Citation for published version:

Fiestras-Janeiro, M.G., García-Jurado, I., Meca, A. *et al.* On benefits of cooperation under strategic power. *Ann Oper Res* **288**, 285–306 (2020). https://doi.org/10.1007/s10479-019-03495-6

Peer reviewed version

DOI: 10.1007/s10479-019-03495-6

Annals of Operations Research manuscript No.
(will be inserted by the editor)

On benefits of cooperation under strategic power

M. G. Fiestras-Janeiro · I. García-Jurado · A. Meca · M. A. Mosquera

Received: date / Accepted: date

Abstract We introduce a new model involving TU-games and exogenous structures. Specifically, we consider that each player in a population can choose an element in a strategy set and that, for every possible strategy profile, a TU-game is associated with the population. This is what we call a *TU-game with strategies*. We propose and characterize the maxmin

M. G. Fiestras-Janeiro

Departamento de Estatística e Investigación Operativa. Universidade de Vigo, Spain.

E-mail: fiestras@uvigo.es

I. García-Jurado

CITIC & Departamento de Matemáticas. Universidade da Coruña, Spain.

E-mail: ignacio.garcia.jurado@udc.es

A. Meca

I. U. Centro de Investigación Operativa. Universidad Miguel Hernández de Elche, Spain.

E-mail: ana.meca@umh.es

M. A. Mosquera

Corresponding author.

Departamento de Estatística e Investigación Operativa. Universidade de Vigo, Spain.

E-mail: mamrguez@uvigo.es

procedure to map every game with strategies to a TU-game. We also study whether or not the relevant properties of TU-games are transmitted by applying the maxmin procedure. Finally, we examine two relevant classes of TU-games with strategies: airport and simple games with strategies.

Keywords game theory · cooperative games · maxmin procedure · strategies · airport games · simple games

1 Introduction

After its introduction in von Neumann and Morgenstern (1944), the model of TU-games has been widely used to analyze cooperation in multi-agent decision problems. Just to give few examples, Çetiner and Kimms (2013) applied TU-games in revenue sharing for airline alliances, Fiestras-Janeiro et al. (2015) in inventory control, Lozano et al. (2015) in production processes, Naber et al. (2015) in allocating CO_2 emission, Kimms and Kozeletskyi (2016) in logistics, Li et al. (2016) in efficiency evaluation, Goyal and Kaushal (2017) in traffic management, and Balog et al. (2017) in finance. In the seventies, several authors dealt with cooperative situations in which the cooperation was somewhat restricted by exogenous structures. For instance, Aumann and Dreze (1974) treated TU-games with a coalition structure, Myerson (1977) studied TU-games with graph-restricted communication, and Owen (1977) considered TU-games with a priori unions. Since then, a number of models and solutions for TU-games with structures have been studied in the literature. A couple of recent papers on this topic are Alonso-Meijide et al. (2015) and Fernández et al. (2016). Bilbao (2000) is a book devoted to this topic.

In this paper, we introduce a novel model involving TU-games and exogenous structures. Specifically, we consider that each player in a population chooses one of the strategies in a strategy set. Next, a TU-game that depends on the selected strategy profile is associated with the population. This is what we call a *TU-game with strategies*. This model can be useful in a number of practical situations. For instance, consider a set of agents that have to divide an amount of money; they can negotiate directly or, on the contrary, they can previously take some (costly) actions that will modify their negotiation power. Any situation of this type can be modeled as a TU-game with strategies. A reasonable recommendation for the players involved in one such process is that they negotiate directly, avoiding the costly actions, but taking into account their capacities for changing the negotiation power. This is the main idea of our approach to TU-games with strategies: to associate each TU-game with strategies to a new TU-game that appropriately reflects the bargaining coalitional power of the involved players.

As far as we know, this approach is not used in the literature to analyze situations involving strategic and cooperative environments. Instead, the existing literature tried to reduce the initial model to a strategic game in order to study its values and solutions. For instance, Ui (2000) introduced the model of TU-games with action choices. This mathematical model is similar to our model but the value of a coalition in any TU-game is not affected by the strategies of players outside that coalition. Moreover, its analysis focused on reducing a TU-game with action choices to a strategic game where the payoff of each strategy profile is determined by the Shapley value of the corresponding TU-game. In the same line, Brandenburger and Stuart (2007) introduced the class of biform games where the value of each coalition depends on the strategies of all players, like in our model. Nevertheless, their analysis is similar to that in Ui (2000), in the sense that they reduce a biform game to a strategic game, where the payoff of each strategy profile is determined by a particular element in the core of the corresponding TU-game. Platkowski (2016) introduced the class of evolutionary coalitional games. Again, the initial situation is transformed into a strategic game where

NR	NL	L
NL	u	u
L	u	v

R	NL	L
NL	u	u
L	u	w

Fig. 1 The TU-game with strategies in the three heirs willing to divide the inheritance.

the payoffs are obtaining by redistributing the sum of the initial individual payoffs using a well-known solution concept for TU-games. On the other hand, some literature is devoted to the study of uncertainty in TU-games. Habis and Herings (2011) considered a model where agents are involved in a particular TU-game depending on the state of nature chosen from a finite set. In this model, Habis and Herings (2011) are interested in finding stable allocations in each state. Ertemel and Kumar (2018) studied rationing problems where the claims are state dependent. They proposed an extension of the proportional rule as an allocation rule.

