(a,b)$. By Theorem 1.3, page 36, of [14], we can find polynomials $P_1, \ldots, P_m$ such that $\mathrm{H}(P_1, \ldots, P_m, x)$ has a simple zero at each point of $(a,b)$ where $Q_{\mathbf{n}}$ changes sign and no other zero on that interval. Then $Q_{\mathbf{n}}(x)\mathrm{H}(P_1, \ldots, P_m, x)$ has constant sign on $[a,b]$ whereas by orthogonality

$$\int_a^b Q_{\mathbf{n}}(x)\mathrm{H}(P_1, \ldots, P_m, x)\, \mathrm{d}\sigma(x) = 0.$$

Since the support of $\sigma$ contains infinitely many points this is impossible.

There are integral formulas for the remainders and the polynomials $P_{\mathbf{n},k}$ in terms of $Q_{\mathbf{n}}$ and $\alpha_{\mathbf{n},k}$.

**Proposition 2** *Let* $q$ *be an arbitrary polynomial of degree* $\leq n_k$, *then*

$$\frac{Q_{\mathbf{n}}(z)\hat{s}_k(z) - P_{\mathbf{n},k}(z)}{\alpha_{\mathbf{n},k}(z)} = \int_a^b \frac{q(x)Q_{\mathbf{n}}(x)}{q(z)(z-x)}\, \mathrm{d}s_{\mathbf{n},k}(x). \tag{8}$$

$$P_{\mathbf{n},k}(z) = \int_a^b \frac{Q_{\mathbf{n}}(z)\alpha_{\mathbf{n},k}(x) - Q_{\mathbf{n}}(x)\alpha_{\mathbf{n},k}(z)}{z-x}\, \mathrm{d}s_{\mathbf{n},k}(x), \tag{9}$$

*where* $\mathrm{d}s_{\mathbf{n},k}(x) = w_k(x)\mathrm{d}\sigma(x)/\alpha_{\mathbf{n},k}(x)$, $k = 1, \ldots, m$.

*Proof* Let $Q_{\mathbf{n}}$ be the Hermite–Padé polynomial associated to $(S, (\alpha_{\mathbf{n}})_{k=1}^m)$. Define $P_{\mathbf{n},k}$ according to (9) (not from the definition of Hermite–Padé approximant). Rearranging (9) we obtain (8) for $q \equiv 1$. Using (4) we have that for any $q, \deg q \le n_k$,

$$\int_a^b \frac{q(z) - q(x)}{z - x} Q_{\mathbf{n}}(x)\, ds_{\mathbf{n},k}(x) = 0.$$

Combining this with (8) for $q \equiv 1$ we obtain (8) in general. Taking $q(z) = z^{n_k}$ one sees that the right hand side of (8) satisfies (7). To conclude we must show that $\deg P_{\mathbf{n},k} \le |\mathbf{n}| - 1$ when $P_{\mathbf{n},k}$ is given by (9).

In fact, factorize $\alpha_{\mathbf{n},k}(x)$ as follows

$$\alpha_{\mathbf{n},k}(x) = \widetilde{\alpha}_{\mathbf{n},k}(x)\widetilde{\widetilde{\alpha}}_{\mathbf{n},k}(x), \quad \deg \widetilde{\alpha}_{\mathbf{n},k} \le n_k, \quad \deg \widetilde{\widetilde{\alpha}}_{\mathbf{n},k} \le |\mathbf{n}|\,.$$

(Any such factorization will do.) We can rewrite (9) as follows

$$P_{\mathbf{n},k}(z) = \int_a^b \frac{(Q_{\mathbf{n}}(z)\alpha_{\mathbf{n},k}(x) \pm Q_{\mathbf{n}}(x)\widetilde{\alpha}_{\mathbf{n},k}(x)\widetilde{\widetilde{\alpha}}_{\mathbf{n},k}(z) - Q_{\mathbf{n}}(x)\alpha_{\mathbf{n},k}(z))}{(z - x)}\, ds_{\mathbf{n},k}(x).$$

Separating conveniently the last integral in two and using (4) we obtain

$$P_{\mathbf{n},k}(z) = \int_a^b \widetilde{\alpha}_{\mathbf{n},k}(x) \left( \frac{Q_{\mathbf{n}}(z)\widetilde{\widetilde{\alpha}}_{\mathbf{n},k}(x) - Q_{\mathbf{n}}(x)\widetilde{\widetilde{\alpha}}_{\mathbf{n},k}(z)}{z - x} \right) ds_{\mathbf{n},k}(x)$$

which is obviously a polynomial in $z$ of degree $\le |\mathbf{n}| - 1$ as needed. □

We are ready to produce the simultaneous quadrature formulas that we will use. We assume that the multi-index $\mathbf{n}$ is strongly normal with respect to $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$. Then, $Q_{\mathbf{n}}$ has $|\mathbf{n}|$ simple zeros $x_{\mathbf{n},1}, \ldots, x_{\mathbf{n},|\mathbf{n}|}$ lying on $[a, b]$; therefore,

$$R_{\mathbf{n},k}(z) = \frac{P_{\mathbf{n},k}(z)}{Q_{\mathbf{n}}(z)} = \sum_{j=1}^{|\mathbf{n}|} \frac{\lambda_{\mathbf{n},k,j}}{z - x_{\mathbf{n},j}}\,, \quad k = 1, \ldots, m\,, \tag{10}$$

and using (9),

$$\begin{aligned} \lambda_{\mathbf{n},k,j} &= \lim_{z \to x_{\mathbf{n},j}} (z - x_{\mathbf{n},j}) R_{\mathbf{n},k}(z) \\ &= \alpha_{\mathbf{n},k}(x_{\mathbf{n},j}) \int_a^b \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})}\, ds_{\mathbf{n},k}(x)\,, \end{aligned} \tag{11}$$

for $j = 1, \ldots, |\mathbf{n}|$, $k = 1, \ldots, m$. The coefficients defined by (11) will be called Nikishin–Christoffel coefficients.

**Theorem 1** *Assume that* $\mathbf{n}$ *is strongly normal with respect to* $(S, (\alpha_{\mathbf{n},k})_{k=1}^m)$. *Then, for each* $k = 1, \ldots, m$,

$$\int_a^b \frac{p(x)}{\alpha_{\mathbf{n},k}(x)} \, \mathrm{d}s_k(x) = \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,j} \frac{p(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},k}(x_{\mathbf{n},j})}, \quad p \in \mathrm{P}_{|\mathbf{n}|+n_k-1}. \tag{12}$$

*Proof* Let $p \in \mathrm{P}_{|\mathbf{n}|+n_k-1}$ and L be the Lagrange polynomial which interpolates $p$ at the zeros of $Q_{\mathbf{n}}$. Then, $p - \mathrm{L} = qQ_{\mathbf{n}}, \deg q \leq n_k - 1$. By (4)

$$\int_a^b (p - \mathrm{L})(x) \, \mathrm{d}s_{\mathbf{n},k}(x) = 0 \, .$$

Since

$$\mathrm{L}(x) = \sum_{j=1}^{|\mathbf{n}|} \frac{Q_{\mathbf{n}}(x) p(x_{\mathbf{n},j})}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} \, ,$$

we obtain

$$\int_a^b \frac{p(x)}{\alpha_{\mathbf{n},k}(x)} \, \mathrm{d}s_k(x) = \sum_{j=1}^{|\mathbf{n}|} p(x_{\mathbf{n},j}) \int_a^b \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} \, \mathrm{d}s_{\mathbf{n},k}(x)$$

and (12) follows from (11). □

When $\deg \alpha_{\mathbf{n},k} \leq |\mathbf{n}| + n_k - 1$, (12) gives a mixed polynomial-rational rule. This means that in each row of $\mathbf{A}_k$ we place $\infty$ at least one time. In particular, it allows the constants to be in the space of functions for which the quadrature rule is exact. This is very convenient, and in practice imposes no real restriction. So, in the sequel, we will suppose that our tables $\mathbf{A}_k$ satisfy this condition.

Assume that $\Lambda \subset \mathbb{Z}_+^m$ is a sequence of distinct multi-indices such that each $\mathbf{n} \in \Lambda$ is strongly normal with respect to $(S, (\alpha_{\mathbf{n}})_{k=1}^m)$. Fix $k \in \{1, \ldots, m\}$. In what follows, we discuss the role played by the coefficients $(\lambda_{\mathbf{n},k,j})_{j=1}^{|\mathbf{n}|}$ in the convergence properties (cf. [7]) of the quadrature rules assuming one of the following conditions.

(A) For each $\mathbf{n} \in \Lambda$ all $\lambda_{\mathbf{n},k,j}$, $j = 1, \ldots, |\mathbf{n}|$, have the same sign as $s_k$.
(B) $\sup_{\mathbf{n} \in \Lambda} \sum_{j=1}^{|\mathbf{n}|} |\lambda_{\mathbf{n},k,j}| \leq C < \infty$.
(C) $\sum_{j=1}^{|\mathbf{n}|} |\lambda_{\mathbf{n},k,j}| \leq \kappa(n)$, where $\lim_{\mathbf{n} \in \Lambda} \kappa(n)^{1/(|\mathbf{n}|+n_k)} = 1$.

Obviously (A) ⇒ (B) ⇒ (C). In (A) ⇒ (B) we use that $\deg \alpha_{\mathbf{n},k} \leq |\mathbf{n}| + n_k - 1$ and taking $p = \alpha_{\mathbf{n},k}$ in (12), we have $\int_a^b \mathrm{d}s_k(x) = \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,j}$. The question is, under which conditions do we have

$$\lim_{\mathbf{n}\in\Lambda,\ |\mathbf{n}|\to\infty} \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,j} f(x_{\mathbf{n},j}) = \int_a^b f(x) w_k(x)\, \mathrm{d}\sigma(x) \tag{13}$$

for every $f$ in a certain general class of functions? This problem can be solved depending on the assumptions on $f$, $\mathbf{n}$, $\mathbf{A}_k$, and the behavior of $\sum_j |\lambda_{\mathbf{n},k,j}|$.