Let us see now a few examples that fit our model and can be used to motivate our approach. Consider a situation in which three heirs, at first symmetrical, share an inheritance of three million euros and want to divide it. The heirs 1 and 2 can take legal actions that cost 0.25 million euros for each of the three. If heirs 1 and 2 take these actions, they will keep the entire inheritance. However, heir 3 can respond with new legal actions; if heir 3 responds, all heirs will have an additional cost of 0.25 million euros each and the final result will be that heirs 1 and 2 keep two millions and the remaining million is divided among the three. This situation can be modeled as the TU-game with strategies given in Figure 1. In this game heir 1 chooses row, heir 2 chooses column, and heir 3 chooses box. NL and L stand for "no legal actions" and "legal actions", respectively; NR and R stand for "no react" and "react", respectively. Finally, $N = \{1,2,3\}$ and u, v, w are TU-games with set of players N given by:

- $u(S) = 0$ for all $S \subsetneq N$, $u(N) = 3$.
- $v(1) = v(2) = v(3) = -0.25$, $v(12) = 2.5$, $v(13) = v(23) = -0.5$, $v(N) = 2.25$.

- $w(1) = w(2) = w(3) = -0.5$, $w(12) = 1$, $w(13) = w(23) = -1$, $w(N) = 1.5$.

A sensible advise for the heirs can be the following: do not litigate, negotiate, but take into account in your negotiation the litigation options that you have. This advise is incorporated in our approach to TU-games with strategies because we build a new TU-game that takes into account the litigation power of the involved players and, then, we suggest that they use such a new game to allocate the inheritance. Since the main feature here is that heirs 1 and 2 can guarantee one million, the TU-game that seems to reflect the real power of the heirs is $\bar{u}$ given by $\bar{u}(12) = 1$, $\bar{u}(S) = u(S)$ for all other $S \subset N$. Using the Shapley value,[1] Φ, to allocate the inheritance, the results is $\Phi(\bar{u}) = (7/6, 7/6, 4/6)$ which takes into account the negotiation power of heirs 1 and 2.

Consider now a cost allocation situation with three companies, $N = \{1,2,3\}$, involved in the realization of three projects. Each of these projects is an extension of the previous and has a cost of 100, 200, or 300 millions of euros. This means, for example, that project 2, which costs 200 millions, is an extension of project 1, with a cost of 100 millions. There is a first stage in which company 1 can choose between applying for a subsidy (a), or not applying for a subsidy (b), for the realization of all the projects. If such a subsidy is granted, the cost of each project is reduced by 10 millions of euros. Depending on the choice of company 1, the resulting cost allocation problems (in millions of euros) are respectively described by the following airport games (Littlechild and Owen 1973):

- $c_a(1) = 90$, $c_a(2) = 190$, $c_a(3) = 290$, $c_a(12) = 190$, $c_a(13) = c_a(23) = c_a(N) = 290$,
- $c_b(1) = 100$, $c_b(2) = 200$, $c_b(3) = 300$, $c_b(12) = 200$, $c_b(13) = c_b(23) = c_b(N) = 300$.

Assume that, the three companies decide to cooperate and to allocate the costs using the Shapley value, Φ. It seems reasonable to conclude that company 1 will choose to apply

[1] For an introduction to cooperative games and to the Shapley value, González-Díaz et al. (2010) can be consulted.

for the subsidy (a) and the cost allocation problem will be c_a. However, the Shapley value of c_a is $\Phi(c_a) = (30, 80, 180)$ and it does not take into account the special contribution of company 1. Notice that in this example we are dealing with cost games. We propose the following alternative approach. A new cost game $\bar{c}$ is built; it associates with every coalition S the minimum cost that S can guarantee making the right choices under its control. So,

$$\bar{c}(S) = \begin{cases} c_a(S) & \text{if } 1 \in S \\ c_b(S) & \text{if } 1 \notin S, \end{cases}$$

i.e., $\bar{c}(1) = 90$, $\bar{c}(2) = 200$, $\bar{c}(3) = 300$, $\bar{c}(12) = 190$, $\bar{c}(23) = 300$, $\bar{c}(13) = \bar{c}(N) = 290$. Observe that $\Phi(\bar{c}) = (23.33, 83.33, 183.33)$ seems to be a more reasonable allocation of the total cost in this situation, due to the special contribution of company 1. Indeed, the Shapley value for this alternative situation divides equally the subsidy of 10 millions, i.e. 10/3, and adds this amount to the previous costs of companies 2 and 3, while reducing in 20/3 the cost of company 1.

Finally we present a political example. The composition of the Spanish Congress of Deputies (the highest legislative body in Spain) arising from the general elections held on 28 April 2019 is as shown in Table 1 below.[2] The Spanish Congress of Deputies consists of 350 seats, so the simple majority is reached with 176 votes. We denote by N the set of parties represented in the Congress of Deputies and by v_0 the simple TU-game resulting from it.[3] Taking into account the current situation of Spanish politics, most analysts agree that there are two possible coalitions on which the governance of Spain can be based. One is the coalition of the socialists (PSOE) and the centre-right liberal party (Cs). If this coalition is formed, the resulting voting problem is described by the simple game v_1 in which the

[2] In order not to lengthen with unnecessary details, we will only give a brief description of the political parties corresponding with the acronyms when relevant to the present example.

[3] It is the weighted majority game given by the players in N, the quota 176 and the weights $(123, 66, 57, 42, 24, 15, 7, 6, 4, 2, 2, 1, 1)$.

Scenario {PSOE, Cs}

A	A	N
A	v_1	v_1
N	v_1	v_1

N	A	N
A	v_0	v_0
N	v_0	v_0

Scenario {PSOE, UP, ERC}

A	A	N
A	v_2	v_0
N	v_0	v_0

N	A	N
A	v_2	v_0
N	v_0	v_0

Fig. 2 The TU-game with strategies in the Congress of Deputies.

set of players is N and the unique minimal winning coalition is $\{\text{PSOE}, \text{Cs}\}$. The other is formed by the socialists (PSOE), the left-wing party (UP) and the Catalonian left-wing, republican and pro-independence party ERC. If this coalition is formed, the resulting voting problem is described by the simple game v_2 in which the set of players is N and the unique minimal winning coalition is $\{\text{PSOE}, \text{UP}, \text{ERC}\}$. As the King of Spain usually entrusts the formation of government to the most voted party, the PSOE can turn to Cs or UP-ERC to form a coalition government; note that later, Cs, UP and ERC may or may not accept the PSOE proposal. This situation can be modeled as the TU-game with strategies in Figure 2 in which PSOE chooses scenario, Cs chooses box, UP chooses row, and ERC chooses column; A and N stand for "accept the PSOE proposal" and "do not accept the PSOE proposal", respectively.

In this scenario, if we want to measure the power of each party, we propose the following approach. Like in the examples above, a new game $\bar{v}$ is built; it associates with every coalition S the power that S can guarantee making the right choices under its control. Observe that this example is more complex than the two others and, in this case, it is not clear how to build $\bar{v}$. We do it later, in Example 3.

These three examples illustrate situations that can be modeled as TU-games with strategies. In each of these games, there is a set of players, a set of strategies for every player, and a map that associates a TU-game with every possible strategy profile. In order to allocate the benefits arising from cooperation to the players, we build a new TU-game that maps every possible coalition to the optimal value that it can guarantee based on the strategies selected by its members. The reader may notice that the game under study is a TU-game with strategies and we then apply a procedure to transform it into a TU-game that appropriately reflects the bargaining coalitional power of the involved players. This procedure to build a TU-game is also used in other environments like strategic games. For instance, Carpente et al. (2005) analyzed strategic games where players can cooperate and used this procedure to build a TU-game from the strategic game in order to define a value for it. Differently, Kalai and Kalai (2013) used cooperation to define values for strategic games by decomposing each strategic game to cooperative and competitive components.

In the next section, we formally introduce and analyze the procedure used to map a TU-game with strategies to a TU-game. We also provide an axiomatic characterization of this procedure. In Section 3, we determine which properties of the TU-games defining a TU-game with strategies are transmitted through the procedure. In Section 4, we analyze the applications of TU-games with strategies for airport games and simple games. Finally, Section 5 presents our conclusions.

2 TU-games with strategies

Let N be any finite set of players. A transferable utility game (in short a TU-game) is a function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. We denote by $G(N)$ the set of TU-games with player set N and by G the set of all TU-games with a finite set of players.

Definition 1 A TU-game with strategies with player set N is a pair (X,V) such that:

- $X = \prod_{i\in N} X_i$ is a finite set of strategy profiles, being X_i the player's i strategy set for every $i \in N$, and
- $V : X \to G(N)$ is a map that associates a TU-game $V(x)$ with every $x \in X$.

We denote by $SG(N)$ the set of TU-games with strategies and player set N and by SG the set of all TU-games with strategies with a finite set of players.

Notice that a TU-game with strategies is the same mathematical model as a biform game, introduced in Brandenburger and Stuart (2007). However, the interpretation of the model is completely different in this paper. Here, we assume that players can coordinate their strategies and cooperate, and that the grand coalition aims to allocate among its members the benefits of the cooperation by considering their strategic power. So, we focus on a procedure to transform a TU-game with strategies into a TU-game. We do it using an axiomatic approach, i.e. we first identify some properties that are appropriate for such a procedure and then we prove that there exists only one procedure satisfying those properties. The properties are inspired by analogous ones introduced in Carpente et al. (2005).

Definition 2 A procedure to transform a TU-game with strategies into a TU-game (in brief a *procedure*) is a map $\phi : SG \to G$ that associates a TU-game $\phi(X,V) \in G(N)$ with every TU-game with strategies $(X,V) \in SG(N)$.

Let us state some properties that are appropriate for a procedure ϕ.

Individual objectivity. For every $(X,V) \in SG(N)$, if a player $i \in N$ is such that $V(x)(i) = c$, for all $x \in X$, then $\phi(X,V)(i) = c$.

Individual objectivity states that if the coalition consisting of player i receives the same utility c for every possible strategy profile, the TU-game resulting from the procedure associates c to coalition $\{i\}$.

Take $(X,V) \in SG(N)$, $i \in N$, and $S \subset N$ with $i \in S$. A strategy $x_i \in X_i$ of player i is *weakly dominated* in S if there exists a strategy $x_i' \in X_i$, $x_i' \neq x_i$, such that $V(\bar{x}_{-i}, x_i')(S) \geq V(\bar{x}_{-i}, x_i)(S)$ for all $\bar{x}_{-i} \in \prod_{j \in N \setminus \{i\}} X_j$. Moreover, (X^{-x_i}, V) denotes the TU-game with strategies that is obtained from (X,V) by deleting strategy x_i.

Irrelevance of weakly dominated strategies. For any $(X,V) \in SG(N)$, $i \in N$, and $S \subset N$ with $i \in S$, if strategy $x_i \in X_i$ is weakly dominated in S, then $\phi(X,V)(S) = \phi(X^{-x_i}, V)(S)$.