**Definition 4** Let $\mathbf{A}_k = \{z_{k,\mathbf{n},j} : \mathbf{n} \in \Lambda \subset \mathbb{Z}_+^m, j = 1, \ldots, K_{\mathbf{n},k} = |\mathbf{n}| + n_k\}$. We say that $\mathbf{A}_k$ has the density property with respect to the interval $[a, b]$, if for each $f \in \mathrm{C}[a, b]$ (the space of complex continuous functions on $[a, b]$ equipped with the uniform norm) there exists a sequence of polynomials $\{p_{\mathbf{n}}\}_{\mathbf{n}\in\Lambda}$ such that

$$\lim_{\mathbf{n}\in\Lambda} \frac{p_{\mathbf{n}}}{\alpha_{\mathbf{n},k}} = f$$

uniformly in $[a, b]$

*Remark 1* 1. It is well known that a table $\mathbf{A}_k$ has the density property with respect to the interval $[a, b]$ if it is contained in a compact subset of $\overline{\mathbb{C}} \setminus [a, b]$. Moreover, let $\varphi$ denote the conformal representation of $\overline{\mathbb{C}} \setminus [a, b]$ onto the unit disk such that $\varphi(\infty) = 0, \varphi'(\infty) > 0$. The density property takes place (see (16) and Corollary 1 in [3]) if

$$\lim_{\mathbf{n}\in\Lambda} \sum_{j=1}^{K_{\mathbf{n},k}} (1 - |\varphi(z_{k,\mathbf{n},j}|) = \infty\,.$$

2. The table $\mathbf{A}_k$ is called newtonian if for each $\mathbf{n} \in \Lambda$ we have that $\alpha_{\mathbf{n},k}$ divides $\alpha_{\mathbf{n}_+,k}$ where $\mathbf{n}_+$ denotes the multi-index in $\Lambda$ which immediately follows $\mathbf{n}$. In other words, in a newtonian table each row contains the points in the previous one and, therefore, $\mathbf{A}_k$ can be identified with a sequence of points $(z_N)_{N\in\mathbb{Z}_+}$, and an increasing sequence of numbers $K_{\mathbf{n},k}, \mathbf{n} \in \Lambda$, so that the $\mathbf{n}$th row in $\mathbf{A}_k$ is formed by the first $K_{\mathbf{n},k}$ points of the sequence of points. An immediate consequence of the preceding remark is that a newtonian table $\mathbf{A}_k$ has the density property with respect to the interval $[a, b]$ if $\lim_{N\to\infty} z_N \in \overline{\mathbb{C}} \setminus [a, b]$.

There are two basic ways of obtaining convergence for these quadrature formulas. Before stating the corresponding results let us introduce some notation. By

$$\mathrm{E}_{\mathbf{n},k}(f) = \left| \int_a^b f(x)\, \mathrm{d}s_k(x) - \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,j} f(x_{\mathbf{n},j}) \right|,$$

we denote the quadrature error, and

$$\mathbf{R}_{|\mathbf{n}|+n_k-1}(f) = \inf_{p\in \mathrm{P}_{|\mathbf{n}|+n_k-1}} \left\| f - \frac{p}{\alpha_{\mathbf{n},k}} \right\|_{[a,b]}$$

denotes the best uniform approximation in the given class of rational functions.

**Proposition 3** *Let $S$=$(s_1,\ldots,s_m)$ be a system of measures with constant sign on [a,b] and $\Lambda \subset \mathbb{Z}_+^m$ a sequence of distinct multi-indices such that each $\mathbf{n} \in \Lambda$ is strongly normal with respect to $(S,(\alpha_{\mathbf{n},k})_{k=1}^m)$. Fix $k \in \{1,\ldots,m\}$. If f is defined on [a,b] and integrable with respect to $s_k$, then*

$$\left|\mathrm{E}_{\mathbf{n},k}(f)\right| \le \left( \left| \int_a^b \mathrm{d}s_k(x) \right| + \sum_{j=1}^{|\mathbf{n}|} |\lambda_{\mathbf{n},k,j}| \right) \mathbf{R}_{|\mathbf{n}|+n_k-1}(f)\,, \tag{14}$$

*In particular, if (B) takes place and $\mathbf{A}_k$ has the density property on [a,b], then (13) holds for all f $\in$ C[a,b].*

*Proof* In fact, since the multi-indices are strongly normal, and $f$ is defined and integrable, the quadrature formula makes sense. Let $p \in \mathrm{P}_{|\mathbf{n}|+n_k-1}$ be given. Using (12), it follows that

$$\left|\mathrm{E}_{\mathbf{n},k}(f)\right| \le \int_a^b \left| f(x) - \frac{p(x)}{\alpha_{\mathbf{n},k}(x)} \right| |\mathrm{d}s_k(x)| + \sum_{j=1}^{|\mathbf{n}|} \left| \lambda_{\mathbf{n},k,j} \left( f(x_{\mathbf{n},j}) - \frac{p(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},k}(x_{\mathbf{n},j})} \right) \right| .$$

Taking the sup norm and then the infimum over all $p \in \mathrm{P}_{|\mathbf{n}|+n_k-1}$, we obtain (14). If (B) takes place, the expression under the parenthesis in (14) is uniformly bounded and convergence follows from the assumption that $\mathbf{A}_k$ has the density property. □

In order to obtain estimates of the rate of convergence for different classes of functions we have to analyze the behavior of $\mathbf{R}_{|\mathbf{n}|+n_k-1}(f), \mathbf{n} \in \Lambda$. Some results in this direction may be found in [22].

The second approach is fit for functions which are analytic on a neighborhood of the interval $[a,b]$.

**Proposition 4** *Let $S = (s_1,\ldots,s_m)$ be a system of measures with constant sign on $[a,b]$ and $\Lambda \subset \mathbb{Z}_+^m$ a sequence of distinct multi-indices such that each $\mathbf{n} \in \Lambda$ is strongly normal with respect to $(S,(\alpha_{\mathbf{n},k})_{k=1}^m)$. Fix $k \in \{1,\ldots,m\}$. If $f \in \mathrm{H}(V)$, the space of analytic functions in a neighborhood V of $[a,b]$, and $\Gamma$ is a smooth*

*Jordan curve contained in V which surrounds* $[a,b]$, *then*

$$\left|\mathrm{E}_{\mathbf{n},k}(f)\right| \leq \frac{\ell_\Gamma}{2\pi}\left\|f(\hat{s}_k - R_{\mathbf{n},k})\right\|_\Gamma \leq \frac{\ell_\Gamma}{2\pi}\|f\|_\Gamma\left\|\hat{s}_k - R_{\mathbf{n},k}\right\|_\Gamma, \tag{15}$$

*where* $\ell_\Gamma$ *denotes the length of* $\Gamma$ *and* $R_{\mathbf{n},k}$ *is the k-th component of the simultaneous multi-point Hermite–Padé approximant. In particular, if C) takes place and* $\mathbf{A}_k \subset F$ *where F is a compact subset of* $\overline{\mathbb{C}} \setminus [a,b]$*, then there exists* $\delta(V), 0 < \delta(V) < 1$, *such that for all* $f \in \mathrm{H}(V)$

$$\limsup_{\mathbf{n}\in\Lambda}\left|\mathrm{E}_{\mathbf{n},k}(f)\right|^{1/(|\mathbf{n}|+n_k)} \leq \delta(V).$$

*Proof* L Let $\Gamma$ be as described above. Using Cauchy's integral formula and Fubini's Theorem, we have

$$\left|\mathrm{E}_{\mathbf{n},k}(f)\right| = \frac{1}{2\pi}\left|\int_\Gamma f(z)\left(\int_a^b \frac{\mathrm{d}s_k(x)}{z-x} - \sum_{j=1}^{|\mathbf{n}|}\frac{\lambda_{\mathbf{n},k,j}}{z - x_{\mathbf{n},j}}\right)\mathrm{d}z\right|. \tag{16}$$

Therefore, (see (10))

$$\left|\mathrm{E}_{\mathbf{n},k}(f)\right| \leq \frac{\ell_\Gamma}{2\pi}\sup_{z\in\Gamma}|f(z)(\widehat{s}_k - R_{\mathbf{n},k})(z)| \leq \frac{\ell_\Gamma}{2\pi}\sup_{z\in\Gamma}|f(z)|\sup_{z\in\Gamma}|(\widehat{s}_k - R_{\mathbf{n},k})(z)|,$$

which is (15).

Take $\rho < 1$ sufficiently close to 1 so that $\Gamma_\rho = \{z : |\varphi(z)| = \rho\} \subset V$ and $F$ lies in the unbounded component of the complement of $\Gamma_\rho$, where $\varphi$ is as defined in Remark 1. Using the notation in C), it follows that

$$\left\|\hat{s}_k - R_{\mathbf{n},k}\right\|_{\Gamma_\rho} \leq \left\|\hat{s}_k\right\|_{\Gamma_\rho} + \left\|R_{\mathbf{n},k}\right\|_{\Gamma_\rho} \leq \frac{|s_k| + \kappa(\mathbf{n})}{d([a,b],\Gamma_\rho)} := C_\rho(\mathbf{n}), \tag{17}$$

where $|s_k|$ denotes the total variation of $s_k$ and $d([a,b],\Gamma_\rho)$ the distance between $[a,b]$ and $\Gamma_\rho$.