Irrelevance of weakly dominated strategies states that if a player loses the ability to use a weakly dominated strategy (for a coalition S) in the original game with strategies, this should not affect the final result for S.

Take $(X,V) \in SG(N)$, $j \in N$, and $S \subset N \setminus \{j\}$. A strategy $x_j \in X_j$ of player j is a *weakly dominated threat* to coalition S if for every $\bar{x}_{-j} \in \prod_{k \in N \setminus \{j\}} X_k$ there exists a strategy $x_j' \in X_j$, $x_j' \neq x_j$, such that $V(\bar{x}_{-j}, x_j')(S) \leq V(\bar{x}_{-j}, x_j)(S)$. In words, a strategy of player j is a weakly dominated threat to coalition S if there exists a different strategy of player j that can further harm coalition S.

Irrelevance of weakly dominated threats. For any $(X,V) \in SG(N)$, player $j \in N$, and $S \subset N \setminus \{j\}$, if strategy $x_j \in X_j$ is a weakly dominated threat to coalition S, then $\phi(X,V)(S) = \phi(X^{-x_j}, V)(S)$.

Irrelevance of weakly dominated threats states that if a player j loses the ability to use a weakly dominated threat to a coalition S in the original game with strategies, this should not affect the final result for S.

Take $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$. Denote by N^S the set $\{[S]\} \cup N \setminus S$, i.e. the set of $|N| - |S| + 1$ players in which the coalition S is considered as a single player, and by (X^S, V^S)

the TU-game with strategies that is obtained from (X,V) by considering the coalition S as a single player.[4]

Merge invariance. Take $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$, then, for all $T \subset N \setminus S$, $\phi(X,V)(T) = \phi(X^S,V^S)(T)$ and $\phi(X,V)(T \cup S) = \phi(X^S,V^S)(T \cup \{[S]\})$.

Merge invariance states that a coalition of players cannot influence its worth by merging and acting as one player.

Now we propose a specific procedure that, as we see later, is characterized by the properties introduced in this section.

Definition 3 The *maxmin procedure* is the map $\psi : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by:

$$\psi(X,V)(S) = \max_{x_S \in \prod_{i \in S} X_i} \min_{x_{N \setminus S} \in \prod_{i \in N \setminus S} X_i} V(x_S, x_{N \setminus S})(S)$$

for all $S \subset N$.

The next result provides an axiomatic characterization of ψ. It is an extension and an improvement of Theorem 7 in Carpente et al. (2005). It is an extension because every finite strategic game $(\{X_i\}_{i \in N}, \{H_i\}_{i \in N})$ can be seen as the game with strategies (X,V) with $X = \prod_{i \in N} X_i$ and $V(x)(S) = \sum_{i \in S} H_i(x)$, for all $x \in X$ and all $S \subset N$. It is an improvement because it does not use a monotonicity property.

Theorem 1 *The maxmin procedure is the unique procedure to transform a TU-game with strategies into a TU-game that satisfies individual objectivity, irrelevance of weakly dominated strategies, irrelevance of weakly dominated threats and merge invariance.*

[4] (X^S,V^S) is given by $X^S = X_{[S]} \times (\prod_{i \in N \setminus S} X_i)$ with $X_{[S]} = \prod_{i \in S} X_i$, and for every $x^S \in X^S$ and every $T \subset N \setminus S$, $V^S(x^S)(T) = V(x)(T)$, and $V^S(x^S)(T \cup [S]) = V(x)(T \cup S)$.

Proof First, we show that ψ satisfies the four properties. Let $(X,V) \in SG(N)$ and $i \in N$ be such that $V(x)(i) = c$, for all $x \in X$. Then it is clear that $\psi(X,V)(i) = c$, which shows that ψ satisfies individual objectivity.

To see that ψ satisfies irrelevance of weakly dominated strategies notice that, if strategy x_i for player i is weakly dominated in S ($S \subset N$ with $i \in S$) for $(X,V) \in SG(N)$, then

$$\begin{aligned}
\psi(X,V)(S) &= \max_{x_S \in \prod_{j\in S} X_j} \min_{x_{N\setminus S} \in \prod_{j\in N\setminus S} X_j} V(x_S, x_{N\setminus S})(S) \\
&= \max_{x_S \in \prod_{j\in S\setminus\{i\}} X_j \times X_i \setminus \{x_i\}} \min_{x_{N\setminus S} \in \prod_{j\in N\setminus S} X_j} V(x_S, x_{N\setminus S})(S) \\
&= \psi(X^{-x_i}, V)(S)
\end{aligned}$$

To check that ψ satisfies irrelevance of weakly dominated threats, notice that if strategy $x_j \in X_j$ of a player j is a weakly dominated threat to coalition $S \subset N \setminus \{j\}$ for $(X,V) \in SG(N)$, then

$$\begin{aligned}
\psi(X,V)(S) &= \max_{x_S \in \prod_{k\in S} X_k} \min_{x_{N\setminus S} \in \prod_{k\in N\setminus S} X_k} V(x_S, x_{N\setminus S})(S) \\
&= \max_{x_S \in \prod_{k\in S} X_k} \min_{x_{N\setminus S} \in \prod_{k\in N\setminus (S\cup\{j\})} X_k \times X_j \setminus \{x_j\}} V(x_S, x_{N\setminus S})(S) \\
&= \psi(X^{-x_j}, V)(S)
\end{aligned}$$

Finally, it is clear that ψ satisfies merge invariance.

We now proceed to show that any procedure satisfying the four properties must coincide with ψ. Let $\phi : SG \to G$ be a procedure satisfying individual objectivity, irrelevance of weakly dominated strategies, irrelevance of weakly dominated threats, and merge invariance. Take $(X,V) \in SG(N)$, we have to prove that $\phi(X,V)(S) = \psi(X,V)(S)$ for all $S \subset N$. We do it by induction in $|N|$, the number of elements of N.

Assume first that $|N| = 1$, i.e., that $N = \{i\}$. Since ϕ satisfies irrelevance of weakly dominated strategies and individual objectivity, and by definition of ψ, we have that

$$\phi(X,V)(\{i\}) = \phi(\{x^V\},V)(\{i\}) = \max_{x\in X} V(x)(\{i\}) = \psi(X,V)(\{i\}) \tag{1}$$

where x^V is a strategy profile such that $V(x^V)(\{i\}) = \max_{x\in X} V(x)(\{i\})$.

Second, we prove the result when $|N| = 2$ and consider $N = \{1,2\}$ without loss of generality. Take $(X^{\{1,2\}},V^{\{1,2\}}) \in SG(N^{\{1,2\}})$. Notice that $|N^{\{1,2\}}| = 1$ and then, by (1)

$$\phi(X^{\{1,2\}},V^{\{1,2\}})(N^{\{1,2\}}) = \psi(X^{\{1,2\}},V^{\{1,2\}})(N^{\{1,2\}}). \tag{2}$$

Since ϕ and ψ satisfy merge invariance, it is clear that (2) implies

$$\phi(X,V)(N) = \psi(X,V)(N).$$

Now we prove the result for individual coalitions $\{1\}$ and $\{2\}$. We do it for $\{1\}$; it can be done analogously for $\{2\}$.

Take a new strategy y_2 for player 2 and define $(X',V') \in SG(N)$ given by:

- $X' = X_1 \times (X_2 \cup \{y_2\})$
- $V'(x) = V(x)$ for all $x \in X$.
- $V'(x_1,y_2)(S) = \min_{x_2\in X_2} V(x_1,x_2)(S)$ for all $x_1 \in X_1$ and for all $S \subset N$.

Notice that y_2 is a weakly dominated threat to $\{1\}$ in (X',V') and that, at the same time, every $x_2 \in X_2$ is a weakly dominated threat to $\{1\}$ in (X',V'). Hence, since ϕ satisfies irrelevance of weakly dominated threats,

$$\phi(X',V')(\{1\}) = \phi(X,V)(\{1\})$$

and, at the same time,

$$\phi(X',V')(\{1\}) = \phi(X_1 \times \{y_2\},V')(\{1\}).$$

Thus, $\phi(X,V)(\{1\}) = \phi(X_1 \times \{y_2\}, V')(\{1\})$. Now, taking into account that ϕ satisfies irrelevance of weakly dominated strategies and individual objectivity, and by definition of ψ, it is clear that

$$\phi(X_1 \times \{y_2\}, V')(\{1\}) = \max_{x_1 \in X_1} V'(x_1, y_2)(\{1\}) = \max_{x_1 \in X_1} \min_{x_2 \in X_2} V(x_1, x_2)(\{1\}) = \psi(X,V)(\{1\}).$$

Therefore, $\phi(X,V)(\{1\}) = \psi(X,V)(\{1\})$.

Assume now that the result is true when $|N| < n$. Take N with $|N| = n \geq 3$ and $i, j \in N$ $(i \neq j)$. Take also $(X^{\{i,j\}}, V^{\{i,j\}}) \in SG(N^{\{i,j\}})$. Notice that $|N^{\{i,j\}}| = n-1$ and then, by induction hypothesis

$$\phi(X^{\{i,j\}}, V^{\{i,j\}})(T) = \psi(X^{\{i,j\}}, V^{\{i,j\}})(T) \tag{3}$$

for all $T \subset N^{\{i,j\}}$. Since ϕ and ψ satisfy merge invariance, it is clear that (3) implies

$$\phi(X,V)(S) = \psi(X,V)(S)$$

for all $S \subset N \setminus \{i,j\}$ and for all $S \subset N$ with $\{i,j\} \subset S$. So, the equality has been proved for all the coalitions in the family $\mathscr{F}_{\{i,j\}}$ given by

$$\mathscr{F}_{\{i,j\}} = \{S \in 2^N \ : \ S \cap \{i,j\} = \emptyset \text{ or } S \cap \{i,j\} = \{i,j\}\}.$$

Notice that $\bigcup_{i \in N} \bigcup_{j \in N \setminus \{i\}} \mathscr{F}_{\{i,j\}} = 2^N$ for all N with $|N| \geq 3$. Hence, the proof is concluded.

□

Remark 1 The axiomatic characterization in Theorem 1 is tight in the sense that all the properties in the theorem are independent. We provide the proof in the Appendix.

Notice that this procedure is the one that we used in the introduction of this paper to tackle the three examples presented there. In order to verify the procedure, we will now revisit the same three examples.

Example 1 We consider the case of the three heirs willing to divide the inheritance. That situation can be modeled as a TU-game with strategies (X,V) such that:

- $X = X_1 \times X_2 \times X_3$, where $X_1 = X_2 = \{NL, L\}$, $X_3 = \{NR, R\}$.
- V is given by $V(L,L,NR) = v$, $V(L,L,R) = w$ and $V(x) = u$ for all other $x \in X$.

It is easy to check that $\psi(X,V)(S) = \bar{u}(S)$ for all $S \subset N$.