Set

$$\varphi(z,\zeta) = \frac{\varphi(z) - \varphi(\zeta)}{1 - \overline{\varphi(\zeta)}\varphi(z)}, \quad z,\zeta \in \overline{\mathbb{C}}\setminus[a,b].$$

For each fixed $\zeta \in \overline{\mathbb{C}}\setminus[a,b]$, $\varphi(z,\zeta)$ represents conformally $\overline{\mathbb{C}}\setminus[a,b]$ onto the unit circle. Since $\varphi(z)$ extends continuously onto $\overline{\mathbb{C}}$ (considering the interval $[a,b]$ as having two distinct sides depending on whether we approach its points from the upper or lower half planes), the function $|\varphi(z,\zeta)|$ of the two variables $z$ and $\zeta$, can also be extended continuously onto $\overline{\mathbb{C}} \times (\overline{\mathbb{C}} \setminus [a,b])$. This function vanishes only when $z = \zeta$; therefore,

$$\delta_{\inf}(\Gamma_\rho, F) := \inf\{|\varphi(z,\zeta)| : z \in \Gamma_\rho, \zeta \in F\} > 0\,,$$

and, taking into consideration (17),

$$\left\| (\widehat{s}_k(z) - R_{\mathbf{n},k}(z)) / \prod_{j=1}^{|\mathbf{n}|+n_k} \varphi(z, z_{\mathbf{n},j}) \right\|_{\Gamma_\rho} \leq C_\rho(\mathbf{n}) / \delta_{\inf}(\Gamma_\rho, F)^{|\mathbf{n}|+n_k}\,.$$

From the interpolation condition (7) it follows that

$$(\widehat{s}_k(z) - R_{\mathbf{n},k}(z)) / \prod_{j=1}^{|\mathbf{n}|+n_k} \varphi(z, z_{\mathbf{n},j}) \in \mathrm{H}(\overline{\mathbb{C}} \setminus [a,b])\,.$$

Using the maximum modulus principle for analytic functions, for all $z$ in the unbounded component of the complement of $\Gamma_\rho$, we have

$$|\widehat{s}_k(z) - R_{\mathbf{n},k}(z)| \leq C_\rho(\mathbf{n}) / \delta_{\inf}(\Gamma_\rho, F)^{|\mathbf{n}|+n_k} \prod_{j=1}^{|\mathbf{n}|+n_k} |\varphi(z, z_{\mathbf{n},j})|\,.$$

Let

$$\delta_{\sup}(\Gamma, F) := \sup\{|\varphi(z,\zeta)| : z \in \Gamma, \zeta \in F\}\,.$$

Select $\tau, 0 < \tau < \rho$, such that $\Gamma_\tau \subset V$. Then $\delta_{\sup}(\Gamma_\tau, F) < 1$ because $|\varphi(z,\zeta)| = 1$ if and only if $z \in [a,b]$. Therefore,

$$\begin{aligned} \|\widehat{s}_k(z) - R_{\mathbf{n},k}(z)\|_{\Gamma_\tau} &\leq \frac{C_\rho(\mathbf{n})}{\delta_{\inf}(\Gamma_\rho, F)^{|\mathbf{n}|+n_k}} \prod_{j=1}^{|\mathbf{n}|+n_k} \|\varphi(z, z_{\mathbf{n},j})\|_{\Gamma_\tau} \\ &\leq C_\rho(\mathbf{n}) \left( \frac{\delta_{\sup}(\Gamma_\tau, F)}{\delta_{\inf}(\Gamma_\rho, F)} \right)^{|\mathbf{n}|+n_k}\,. \end{aligned}$$

Using (15) and C), we obtain

$$\limsup_{\mathbf{n} \in \Lambda} \left| \mathrm{E}_{\mathbf{n},k}(f) \right|^{1/(|\mathbf{n}|+n_k)} \leq \frac{\delta_{\sup}(\Gamma_\tau, F)}{\delta_{\inf}(\Gamma_\rho, F)}\,.$$

By the continuity of $|\varphi(z,\zeta)|$ on $\overline{\mathbb{C}} \times (\overline{\mathbb{C}} \setminus [a,b])$, it follows that $\lim_{\rho \to 1} \delta_{\inf}(\Gamma_\rho, F) = 1$. Since the left hand side of the previous inequality does not depend on $\rho$ or $\tau$, making $\rho$ tend to 1, we obtain

$$\limsup_{\mathbf{n}\in\Lambda} \left|\mathrm{E}_{\mathbf{n},k}(f)\right|^{1/(|\mathbf{n}|+n_k)} \leq \inf\{\delta_{\sup}(\Gamma_\tau, F) : \Gamma_\tau \subset V\} := \delta(V) < 1\,.$$

With this we conclude the proof.

*Remark 2* 1. The proof of the previous proposition is based on the rate of convergence to $\widehat{s}_k$ of the corresponding Hermite–Padé approximants. In this respect, more precise estimates from the ones above may be obtained for Nikishin systems of measures using results from [5,6].

2. The contribution of $\alpha_{\mathbf{n}}$ to the rate of convergence can be quite effective. Notice that $f(z)(\widehat{s}_k(z) - R_{\mathbf{n},k}(z))$ is under the integral sign of (16). If we interpolate $\widehat{s}_k$ at the poles of $f$ close to $[a, b]$ to a sufficient order, we annihilate their effect. The neighborhood $V$ on which we can integrate becomes larger, we can take $\Gamma_\rho$ further away from $[a, b]$ and $\delta(V)$ becomes smaller. This is used in Sect. 5 (see also [9,10]).

## 3 A subordination condition for a pair of weight functions

As seen above, the convergence of the $k$-th rational quadrature formula of a simultaneous system depends on the behavior of $L_{\mathbf{n},k} = \sum_{j=1}^{|\mathbf{n}|} |\lambda_{\mathbf{n},k,j}|$, $|\mathbf{n}| \to \infty$. Here, we show how some convergence results can be extended to rational quadrature formulas on the basis of a subordination condition imposed on a pair of weight functions. This approach was introduced for polynomial formulas by Sloan and Smith [23] and used recently by the authors in [7]. Using this technique, one can estimate the growth of $L_{\mathbf{n},k}$, $|\mathbf{n}| \to \infty$, corresponding to a non-Gaussian rational quadrature formula, in terms of the analogous sequence of an associated Gaussian rational quadrature formula. Let $k \in \{1, \ldots, m\}$ be fixed. The sequence of strongly normal multi-indices $\mathbf{n}$ to be considered is that for which the $k$-th component $n_k = N$ is a positive integer, and $n_j = 0$ for $j \neq k$.

Without loss of generality, the problem can be reduced to investigating the simultaneous convergence of two rational quadrature formulas ($m = 2$) for multi-indices of the form $\mathbf{n} = (N, 0)$, $N \in \mathbb{N}$. The first rational quadrature formula will be of Gaussian type and the second interpolatory. We consider two sequences of polynomials $\alpha_{\mathbf{n},k}, k = 1, 2, N \in \mathbb{N}, \deg \alpha_{\mathbf{n},1} < 2N, \deg \alpha_{\mathbf{n},2} < N$, associated with the first and second quadrature formulas, respectively.

In the rest of this section $S = (w_1 d\sigma, w_2 d\sigma)$, and the weights $w_1, w_2$ on $[a, b]$ verify

$$C_0 := \int_a^b \frac{|w_2(x)|^2}{|w_1(x)|}\,\mathrm{d}\sigma(x) = \int_a^b \left(\frac{w_2(x)}{w_1(x)}\right)^2 |w_1(x)|\,\mathrm{d}\sigma(x) < \infty. \tag{18}$$

The common denominator $Q_{\mathbf{n}}$ is taken to be the $N$-th orthogonal polynomial with respect to $(w_1/\alpha_{\mathbf{n},1})\mathrm{d}\sigma$. Thus the Christofel–Nikishin coefficients associated with the first measure satisfy A). On the other hand, (18) allows to prove

(B) for the Christoffel–Nikishin coefficients associated with the second measure. The polynomials $P_{\mathbf{n},k}$, $k = 1, 2$, (and consequently $R_{\mathbf{n},k}$) are uniquely determined and

$$R_{\mathbf{n},k} = \frac{P_{\mathbf{n},k}(z)}{Q_{\mathbf{n}}(z)} = \sum_{j=1}^{N} \frac{\lambda_{\mathbf{n},k,j}}{z - x_{\mathbf{n},j}}, \quad k = 1, 2, \tag{19}$$

where $Q_{\mathbf{n}}(z) = \prod_{j=1}^{N}(z - x_{\mathbf{n},j}), x_{\mathbf{n},j} \in (a, b)$, and $\lambda_{\mathbf{n},k,j}$ is given by (11). In particular,

$$\int_a^b \frac{p(x)}{\alpha_{\mathbf{n},1}(x)} w_1(x)\, \mathrm{d}\sigma(x) = \sum_{j=1}^{N} \lambda_{\mathbf{n},1,j} \frac{p(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},1}(x_{\mathbf{n},j})}, \quad p \in \mathrm{P}_{2N-1}, \tag{20}$$

and

$$\int_a^b \frac{p(x)}{\alpha_{\mathbf{n},2}(x)} w_2(x)\, \mathrm{d}\sigma(x) = \sum_{j=1}^{N} \lambda_{\mathbf{n},2,j} \frac{p(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},2}(x_{\mathbf{n},j})}, \quad p \in \mathrm{P}_{N-1}, \tag{21}$$

where each $\alpha_{\mathbf{n},k}$, $k = 1, 2$, is defined according to (5).

Assume that $\mathbf{A}_k$ is the table associated with $\alpha_{\mathbf{n},k}, k = 1, 2$, and there exist positive constants $C_1$, $C_2$, such that

$$C_1\, |\alpha_{\mathbf{n},2}(t)|^2 \le |\alpha_{\mathbf{n},1}(t)| \le C_2\, |\alpha_{\mathbf{n},2}(t)|^2, \quad t \in [a, b], \;\; N \in \mathbb{N}. \tag{22}$$

*Remark 3* The relation $|\alpha_{\mathbf{n},2}(t)|^2 = |\alpha_{\mathbf{n},1}(t)|$, trivially implies (22) but it is too restrictive. Other examples are easily constructed. For example, assume that for each $N \in \mathbb{N}, \alpha_{\mathbf{n},2}$ has $\kappa_N (< N)$ simple zeros on an interval $[c, d], [a, b] \cap [c, d] = \emptyset$. Between two consecutive zeros of $\alpha_{\mathbf{n},2}$ take two distinct points and assign all these points as the zeros of $\alpha_{\mathbf{n},1}$. Because of the interlacing property it is easy to prove that (22) takes place.