Example 2 We now consider the case of the three companies involved in the realization of three projects whose costs depend on the company 1's choice between applying for a subsidy (a) or not applying for a subsidy (b). Recall that the cost functions are given by

- $c_a(1) = 90$, $c_a(2) = 190$, $c_a(3) = 290$, $c_a(12) = 190$, $c_a(13) = c_a(23) = c_a(N) = 290$,
- $c_b(1) = 100$, $c_b(2) = 200$, $c_b(3) = 300$, $c_b(12) = 200$, $c_b(13) = c_b(23) = c_b(N) = 300$.

This situation can be modeled as the TU-game with strategies (X,C) such that:

- $X = X_1 \times X_2 \times X_3$, where $X_1 = \{a, b\}$, $X_2 = \{\alpha\}$, $X_3 = \{\beta\}$.
- C is given by $C(a,\alpha,\beta) = c_a$ and $C(b,\alpha,\beta) = c_b$.

Notice that c_a and c_b are cost TU-games. Then, for preserving the philosophy of the maxmin procedure we have to interchange maximum and minimum in its definition; with this in mind, it is easy to check that $\psi(X,C)(S) = \bar{c}(S)$, for all $S \subset N$.

Example 3 Finally we consider the Spanish Congress of Deputies described in the Introduction. This situation can be modeled as the TU-game with strategies (X,V) such that:

- $X = \prod_{i \in N} X_i$, where N is the set of parties, and
  - $X_{PSOE} = \{\text{Cs}, \text{UP-ERC}\}$,
  - $X_{Cs} = \{\text{Yes}, \text{No}\}$,
  - $X_{UP} = \{\text{Yes}, \text{No}\}$,
  - $X_{ERC} = \{\text{Yes}, \text{No}\}$,

– $X_i = \{\alpha_i\}$, for all other i in N.

– V is given by:

$$V(x) = \begin{cases} v_1 & \text{if } x_{PSOE} = \text{Cs and } x_{Cs} = \text{Yes}, \\ v_2 & \text{if } x_{PSOE} = \text{UP-ERC}, x_{UP} = \text{Yes and } x_{URC} = \text{Yes}, \\ v_0 & \text{in any other case.} \end{cases}$$

Notice that in the informal description of the TU-game with strategies in the Introduction, we only took into account the players with more than one strategy (i.e., PSOE, Cs, UP and ERC). However, rigorously, the TU-game with strategies according to Definition 1 that models the situation described in this example is the one we have just indicated. Now, we can define $\bar{v}(S) = \psi(X,V)(S)$ for all $S \subset N$. Then we use the Shapley value Φ as a power index. Table 1 displays $\Phi(v_0)$ and $\Phi(\bar{v})$. We propose to use $\Phi(\bar{v})$ as a measure of the power of each party in this scenario. Note that Cs, UP and ERC have more power (measured by $\Phi(\bar{v})$) than they would if we only took into account the number of seats (measured by $\Phi(v_0)$), PSOE and PP (the conservative party) have less power, and all the other parties do not change their power. It may seem a little surprising that the PSOE loses power because it is responsible for taking the initiative to form a government; however, the fact that its possible government partners may turn their backs on it seems to provoke a certain weakness in the PSOE and a certain strength in such potential partners. The PP's loss of power is due to its lack of strategic capacity despite its high number of seats; this lack of strategic capacity has imperceptible effects on parties with few seats whose marginal power remains unchanged.

3 Inheritance of properties

In this section we analyze which properties are transmitted by the maxmin procedure. We want to know if for every TU-game with strategies (X,V) such that the TU-game $V(x)$

Parties	PSOE	PP	Cs	UP	VOX	ERC	JxC	PNV	EH Bildu	CCa	NA+	Comp	PRC
Seats	123	66	57	42	24	15	7	6	4	2	2	1	1
$\Phi(v_0)$	0.462	0.167	0.167	0.1	0.029	0.029	0.016	0.012	0.009	0.003	0.003	0.002	0.002
$\Phi(\bar{v})$	0.412	0.117	0.2	0.133	0.029	0.062	0.016	0.012	0.009	0.003	0.003	0.002	0.002

Table 1 The power in the Spanish Congress of Deputies.

satisfies some property for every strategy profile x, it is true that the TU-game $\psi(X,V)$ also satisfies this property. In particular, we study the properties of superadditivity, monotonicity, and balancedness. We recall the definition of these properties in the context of TU-games. Let $v \in G(N)$. We say that v is *superadditive* if $v(S\cup T) \geq v(S)+v(T)$ for every $S,T \subset N$ with $S\cap T = \emptyset$. We say that v is *monotone* if $v(S) \leq v(T)$ whenever $S \subset T$. We say that v is *balanced* if $Core(v) \neq \emptyset$ being

$$Core(v) = \{x \in \mathbb{R}^N : \sum_{i\in S} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geq v(S) \text{ for every } S \subset N\}.$$

Next theorem shows that the maxmin procedure transmits the superadditivity property.

Theorem 2 *Let $(X,V) \in SG(N)$. If $V(x)$ is superadditive for all $x \in X$, then $\psi(X,V)$ is also superadditive.*

Proof Assume that $V(x)$ is superadditive for all $x \in X$. Consider the TU-game obtained by the maxmin procedure:

$$\psi(X,V)(S) = \max_{x_S\in\prod_{i\in S} X_i} \min_{x_{N\setminus S}\in\prod_{i\in N\setminus S} X_i} V(x_S,x_{N\setminus S})(S),$$

for every $S \subset N$.

Assume that $\psi(X,V)$ is not superadditive. Then, there exist two non-empty coalitions $S,T \subset N$ with $S \cap T = \emptyset$ such that

$$\psi(X,V)(S \cup T) < \psi(X,V)(S) + \psi(X,V)(T).$$

For each $S \subset N$, let $x^S \in X$ be such that $\psi(X,V)(S) = V(x^S)(S)$. Take $\bar{x}_{N\setminus(S\cup T)} \in X_{N\setminus(S\cup T)}$ such that

$$V(x_S^S, x_T^T, \bar{x}_{N\setminus(S\cup T)})(S \cup T) = \min_{x_{N\setminus(S\cup T)} \in \Pi_{i\in N\setminus(S\cup T)} X_i} V(x_S^S, x_T^T, x_{N\setminus(S\cup T)})(S \cup T).$$

Then,

$$\begin{aligned}
\psi(X,V)(S\cup T) &< \psi(X,V)(S) + \psi(X,V)(T) \\
&\leq V(x_S^S, x_T^T, \bar{x}_{N\setminus(S\cup T)})(S) + V(x_S^S, x_T^T, \bar{x}_{N\setminus(S\cup T)})(T) \\
&\leq V(x_S^S, x_T^T, \bar{x}_{N\setminus(S\cup T)})(S\cup T) \\
&= \min_{x_{N\setminus(S\cup T)} \in X_{N\setminus(S\cup T)}} V(x_S^S, x_T^T, x_{N\setminus(S\cup T)})(S\cup T)
\end{aligned}$$

where the first inequality follows by assumption, the second one follows by definition of $\psi(X,V)$ and the choice of x^S and x^T, the third one follows by the superadditivity of each $V(x)$, and the last one follows by definition of $\bar{x}_{N\setminus(S\cup T)}$.

But, this is a contradiction since

$$\psi(X,V)(S\cup T) = \max_{x_{S\cup T} \in X_{S\cup T}} \min_{x_{N\setminus(S\cup T)} \in X_{N\setminus(S\cup T)}} V(x_{S\cup T}, x_{N\setminus(S\cup T)})(S).$$

□

Next result shows that the maxmin procedure transmits the monotonicity property.

Theorem 3 *Let $(X,V) \in SG(N)$. If $V(x)$ is monotone for all $x \in X$, then $\psi(X,V)$ is also monotone.*

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
V(U,L,F)	1	3	1	7	6	1	9
V(U,R,F)	1	1	1	2	7	9	10

Table 2 The TU-games associated with each strategy profile in Example 4.

Proof Assume that $V(x)$ is monotone for all $x \in X$. Consider the TU-game obtained by the maxmin procedure:

$$\psi(X,V)(S) = \max_{x_S \in \prod_{i\in S} X_i} \min_{x_{N\setminus S} \in \prod_{i\in N\setminus S} X_i} V(x_S, x_{N\setminus S})(S),$$

for every $S \subset N$. Let $S \subset T \subset N$. Then,

$$\begin{aligned}
\psi(X,V)(S) &= \max_{x_S \in \prod_{i\in S} X_i} \min_{x_{N\setminus S} \in \prod_{i\in N\setminus S} X_i} V(x_S, x_{N\setminus S})(S), \\
&= \max_{x_S \in \prod_{i\in S} X_i} \min_{x_{T\setminus S} \in \prod_{i\in T\setminus S} X_i} \min_{x_{N\setminus T} \in \prod_{i\in N\setminus T} X_i} V(x_S, x_{T\setminus S}, x_{N\setminus T})(S) \\
&\leq \max_{x_S \in \prod_{i\in S} X_i} \max_{x_{T\setminus S} \in \prod_{i\in T\setminus S} X_i} \min_{x_{N\setminus T} \in \prod_{i\in N\setminus T} X_i} V(x_S, x_{T\setminus S}, x_{N\setminus T})(T) \\
&= \psi(X,V)(T).
\end{aligned}$$

□

In general, the maxmin procedure does not transmit the balanced property as we see next.

Example 4 Let us take $N = \{1,2,3\}$, $X_1 = \{U\}$ $X_2 = \{L,R\}$ $X_3 = \{F\}$. The TU-games associated with each strategy profile are given in Table 2. It is easy to check that both games have non-empty cores. Nevertheless, the maxmin procedure provides the TU-game given in Table 3. Let us check now that $\psi(X,V)$ is not balanced. If the allocation (x_1,x_2,x_3) belongs to the core of $\psi(X,V)$ we have $x_1 = 10 - x_2 - x_3$, $x_1 \geq 1$, and $x_2 + x_3 \geq 9$. Then, $1 \leq x_1 \leq 1$. Since $x_1 = 1$, $x_1 + x_2 \geq 7$, and $x_1 + x_3 \geq 6$, we obtain $x_2 \geq 6$ and $x_3 \geq 5$. Thus, $x_1 + x_2 + x_3 > 10$ and we get a contradiction. Then, the core of the game $\psi(X,V)$ is empty.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$\psi(X,V)$	1	3	1	7	6	9	10

Table 3 The TU-game provided by the maxmin procedure for Example 4.

The fact that the balanced property is not transmitted by the maxmin procedure can be a drawback in some cases. Therefore, below we give a result that may be helpful to study the core of $\psi(X,V)$ using information from the original TU-games with strategies.