**Theorem 2** *Assume that* (18) *holds. Let* $\Lambda$ *be the sequence of multi-indices* $\mathbf{n} = (N, 0)$*,* $N \in \mathbb{N}$*. Let* $\alpha_{\mathbf{n},k}$*,* $k = 1, 2$*, be polynomials which satisfy* (22)*. Then*

$$\sum_{j=1}^{N} |\lambda_{\mathbf{n},2,j}| \le \sqrt{\frac{C_2 C_0}{C_1}} \left| \int_a^b w_1(x)\, \mathrm{d}\sigma(x) \right|^{1/2}$$

*Proof* All multi-indices in $\Lambda$ are strongly normal because $Q_{\mathbf{n}}$ is the $N$-th monic orthogonal polynomial with respect to the measure $(w_1/\alpha_{\mathbf{n},1})\mathrm{d}\sigma$ on $[a, b]$. Condition (18) means that $w_2/w_1 \in L^2(w_1\, \mathrm{d}\sigma)$. Fix $\mathbf{n} \in \Lambda$. Let $(Q_{\mathbf{n},\nu})_{\nu=1}^{\infty}$ be the sequence of monic orthogonal polynomials associated to the measure $(w_1/\alpha_{\mathbf{n},1})\mathrm{d}\sigma$. In particular, $Q_{\mathbf{n},1,N} = Q_{\mathbf{n}}$.

Let $S_{\mathbf{n},N-1}(x)$ be the $N$-th partial sum of the Fourier series of $w_2\alpha_{\mathbf{n},1}/w_1\alpha_{\mathbf{n},2} \in L_2(w_1 \mathrm{d}\sigma)$. Using (11), the $L_2(w_1 \mathrm{d}\sigma)$ convergence of the Fourier series, and orthogonality (notice that $\deg Q_{\mathbf{n}}(x)/(x - x_{\mathbf{n},j}) \le N - 1$), we have

$$\begin{aligned}\lambda_{\mathbf{n},2,j} &= \alpha_{\mathbf{n},2}(x_{\mathbf{n},j}) \int_a^b \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} \frac{w_2(x)\, \mathrm{d}\sigma(x)}{\alpha_{\mathbf{n},2}(x)} \\ &= \alpha_{\mathbf{n},2}(x_{\mathbf{n},j}) \int_a^b \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} \frac{\alpha_{\mathbf{n},1}(x) w_2(x)}{\alpha_{\mathbf{n},2}(x) w_1(x)} \frac{w_1(x)\, \mathrm{d}\sigma(x)}{\alpha_{\mathbf{n},1}(x)} \\ &= \alpha_{\mathbf{n},2}(x_{\mathbf{n},j}) \int_a^b \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} S_{\mathbf{n},N-1}(x) \frac{w_1(x)\, \mathrm{d}\sigma(x)}{\alpha_{\mathbf{n},1}(x)}.\end{aligned}$$

Since $\deg(S_{\mathbf{n},N-1}(x) - S_{\mathbf{n},N-1}(x_{\mathbf{n},j}))/(x - x_{\mathbf{n},j}) = N - 2$, using once more the orthogonality relations, we find that

$$\int_a^b Q_{\mathbf{n}}(x) \frac{S_{\mathbf{n},N-1}(x) - S_{\mathbf{n},N-1}(x_{\mathbf{n},j})}{(x - x_{\mathbf{n},j})} \frac{w_1(x)\, \mathrm{d}\sigma(x)}{\alpha_{\mathbf{n},1}(x)} = 0$$

which combined with the previous equalities and the formula for $\lambda_{\mathbf{n},1,j}$ gives us

$$\lambda_{\mathbf{n},2,j} = \frac{\alpha_{\mathbf{n},2}(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},1}(x_{\mathbf{n},j})} S_{\mathbf{n},N-1}(x_{\mathbf{n},j}) \lambda_{\mathbf{n},1,j}\,,$$

which is the main relation.

On account of (22), it follows that

$$|\lambda_{\mathbf{n},2,j}| \leq \frac{1}{\sqrt{C_1}}\, |S_{\mathbf{n},N-1}(x_{\mathbf{n},j})| \left( \frac{|\lambda_{\mathbf{n},1,j}|}{|\alpha_{\mathbf{n},1}(x_{\mathbf{n},j})|} \right)^{1/2} |\lambda_{\mathbf{n},1,j}|^{1/2}\,. \tag{23}$$

Summing both sides of (23) for $j = 1, \ldots, N$, using that the $\lambda_{\mathbf{n},1,j}$ have constant sign, the Cauchy–Schwartz inequality, that the quadrature formula is exact for $S^2_{\mathbf{n},N-1}$, the Bessel inequality, (18), and (22), we obtain that

$$\begin{aligned}\sum_{j=1}^N |\lambda_{\mathbf{n},2,j}| &\leq \frac{1}{\sqrt{C_1}} \left| \sum_{j=1}^N \lambda_{\mathbf{n},1,j} \frac{S^2_{\mathbf{n},N-1}(x_{\mathbf{n},j})}{\alpha_{\mathbf{n},1}(x_{\mathbf{n},j})} \right|^{1/2} \left| \sum_{j=1}^N \lambda_{\mathbf{n},1,j} \right|^{1/2} \\ &= \frac{1}{\sqrt{C_1}} \left| \int_a^b \frac{S^2_{\mathbf{n},N-1}(x)}{\alpha_{\mathbf{n},1}(x)} w_1(x) \mathrm{d}\sigma(x) \right|^{1/2} M_1 \\ &\leq \frac{1}{\sqrt{C_1}} \left| \int_a^b \left( \frac{\alpha_{\mathbf{n},1}(x) w_2(x)}{\alpha_{\mathbf{n},2}(x) w_1(x)} \right)^2 \frac{w_1(x) \mathrm{d}\sigma(x)}{\alpha_{\mathbf{n},1}(x)} \right|^{1/2} M_1 \leq \sqrt{\frac{C_2 C_0}{C_1}} M_1,\end{aligned}$$

where $M_1 = \left| \sum_{j=1}^N \lambda_{\mathbf{n},1,j} \right|^{1/2} = |\int_a^b w_1(x) \mathrm{d}\sigma(x)|^{1/2}$.

## 4 Quadrature formulas for Nikishin systems

Let $\sigma_1$ and $\sigma_2$ be two finite measures with constant sign on the intervals $I_1$ and $I_2$, respectively. If $I_1 \cap I_2 = \emptyset$ we may define

$$\langle \sigma_1, \sigma_2 \rangle(x) = \int_{I_1} \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x) = \widehat{\sigma}_2(x)\, d\sigma_1(x), \tag{24}$$

which gives a new measure with constant sign on $I_1$. Let $\{\sigma_1, \dots, \sigma_m\}$ be a system of measures with constant sign on the intervals $I_k$, $k = 1, \dots, m$, respectively, and $I_i \cap I_{i+1} = \emptyset$, $i = 1, \dots, m-1$. Inductively, we define

$$s_1 = \sigma_1, \quad s_k = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_k \rangle \rangle, \ \ k = 2, \dots, m. \tag{25}$$

Thus, $(s_k)_{i=1}^m$ is a system of measures all of which are supported on the same interval $I_1$. This construction is called the Nikishin system generated by $(\sigma_1, \dots, \sigma_m)$ and it is denoted $(s_1, \dots, s_m) = \mathrm{N}(\sigma_1, \dots, \sigma_m)$.

In [21] the author proves that all multi-indices of the form $(N, \dots, N, N+1, \dots, N+1), N \in \mathbb{Z}_+$, are strongly normal for $\mathrm{N}(\sigma_1, \dots, \sigma_m)$. Recently (see [4,5]), it was shown that for $m = 3$ all multi-indices are strongly normal and, for any $m$, all multi-indices in

$$\mathbb{Z}_+^m(*) = \{\mathbf{n} = (n_1, \dots, n_m) : \nexists 1 \le i < j < k \le m \quad \text{with } n_i < n_j < n_k\}$$

are also strongly normal.

In this section, $\alpha_{\mathbf{n},k} = \alpha_{\mathbf{n}}, k = 1, \dots, m$, where $\alpha_{\mathbf{n}}$ is a polynomial with real coefficients such that

$$\deg\ \alpha_{\mathbf{n}} < |\mathbf{n}| + \min\{n_1, \dots, n_m\}, \tag{26}$$

and $\{z \in \mathbb{C};\ \alpha_{\mathbf{n}}(z) = 0\} \subset F \subset \overline{\mathbb{C}} \setminus I_1$, where $F$ is a compact set.

Given $(s_1, s_2, \dots, s_m) = \mathrm{N}(\sigma_1, \sigma_2, \dots, \sigma_m)$, notice that $ds_k = w_k d\sigma_1$, where $w_1 \equiv 1$ and $w_k, k = 2, \dots, m$, is the Cauchy transform of $\langle \sigma_2, \dots, \sigma_k \rangle$. Fix a multi-index $\mathbf{n}$. Notice that

$$\left( \frac{ds_1}{\alpha_{\mathbf{n}}}, \frac{ds_2}{\alpha_{\mathbf{n}}}, \dots, \frac{ds_m}{\alpha_{\mathbf{n}}} \right) = \left( \frac{d\sigma_1}{\alpha_{\mathbf{n}}}, \frac{w_2 d\sigma_1}{\alpha_{\mathbf{n}}}, \dots, \frac{w_m d\sigma_1}{\alpha_{\mathbf{n}}} \right) = \mathrm{N}\left( \frac{d\sigma_1}{\alpha_{\mathbf{n}}}, \sigma_2, \dots, \sigma_m \right). \tag{27}$$

For convenience, we use here the differential notation of a measure.

**Theorem 3** *Let $S = (s_1, \dots, s_m) = \mathrm{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system. Fix $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m(*)$. Set $M = \max\{n_1 - 1, n_2 - 2, \dots, n_m - 2\}$. Let $k \in \{1, \dots, m\}$ be defined as follows: $k = 1$ if $n_1 - 1 = M$; or $k \in \{2, \dots, m\}$ is the subindex of the first component of $\mathbf{n}$ such that $n_k - 2 = M$. Then, there exists a monic* *polynomial* $\beta_{\mathbf{n},k}$ *of degree* $|\mathbf{n}| - n_k$ *whose zeros are simple and lie in the interior of* $I_2$ *such that*

$$\int_{I_1} x^\nu Q_{\mathbf{n}}(x) \frac{\mathrm{d}s_k(x)}{\alpha_{\mathbf{n}}(x)\beta_{\mathbf{n},k}(x)} = 0, \quad \nu = 0, 1, \ldots, |\mathbf{n}| - 1. \tag{28}$$

$$\int_{I_1} \frac{p(x)}{(\alpha_{\mathbf{n}}\beta_{\mathbf{n},k})(x)} \mathrm{d}s_k(x) = \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,j} \frac{p(x_{\mathbf{n},j})}{(\alpha_{\mathbf{n}}\beta_{\mathbf{n},k})(x_{\mathbf{n},j})}, \quad p \in \mathrm{P}_{2|\mathbf{n}|-1}. \tag{29}$$

*For each* $j = 1, \ldots, |\mathbf{n}|$, *the following equation takes place*

$$\lambda_{\mathbf{n},k,j} = (\alpha_{\mathbf{n}}\beta_{\mathbf{n},k})(x_{\mathbf{n},j}) \int_{I_1} \left( \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},j})(x - x_{\mathbf{n},j})} \right)^2 \frac{\mathrm{d}s_k(x)}{(\alpha_{\mathbf{n}}\beta_{\mathbf{n},k})(x)}. \tag{30}$$

*Proof* It is a direct consequence of [7], Theorem 3, applied to the Nikishin system given by (27).

We wish to underline that for the $k$-th component (where $k$ is as defined in the statement of the Theorem 3) the coefficients $\lambda_{\mathbf{n},k,j}$ are exactly those given in (10),(11). Notice that $Q_{\mathbf{n}}$ satisfies full orthogonality relations with respect to the measure $\mathrm{d}s_k/\alpha_{\mathbf{n}}\beta_{\mathbf{n},k}$ and, consequently, one obtains a Gauss rational quadrature formula with respect to this measure. For example, a multi-index of the form $(N, N, N+1)$ satisfies the conditions of this theorem for the components $k = 1$ and $k = 3$ and, therefore, for those two components (A) is verified.

**Corollary 1** *Let* $S = (s_1, \ldots, s_m) = \mathrm{N}(\sigma_1, \ldots, \sigma_m)$ *be a Nikishin system and let* $k \in \{1, \ldots, m\}$ *be fixed. Let* $\Lambda \subset \mathbb{Z}_+^m(*)$ *be an infinite sequence of distinct multi-indices* $\mathbf{n} = (n_1, \ldots, n_m)$ *such that for all* $\mathbf{n} \in \Lambda$ *the k-th component* $n_k$ *satisfies the hypothesis of Theorem 3. Then the k-th rational quadrature formula of the corresponding simultaneous system converges for all continuous functions on* $I_1$.

*Proof* From (30) we have that the coefficients $\lambda_{\mathbf{n},k,j}$, $\mathbf{n} \in \Lambda$, preserve the same sign as $s_k$ so condition (A) holds. Besides, the quadrature formula is exact for constants which implies condition (B), and the table here considered satisfies the density property (see Remark 1.1). The corollary follows from Proposition 3.

*Remark 4* The sequence $\Lambda \subset \mathbb{Z}_+^m(*)$ defined by $n_j = N, j = 1, \ldots, k-1$, and $n_j = N+1, j = k, \ldots, m, N \in \mathbb{N}$, satisfies the hypothesis of Theorem 3 in the first and the $k$-th component. For the rest of the components we can only prove that most of the Nikishin–Christoffel coefficients have the same sign as the corresponding measure.

**Definition 5** Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. For every $k = 1, \ldots, m$, we construct an associated multi-index $\mathbf{n}^k = (n_1^k, \ldots, n_m^k) \in \mathbb{Z}_+^m$, in the following form (cf. [7]). For $k = 1$

$$n_j^1 = \begin{cases} n_1 & \text{if} \quad j = 1, \\ \min\{n_1 + 1, n_j\} & \text{if} \ 2 \le j \le m. \end{cases} \tag{31}$$

If $k \in \{2, \dots, m\}$

$$n_j^k = \begin{cases} \min\{n_1, \dots, n_j, n_k - 1\} & \text{if } 1 \le j < k, \\ \min\{n_k, n_j\} & \text{if } k \le j \le m. \end{cases} \tag{32}$$

We can derive from Definition 5 that $\mathbf{n} - \mathbf{n}^k \in \mathbb{Z}_+^m$, and $\mathbf{n}^k \in \mathbb{Z}_+^m(*)$. As before $|\mathbf{n} - \mathbf{n}^k| = \sum_{j=1}^m (n_j - n_j^k) = |\mathbf{n}| - |\mathbf{n}^k|$. If $\mathbf{n} \in \mathbb{Z}_+^m(*)$ and $k$ is as defined in Theorem 3, then $\mathbf{n} = \mathbf{n}^k$ and $|\mathbf{n} - \mathbf{n}^k| = 0$.

**Theorem 4** *Let $S = (s_1, \dots, s_m) = \mathrm{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system. Given $\mathbf{n} \in \mathbb{Z}_+^m$ let $\mathbf{n}^i$ be as in Definition 5. Then, for each $i \in \{1, \dots, m\}$ there exists a monic polynomial $\beta_{\mathbf{n},i}$ of degree $|\mathbf{n}^i|$ whose zeros are simple and lie in the interior of $I_2$, such that*

$$\int_{I_1} \frac{p(x)}{(\alpha_{\mathbf{n}}\beta_{\mathbf{n},i})(x)} \, \mathrm{d}s_i(x) = \sum_{j=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},i,j} \frac{p(x_{\mathbf{n},j})}{(\alpha_{\mathbf{n}}\beta_{\mathbf{n},i})(x_{\mathbf{n},j})}, \tag{33}$$

*for all $p \in \mathrm{P}_{|\mathbf{n}|+|\mathbf{n}^i|-1}$.*

*Moreover, at least $(|\mathbf{n}| + |\mathbf{n}^i|)/2$ coefficients $\lambda_{\mathbf{n},i,j}, j = 1, \dots, |\mathbf{n}|$, have the same sign as the measure $s_i$.*

*Proof* This result is contained in Theorem 4 of [7] applied to the Nikishin system (27).

*Remark 5* As a consequence of this theorem, for multi-indices of the form $\mathbf{n} = (N, \dots, N, N+1, \dots, N+1)$, for each $N$ at most $m-1$ coefficients in the quadrature formula may have a sign different from the corresponding measure. In particular, it is easy to verify that the coefficients corresponding to multi-indices of the form $(N, N+1, N+1)$ have the same sign as the corresponding measure and the simultaneous quadrature formulas converge for all continuous functions on $I_1$ for the three measures. We suspect that for all multi-indices of the form $\mathbf{n} = (N, \dots, N, N+1, \dots, N+1)$ and for all $k = 1, \dots, m$ condition (B) is satisfied. The polynomial version of Corollary 1 was given in [7], Corollary 3.

## 5 The numerical method

We restrict our attention to the case $m = 3$ and the sequence of multi-indices

$$\mathbf{n}(r) = \begin{cases} (s, s, s) & \text{if } \quad r = 3s, \\ (s, s, s+1) & \text{if } r = 3s+1, \quad s = 0, 1, 2, \dots \\ (s, s+1, s+1) & \text{if } r = 3s+2. \end{cases} \tag{34}$$

Notice that $r = |\mathbf{n}(r)|$, and $\mathbf{n}(r) \in \mathbb{Z}_+^3(*)$, $r = 0, 1, 2, \dots$.

Let $w_k(x)$, $k = 1, 2, 3$, be three weight functions on the interval $[a, b]$ which form an AT-system. Let $\sigma$ be a measure with constant sign on $[a, b]$ and

$ds_k = w_k d\sigma, k = 1,2,3$. Given $\mathbf{n}(r)$, let $Q_r$ be the $r$-th multiple orthogonal polynomial with respect to $(S,\alpha)$, where $S = \{s_1, s_2, s_3\}$ and $\alpha$ is a suitable polynomial which remains fixed for all $r$.

It is well known (see, for example, [1], Sect. 4.1), that $\{Q_r\}$ satisfies a recurrence formula of $m + 2 = 5$ terms of the form

$$Q_r = (x - a_r)Q_{r-1} - b_r Q_{r-2} - c_r Q_{r-3} - d_r Q_{r-4}, \quad r = 1,2,\ldots \tag{35}$$

Let $z(j) \in \mathbb{R}\backslash[a,b]$, $j = 1,\ldots,K$, and $\zeta(j) \in \mathbb{R}$, $j = 1,\ldots,s$, be the zeros of the polynomials $\alpha$ and $P_s$, respectively, that is

$$\alpha(x) = \frac{x - z(1)}{h_\alpha - z(1)} \cdots \frac{x - z(K)}{h_\alpha - z(K)}, \quad P_s(x) = \frac{x - \zeta(1)}{h_P - \zeta(1)} \cdots \frac{x - \zeta(s)}{h_P - \zeta(s)},$$

where the array $z(\cdot)$ simulate the poles of a given integrand which are considered to be the closest to $[a,b]$. The parameters $h_\alpha$, $h_P$, and the array $\zeta(\cdot)$ can be chosen according to different criteria. The presence of the denominators $h_\alpha - z(j)$ and $h_p - \zeta(j)$ is due to scaling.

Let $\{H_k(s,r)\}$ be a table defined by

$$H_k(s,r) = \int_a^b Q_r(x)P_s(x)\frac{w_k(x)}{\alpha(x)}\,\mathrm{d}\sigma(x), \tag{36}$$

where $r,s = 0,1,2,\ldots$, $k = 1,2,3$, and $P_0 \equiv 1$. Set $H_k(s,r) = 0$ if $r < 0$.

For $r,s = 0,1,2,\ldots$, $k = 1,2,3$, let $E_k(s,r) = E_k(s,r,a_r,b_r,c_r,d_r)$ be the expression given below

$$\begin{aligned} E_k(s,r) = {} & (h_P - \zeta(s))H_k(s+1,r) + \zeta(s)H_k(s,r) - a_r H_k(s,r) \\ & - b_r H_k(s,r-1) - c_r H_k(s,r-2) - d_r H_k(s,r-3). \end{aligned} \tag{37}$$

If we fix $r$ and $s$, from (35) we obtain

$$E_k(s,r) = H_k(s,r+1), \quad k = 1,2,3. \tag{38}$$

There are three types of equations (38). We say that (38) is *basic* when $|H_k(s,r)| + |H_k(s,r-1)| + |H_k(s,r-2)| + |H_k(s,r-3)| \neq 0$ and $H_k(s,r+1) = 0$. If $H_k(s,r+1) \neq 0$ then (38) is a *non-basic* equation. The rest are simply called *trivial*.

Now we assume that $\mathrm{X}_r = [a_r,b_r,c_r,d_r]^t$, is the solution of the linear system $S_r$ formed by the basic linear equations.

Given $r$, it is a fact that the number of basic equations is never greater than four. If $r \geq 4$ then the system $S_r$ is given as follows

$$\begin{aligned}
S_{3s} &= \{E_k(s-1,3s-1)=0,\ k=1,2,3;\ E_1(s-2,3s-1)=0\},\\
S_{3s+1} &= \{E_3(s,3s)=0,\ E_k(s-1,3s)=0,\ k=1,2,3\},\\
S_{3s+2} &= \{E_k(s,3s+1)=0,\ k=2,3;\ E_k(s-1,3s+1)=0,\ k=1,2\}.
\end{aligned}$$

For $r = 1,2,3$ the corresponding systems of basic equations $S_r$ are $S_1 = \{E_3(0,0)=0\}$; $S_2 = \{E_k(0,1)=0,\ k=2,3\}$; $S_3 = \{E_k(0,2)=0,\ k=1,2,3\}$.

We will also denote the linear system $S_r$ by $\mathrm{A}_r\mathrm{X}_r = \mathrm{B}_r$.

To appraise the structure of $\mathrm{A}_r$, let $r = 3s+1$, $r \geq 4$ and $M_\infty$ be the following matrix with four columns and infinitely many rows

$$M_\infty = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
H_3(s,r-1) & H_3(s,r-2) & H_3(s,r-3) & H_3(s,r-4) \\
H_1(s-1,r-1) & H_1(s-1,r-2) & H_1(s-1,r-3) & H_1(s-1,r-4) \\
0 & H_2(s-1,r-2) & H_2(s-1,r-3) & H_2(s-1,r-4) \\
0 & 0 & H_3(s-1,r-3) & H_3(s-1,r-4) \\
0 & 0 & 0 & H_1(s-2,r-4) \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{matrix}
\leftarrow \vdots \\ \leftarrow 0 \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \\ \leftarrow 5 \\ \leftarrow \vdots \\ \leftarrow r
\end{matrix}$$

$\mathrm{A}_r$ is the $4 \times 4$ triangular block of $M_\infty$ whose rows are those labeled by 1, 2, 3 and 4. Some entries are equal to zero because of the orthogonality condition (4). If $r > 4$ then there are exactly $r-4$ rows whose entries are all zero and correspond to trivial equations. The row at level 0 and all those above, correspond to non basic equations. Slight changes must be carried out in $M_\infty$ to illustrate the cases $r = 3s$ and $r = 3s+2$.

Notice that a common characteristic of every system $\mathrm{A}_r\mathrm{X}_r = \mathrm{B}_r$ is the different contribution of the weights to the calculations. This observation leads to a computational strategy which is mainly based on two principles: I) extensive use of data, and II) a balanced participation of the three integration weights in the calculations.

Mathematically speaking, for every $r$ we only need the triangular system $\mathrm{A}_r\mathrm{X}_r = \mathrm{B}_r$ to determine the solution $\mathrm{X}_r$. However, errors occur due to the arithmetic of finite precision which could be made worse because of the complexity of some weight functions. We claim that accuracy may be improved by adding non basic equations to the basic system (Principle I). For example, for $r = 3s$ the first non basic equation to be added is $E_3(s,3s-1) = H_3(s,3s)$. In general, when non basic equations are considered one needs to predict the value of $H_k(s,j)$, for $j \geq r+1$.

We denote by $\mathrm{A}'_r\mathrm{X}_r = \mathrm{B}'_r$ any system formed by consecutive non basic equations. Thus, using Matlab notation, the new system to be solved is $[\mathrm{A}'_r; \mathrm{A}_r]\mathrm{X}_r = [\mathrm{B}'_r; \mathrm{B}_r]$. Taking into account Principle II, we assume in the sequel that the overall number of equations of the augmented system to compute $\mathrm{X}_r$ is a multiple of three. On the other hand, all trivial equations have coefficients which ranges from $1.0 \times 10^{-12}$ to $1.0 \times 10^{-21}$, thus some of them could be included as well.

Once we have found the vectors $X_j, j = 1, \ldots, r$, we can construct the following matrix $M_r$, of order $r$, whose structure is determined by Eq. (35).

$$M_r = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0\,0\,0 \cdots 0 & 0 & 0 & 0 & 0 \\ b_2 & a_2 & 1 & 0 & 0 & 0\,0\,0 \cdots 0 & 0 & 0 & 0 & 0 \\ c_3 & b_3 & a_3 & 1 & 0 & 0\,0\,0 \cdots 0 & 0 & 0 & 0 & 0 \\ d_4 & c_4 & b_4 & a_4 & 1 & 0\,0\,0 \cdots 0 & 0 & 0 & 0 & 0 \\ 0 & d_5 & c_5 & b_5 & a_5 & 1\,0\,0 \cdots 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0\,0\,0 \cdots 0 & d_r & c_r & b_r & a_r \end{pmatrix}$$

If $M_r = VDV^{-1}$ is the spectral decomposition of $M_r$, then the quadrature nodes are given by $(x_{\mathbf{n},j})_{j=1}^r = \operatorname{diag}(D)$. Furthermore, the vector of Christoffel coefficients $\Lambda_{k,r} = [\lambda_{\mathbf{n},k,1}, \ldots, \lambda_{\mathbf{n},k,r}]^t$ fulfills

$$\Lambda_{k,r} = [\alpha(x_{\mathbf{n},1})Y_1, \ldots, \alpha(x_{\mathbf{n},r})Y_r],$$

where $Y = [Y_j]_{j=1}^r$ is the solution of the system $V\,Y = L$, with

$$V = [P_i(x_{\mathbf{n},j})]_{i=0,\ldots,r-1;\, j=1,\ldots,r}$$

and $L = [H_k(i,0)]_{i=0}^{r-1}$.

A lot of computational effort must be carried out to avoid loss of accuracy when the zeros of $\alpha(x)$ are close to the integration interval $[a, b]$. Special techniques to treat these difficult poles have been developed by Gautschi [9,10] (see also [19]). Nevertheless, our method is not based on Gautschi's strategy but on the use of some transformations of the integration interval. The latter approach is also known as the "change of variable method" because it is exactly what is done before applying the integration rule. If $\varphi$ transforms $[a, b]$ onto itself and $\varphi' > 0$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x. \tag{39}$$

Given any integration rule with nodes $x_k \in [a, b]$, and weights $\lambda_k$, $k = 1, \ldots, r$, the rule can be applied to the integral in the right side of (39). This process yields a new rule with nodes $y_k = \varphi(x_k)$, and weights $\Lambda_k = \lambda_k \varphi'(x_k)$ to evaluate the integral in the left side of Eq. (39). Our choice is

$$\varphi_{p,q,a,b}(x) := \frac{(b-a)(x-a)^p}{(x-a)^p + (b-x)^q} + a; \tag{40}$$

$p, q \in \mathbb{N}$, $q, p \geq 1$, $p + q > 2$, which transforms $[a, b]$ onto $[a, b]$.

The transformations (40) have played a major role in solving certain kind of singular integral equations. The non-symmetric case in (40), which occurs when $p \neq q$, was promoted by Monegato and Scuderi when the integrand has only one end-point singularity (cf. [20]). The remarkable property of (40) is that $\varphi'_{p,q,a,b}(x)$ vanishes at $x = a, b$, with multiplicities $p - 1$ and $q - 1$, respectively. Thus, if the integrand has integrable singularities at $x = a, b$, then $\varphi'_{p,q,a,b}$ annihilates them by choosing conveniently the values of $p$ and $q$. Nevertheless, $\varphi_{p,q,a,b}$ does not remove poles.

In order to construct $\varphi_{p,q,a,b}(x)$ we proceed as follows. First, we assume that $(p, q) \in \{(2m, 2m), (2m, 1), (1, 2m)\}$, $1 \leq m \leq 4$. Then, the pair $(p, q)$ is chosen depending on the location of the poles. If one has detected difficult poles in $x < a$ and $x > b$, then $p = q = 2m$. The choice $(p, q) = (2m, 1)$, for example, is expedient when the only poles to take into account are located in $x < a$. Thus, we expect that $\varphi_{p,q,a,b}$ transforms real difficult poles into non real poles which are not difficult, that is, the new poles should not produce instability. Then we fit (40) into the integral (36) with $r = 0$, to be evaluated by means of the well known Gauss-Legendre quadrature rule of polynomial type.

From the numerical and theoretical points of view, interpolatory rational quadrature formulas are nothing else than product integration rules. Possibly their main disadvantage is that every collection of difficult poles requires a specific set of nodes and weights. In [9,10], Gautschi suggests that the zeros of $\alpha$ only have to be close to the poles but he did not establish how much. Thus, one of the most important tasks in this context is the design of low cost algorithms to calculate these formulas depending on which poles must be simulated.

The main stages of the procedure are summarized in the following items.

1. Design $\alpha(x)$ depending on the poles which must be simulated depending on their proximity to $[a, b]$. The selection of the polynomials $P_s$ can be carried out following different criteria. Once $\alpha(x)$ is given, the choice $\zeta(j) = z(j)$, $j = 1, 2, \ldots, s$ and $h_P = h_\alpha$, seems to be convenient from a theoretical perspective. Nevertheless, we suggest that $\zeta(j) = \zeta_0 \leq a < b < h_P, j = 1, \ldots, s$, to derive simplicity and positivity for $P_s$.
2. Calculate the modified moments $H_k(s, 0)$, $s = 0, \ldots, s_0$, where the size of $s_0$ depends on the maximum quadrature order $r$ to be reached, v.g. $24 \leq s_0 \leq 30$ for $r \leq 12$.
3. Obtain $\mathrm{X}_1 = a_1 = \mathrm{A}_1^{-1}\mathrm{B}_1$, where $\mathrm{B}_1 = H_3(1, 0)$, and $\mathrm{A}_1 = \big(\zeta(1)H_3(0, 0) + (h_P - \zeta(1))H_3(1, 0)\big)$. From (38), obtain $H_k(s, 1), s = 0, \ldots, s_0 - 1$, to construct $\mathrm{A}_2$ and $\mathrm{B}_2$.
4. Given $\mathrm{A}_r$, $\mathrm{B}_r$ and the array $H_k(s, j)$ ($j = 0, \ldots, r - 1$; $s = 0, \ldots, s_0 - r + 1$; $k = 1, 2, 3$), calculate the vector $\mathrm{X}_r = \mathrm{A}_r^{-1}\mathrm{B}_r$
   If $r < s_0$ then evaluate $H_k(s, r)$, $s = 0, \ldots, s_0 - r$; $k = 1, 2, 3$ from (37), to construct $\mathrm{A}_{r+1}$ and $\mathrm{B}_{r+1}$ if were necessary.

The next subsections deal with two different AT-systems, the second one is a Nikishin system. For the purpose of comparison with the results in [9,11], we have selected two examples for which the first measure is $w_1(x)\mathrm{d}\sigma(x) = \mathrm{d}x$.

### 5.1 Test 1

The integrals to be evaluated are

$$I_k = \int_{-1}^{1} \frac{(\pi x/\omega)}{\sin(\pi x/\omega)} w_k(x)\, \mathrm{d}\sigma(x), \quad \omega > 1, \ \ k = 1, 2, 3, \tag{41}$$

where $\mathrm{d}\sigma(x) = \mathrm{d}x$, $w_1(x) = 1$, $w_2(x) = \exp(x)$ and $w_3(x) = \exp(x\sqrt{3})$. As we mentioned above this system of weight functions forms an AT-system with respect to any multi-index (cf. [21]).

The integrand in (41) is analytic in a neighborhood of the interval $[-1, 1]$, and has simple real poles at $N\omega$, $N \in \mathbb{Z}$, $N \neq 0$. When $\omega \approx 1$ the most difficult poles are $\pm\omega$, and all the poles can be organized by pairs $\xi_N = \pm N\omega$, $N = 1, \ldots, d$. In order to simulate the $K$ poles closest to $[-1, 1]$ we will assume that the polynomials $\alpha(x)$ have zeros at: $\pm j\omega$, $j = 1, \ldots, d$, hence $K = 2d$. The zero $\xi_{2d+1} = (2d+1)\,\omega$ is included when $K = 2d + 1$. More precisely, we can define $\alpha$ as one of the two expressions given below.

$$\alpha_{2d}(x) = \prod_{k=1}^{d} \left( \frac{x^2 - k\omega)^2}{h_\alpha - (k\omega)^2} \right), \quad \alpha_{2d+1}(x) = \left( \frac{x - (2d+1)\omega}{h_\alpha - (2d+1)\omega} \right) \alpha_{2d}(x),$$

where $h_\alpha \in (-1, 1)$.

Some results are shown in Tables 1, 2 and 3 when $\omega = 1.001$ and $h_\alpha = 0$. Fast convergence of our procedure is observed in Table 1, where the relative errors for the first order values $r = 1, \ldots, 12$ are listed, and $\deg\,\alpha = 2$.

**Table 1** Relative errors obtained in Test 1 when the integrals $I_i(1.001)$, $i = 1, 2, 3$, are evaluated by a simultaneous rational rule ($\mathrm{SRQF}^{(1)}$)

| $w_k$ | $r$ | deg $\alpha$ | $\mathrm{SRQF}^{(1)}$ | $r$ | deg $\alpha$ | $\mathrm{SRQF}^{(1)}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 8.3e−02 | 7 | 2 | 1.6e−07 |
| 2 | | | 1.1e−01 | | | 3.9e−08 |
| 3 | | | 1.3e−01 | | | 6.1e−08 |
| 1 | 2 | 2 | 8.5e−02 | 8 | 2 | 1.2e−08 |
| 2 | | | 1.8e−02 | | | 2.9e−09 |
| 3 | | | 1.6e−02 | | | 4.6e−09 |
| 1 | 3 | 2 | 3.7e−03 | 9 | 2 | 7.3e−10 |
| 2 | | | 1.1e−03 | | | 2.2e−10 |
| 3 | | | 2.1e−03 | | | 3.9e−10 |
| 1 | 4 | 2 | 4.0e−04 | 10 | 2 | 6.1e−11 |
| 2 | | | 8.9e−05 | | | 1.5e−11 |
| 3 | | | 1.3e−04 | | | 2.5e−11 |
| 1 | 5 | 2 | 3.3e−05 | 11 | 2 | 6.9e−12 |
| 2 | | | 7.2e−06 | | | 7.9e−13 |
| 3 | | | 1.0e−05 | | | 4.2e−13 |
| 1 | 6 | 2 | 1.8e−06 | 12 | 2 | 1.2e−12 |
| 2 | | | 5.7e−07 | | | 1.5e−13 |
| 3 | | | 1.0e−06 | | | 6.5e−13 |

**Table 2** Relative errors obtained in Test 1 when the integrals $I_i(1.001)$, $i = 1,2,3$, are evaluated by simultaneous integration rules of rational and polynomial type

| $w_k$ | Rational | | | Polynomial | |
|---|---|---|---|---|---|
| | $r$ | deg $\alpha$ | SRQF$^{(1)}$ | $r$ | SPQF |
| 1 | 2 | 4 | 6.8e−02 | 2 | 7.2e−01 |
| 2 | | | 1.3e−02 | | 7.1e−01 |
| 3 | | | 1.1e−02 | | 7.5e−01 |
| 1 | 4 | 4 | 2.6e−04 | 4 | 5.3e−01 |
| 2 | | | 5.4e−05 | | 5.4e−01 |
| 3 | | | 7.9e−05 | | 5.8e−01 |
| 1 | 5 | 6 | 1.8e−05 | 5 | 4.7e−01 |
| 2 | | | 3.7e−06 | | 4.9e−01 |
| 3 | | | 5.5e−06 | | 5.2e−01 |
| 1 | 6 | 6 | 9.6e−07 | 6 | 4.2e−01 |
| 2 | | | 2.9e−07 | | 4.4e−01 |
| 3 | | | 5.2e−07 | | 4.7e−01 |
| 1 | 7 | 8 | 7.8e−08 | 7 | 3.8e−01 |
| 2 | | | 1.8e−08 | | 4.0e−01 |
| 3 | | | 2.9e−08 | | 4.2e−01 |
| 1 | 8 | 8 | 5.8e−09 | 8 | 3.5e−01 |
| 2 | | | 1.4e−09 | | 3.6e−01 |
| 3 | | | 2.2e−09 | | 3.8e−01 |
| 1 | 9 | 8 | 3.5e−10 | 9 | 3.1e−01 |
| 2 | | | 1.0e−10 | | 3.3e−01 |
| 3 | | | 1.9e−10 | | 3.5e−01 |
| 1 | 10 | 8 | 3.0e−11 | 10 | 2.9e−01 |
| 2 | | | 7.0e−12 | | 3.0e−01 |
| 3 | | | 1.1e−11 | | 3.2e−01 |
| 1 | 11 | 8 | 4.4e−12 | 11 | 2.6e−01 |
| 2 | | | 1.4e−13 | | 2.7e−01 |
| 3 | | | 6.2e−13 | | 2.9e−01 |

**Table 3** Relative errors obtained in Test 1 when $I_1(1.001)$ is evaluated by a simultaneous rational rule (SRQF$^{(1)}$), compared with those obtained in ([9],1999) by a Gaussian rational formula (GRQF)

| $r$ | deg $\alpha$ | GRQF | SRQF$^{(1)}$ | $r$ | deg $\alpha$ | GRQF | SRQF$^{(1)}$ |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 4.1e−01 | 3.7e−03 | 5 | 4 | 2.7e−01 | 2.1e−05 |
| 6 | 4 | 8.4e−03 | 1.1e−06 | 6 | 4 | 8.4e−03 | 1.1e−06 |
| 9 | 6 | 1.1e−04 | 3.7e−10 | 7 | 4 | 3.4e−04 | 9.8e−08 |
| 12 | 8 | 6.0e−07[a] | 5.1e−13 | 9 | 4 | 6.6e−06 | 4.3e−10 |
| | | | | 10 | 4 | 6.1e−06 | 3.6e−11 |
| 2 | 1 | 4.1e−01 | 3.8e−01 | | | | |
| 4 | 2 | 2.9e−02 | 4.0e−04 | 5 | 2 | 2.2e−03 | 3.3e−05 |
| 6 | 3 | 2.1e−03 | 8.1e−07 | 6 | 2 | 1.7e−04[a] | 1.8e−06 |
| 8 | 4 | 1.2e−05[a] | 7.3e−09 | 7 | 2 | 2.0e−05 | 1.6e−07 |
| 10 | 5 | 7.3e−06 | 2.3e−11 | 8 | 2 | 9.3e−06 | 1.2e−08 |
| 12 | 6 | 5.0e−06 | 5.6e−13 | 9 | 2 | 6.3e−08[a] | 7.3e−10 |

[a] This result was obtained from ([11], 2004)

The values of $H_k(s,0)$, $s = 0,\ldots,34$, are computed with a relative error of about 1.0e − 15 applying a composite Gauss–Legendre formula after fitting the smoothing transformations (40) with $p = q = 4$.

Relative errors when we applied simultaneous quadrature formulas of polynomial (SPQF, $K = 0$) and rational type (SRQF$^{(1)}$) are compared in Table 2.

A comparison with the numerical results in [9] is shown in Table 3 where selected output is organized. Here the errors are produced by a Gauss rational quadrature formula (GRQF) and a simultaneous rational quadrature formula (SRQF$^{(1)}$) when the measure is $w_1(x)\mathrm{d}\sigma(x) = \mathrm{d}x$. If deg $\alpha = K$ then $\max\{r - K, 0\}$ indicates polynomial participation in a mixed procedure. Numerical results in column GRQF were extracted from [9].

## 5.2 Test 2

Let $(s_1, s_2, s_3) = \mathrm{N}(\sigma_1, \sigma_2, \sigma_3)$, where $\mathrm{d}\sigma_1(x) = \mathrm{d}x/\sqrt{x}$, $x \in [0, 1]$, $\mathrm{d}\sigma_2(x) = x^2\mathrm{d}x$, $x \in [2, 3]$, and $\mathrm{d}\sigma_3(x) = \mathrm{d}x$, $x \in [4, 5]$. Then $\mathrm{d}s_1 = \mathrm{d}\sigma_1$,

$$\mathrm{d}s_2(x) = \left(-2.5 - x - x^2 \log\left(\frac{3-x}{2-x}\right)\right)\frac{\mathrm{d}x}{\sqrt{x}},$$

$$\mathrm{d}s_3(x) = \left(\int_2^3 \log\left(\frac{4-s}{5-s}\right)\frac{s^2\mathrm{d}s}{x-s}\right)\frac{\mathrm{d}x}{\sqrt{x}}.$$

Next, we describe the experiment of calculating simultaneously the following integrals

$$J_i(\omega) = \int_0^1 \frac{\Gamma(1+x)}{x+\omega}\,\mathrm{d}s_i(x), \quad i = 1, 2, 3. \tag{42}$$

The integral $J_1(\omega)$ is evaluated in [9] (Example 4.2) using a GRQF, and a discretization procedure for the modified inner product based on the recursion coefficients of the Jacobi polynomials with parameters $\alpha = 0$, $\beta = -1/2$.

The common integrand in (42) has poles at $\omega$ and at $x = -1 - j$, $j \in \mathbb{N}$. The factor $1/\sqrt{x}$ with a non polar singularity at $x = 0$ is part of the weight functions but, unlike in ([9], Example 4.2), it does not participate in the definition of the Gaussian quadrature formula to be applied in evaluating (36).

Once more we fit the transformation (40) into the modified moments $H_k(s, 0)$, not into the target integrals (42). After that, we apply a Gauss–Legendre rule to evaluate them. Table 4 shows the relative error produced by a SQRF$^{(2)}$ when (40) is taken with $\omega = 0.001$, $a = 0$, $b = 1$, $p = 6$, $q = 1$, and the integrating measures are now modified by $1/\alpha(x)$, where $\alpha(x) = (x + \omega)\prod_{j=1}^{K}(x + j)$.

The present AT-system requires numerical routines to simulate the measure $ds_3$ in the Nikishin system. This restriction is due to the slow performance of symbolic tools. For that reason we have implemented several routines and found that those based on a quadrature formula seem to be the most stable. However, instability still shows up for $r \geq 7$, so the estimate of $H_k(s, r+1)$ obtained in step 5 should be improved. If $\mathrm{X}_r^{(0)}$ is the solution of $\mathrm{A}_r\mathrm{X}_r = \mathrm{B}_r$ we obtain a prediction

**Table 4** Relative errors obtained in Test 2 when the integrals $J_i(0.001)$, $i = 1,2,3$, with respect to the Nikishin AT-system $\mathrm{N}(\mathrm{d}x/\sqrt{x}, x^2\mathrm{d}x, \mathrm{d}x)$, are evaluated by a simultaneous rational formula

| $w_k$ | $r$ | deg $\alpha$ | SRQF$^{(2)}$ | $r$ | deg $\alpha$ | SRQF$^{(2)}$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1.7e−02 | 6 | 3 | 2.2e−12 |
| 2 | | | 1.1e−02 | | | 3.2e−12 |
| 3 | | | 1.1e−02 | | | 3.4e−12 |
| 1 | 4 | 1 | 2.9e−05 | 8 | 3 | 2.3e−14 |
| 2 | | | 1.7e−05 | | | 6.0e−13 |
| 3 | | | 1.7e−05 | | | 6.2e−13 |
| 1 | 6 | 1 | 4.1e−08 | 4 | 4 | 2.4e−07 |
| 2 | | | 2.5e−08 | | | 2.5e−07 |
| 3 | | | 2.5e−08 | | | 2.7e−07 |
| 1 | 6 | 2 | 1.2e−10 | 6 | 4 | 4.1e−12 |
| 2 | | | 6.1e−11 | | | 2.6e−11 |
| 3 | | | 6.1e−11 | | | 2.7e−11 |
| 1 | 8 | 2 | 1.3e−11 | 8 | 4 | 3.0e−13 |
| 2 | | | 6.3e−12 | | | 3.7e−13 |
| 3 | | | 6.4e−12 | | | 3.9e−13 |
| 1 | 4 | 3 | 7.5e−08 | 9 | 4 | 1.4e−15 |
| 2 | | | 1.8e−07 | | | 1.2e−15 |
| 3 | | | 2.0e−07 | | | 1.3e−15 |

$$H_k^{(0)}(s,r+1) = E_k(s,r,\mathrm{X}_r^{(0)}), \quad k = 1,2,3, \;\; s = 0,\ldots,s_0 - r - 1. \qquad (43)$$

The correction process is carried out by adding non basic equations, that is, we now obtain $\mathrm{X}_r^{(1)}$ from the over determined system $[\mathrm{A}'_r;\mathrm{A}_r]\mathrm{X}_r = [\mathrm{B}'_r;\mathrm{B}_r]$. Finally, we obtain a corrected value $H_k^{(1)}(s,r+1)$ from (43) when $\mathrm{X}_r^{(1)}$ takes the place of $\mathrm{X}_r^{(0)}$. Our calculations have involved matrices $\mathrm{A}'_r$ having up to 39 rows. An estimate $\mathrm{X}_r^{(1)}$ can also be obtained by solving $\mathrm{A}'_r\mathrm{X}_r = \mathrm{B}'_r$.

## 6 Conclusions

Extensive testing confirms the theoretical geometrical rate of convergence of the SRQF for functions analytic on a neighborhood of the interval of inte-

**Table 5** Relative errors obtained in Test 2 when $J_1(0.001)$ is evaluated by a simultaneous rational rule (SRQF$^{(2)}$), compared with those obtained in ([9], 1999) by a rational Gaussian formula (GRQF)

| $r$ | deg $\alpha$ | GRQF | SRQF$^{(2)}$ | $r$ | deg $\alpha$ | GRQF | SRQF$^{(2)}$ |
|---|---|---|---|---|---|---|---|
| 4 | 2 | 2.8e−05 | 5.5e−07 | 2 | 1 | 4.3e−03 | 1.7e−02 |
| 6 | 3 | 5.4e−07 | 2.2e−12 | 3 | 1 | 1.2e−04 | 5.4e−04 |
| 8 | 4 | 1.4e−08 | 3.0e−13 | 4 | 1 | 3.4e−06 | 2.9e−05 |
| 10 | 5 | 9.3e−08 | 1.5e−10 | 5 | 1 | 1.0e−07[a] | 1.5e−06 |
| 3 | 2 | 2.1e−03 | 2.4e−05 | 9 | 6 | 2.1e−06 | 2.4e−09 |
| 6 | 4 | 8.0e−05 | 4.1e−12 | 12 | 8 | 1.4e−08 | 3.4e−12 |

[a] This result was obtained from ([11], 2004)

gration, which we have attained with our numerical approach even when the integrand is meromorphic in a neighborhood of the integration interval with some poles very close to the integration interval.

In principle, the technique of changing the variable can be directly applied to the integrals under interest, and is much more flexible than a GRQF. Nevertheless, the more specific nature of the GRQFs makes them eventually more efficient. An argument in favor of combining both techniques smoothing transformations and rational rules, is that the former seems to be more effective in calculating the modified moments (36) rather than the integral under consideration.

The first principle we have adopted is related to the rank of the matrices considered in this work. An experimental fact is that practically all the matrices $\mathrm{A}_r$, $r = 1, \ldots, 12$, are scarcely full rank up to the tolerance tol $= 1.0\mathrm{e} - 15$, whereas the singular values of $[\mathrm{A}'_r; \mathrm{A}_r]$ are greater than $1.0\mathrm{e} - 6$.

Table 5 suggests that the routines to simulate the rational simultaneous rule used in Test 2 ($\mathrm{SRQF}^{(2)}$) are less stable than those used in Test 1 ($\mathrm{SRQF}^{(1)}$). It can also be observed that for $\deg \alpha \geq 2$ our approach seems to be superior to that presented in [9,11] (see column GRQF of Table 5).

The execution time is one of the most favorable aspects of our procedure. About three seconds are needed to compute and display $H_k(s,r)$, $s = 0, \ldots, 30$, $r = 1, \ldots, 12$, running Matlab® on a computer with an Intel Pentium 4 processor, 2.0 GHz and 512 Mb RAM.

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