Theorem 4 *Let $(X,V) \in SG(N)$. Then, $Core(\psi(X,V)) = \cap_{x\in X} Core(V^x)$ where for every $x \in X$, V^x is the TU-game given by $V^x(N) = \psi(X,V)(N)$ and $V^x(S) = \min_{\bar{x}_{N\setminus S}} V(x_S, \bar{x}_{N\setminus S})(S)$, for every $\emptyset \neq S \subsetneq N$.*

Proof First, we check that $Core(\psi(X,V)) \subset \cap_{x\in X} Core(V^x)$. Let $a \in Core(\psi(X,V))$. By definition,

$$\sum_{i\in S} a_i \geq \psi(X,V)(S), \text{ for every } \emptyset \neq S \subsetneq N,$$
$$\sum_{i=1}^{n} a_i = \psi(X,V)(N).$$

Moreover, since $\psi(X,V)(N) = V^x(N)$ for all $x \in X$ and $\psi(X,V)(S) = \max_{x_S \in \prod_{i\in S} X_i} \min_{\bar{x}_{N\setminus S} \in \prod_{i\in N\setminus S} X_i} V(x_S, \bar{x}_{N\setminus S})(S)$ for every $\emptyset \neq S \subsetneq N$, it is hold, for every $x \in X$,

$$\sum_{i\in S} a_i \geq \min_{\bar{x}_{N\setminus S} \in \prod_{i\in N\setminus S} X_i} V(x_S, \bar{x}_{N\setminus S})(S) = V^x(S), \text{ for every } \emptyset \neq S \subsetneq N,$$
$$\sum_{i=1}^{n} a_i = V^x(N)$$

Then, we obtain that $a \in Core(V^x)$ for every $x \in X$. Therefore, $a \in \cap_{x\in X} Core(V^x)$.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$V^{(U,L,F)}$	1	3	1	7	6	1	10
$V^{(U,R,F)}$	1	1	1	2	6	9	10

Table 4 The TU-games V^x in Example 5.

Second, we prove that $\cap_{x\in X} Core(V^x) \subset Core(\psi(X,V))$. Let $a \in \cap_{x\in X} Core(V^x)$. By definition, for every $x \in X$,

$$\sum_{i\in S} a_i \geq V^x(S) = \min_{\bar{x}_{N\setminus S}\in\prod_{i\in N\setminus S} X_i} V(x_S, \bar{x}_{N\setminus S})(S), \text{ for every } \emptyset \neq S \subsetneq N,$$
$$\sum_{i=1}^{n} a_i = V^x(N).$$

Then,

$$\sum_{i\in S} a_i \geq \max_{x\in X} V^x(S) = \psi(X,V)(S), \text{ for every } \emptyset \neq S \subsetneq N,$$
$$\sum_{i=1}^{n} a_i = \psi(X,V)(N)$$

since $V^x(N) = \psi(X,V)(N)$ for every $x \in X$. Therefore, we obtain that $a \in Core(\psi(X,V))$.

□

We revisit Example 4 to illustrate this result.

Example 5 We have seen that the core of $\psi(X,V)$ is empty in Example 4. Here we analyze the core of each TU-game V^x for every $x \in X$ to illustrate Theorem 4. Notice that the TU-games V^x are given in Table 4. In Figure 3 we depict[5] the core of $V^{(U,L,F)}$ and $V^{(U,R,F)}$ described as

$$Core(V^{(U,L,F)}) = conv\{(4,3,3),(5,4,1),(3,4,3),(6,3,1)\},$$
$$Core(V^{(U,R,F)}) = conv\{(1,1,8),(1,4,5)\}.$$

It is clear that $Core(V^{(U,L,F)}) \cap Core(V^{(U,R,F)}) = \emptyset$.

[5] This figure has been built with the toolbox TUGlab of MATLAB® (Mirás-Calvo and Sánchez-Rodríguez 2008). The web page of TUGlab can be found in `http://eio.usc.es/pub/io/xogos/index.php`.

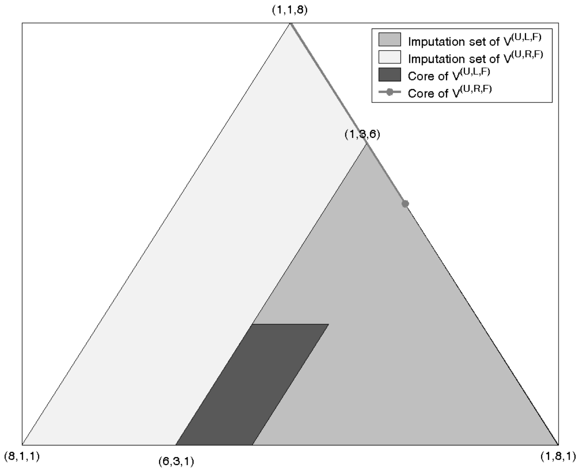

Fig. 3 $Core(V^{(U,L,F)})$ and $Core(V^{(U,R,F)})$ in Example 5.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
V(U,L,F)	2	1	3	4	7	4	9
V(D,L,F)	1	4	2	5	3	6	9

Table 5 The TU-games associated with each strategy profile in Example 6.

A property related to balancedness of a TU-game is convexity since any convex game is balanced. Recall that a TU-game $v \in G(N)$ is *convex* if and only if we have $v(T \cup i) - v(T) \geq v(S \cup i) - v(S)$, for every $i \in N$ and every $S,\ T \subset N \setminus \{i\}$ with $S \subset T$. The following counterexample shows that the maxmin procedure does not preserve the convexity property, in general.

Example 6 Consider $N = \{1,2,3\}, \quad X_1 = \{U,D\} \quad X_2 = \{L\} \quad X_3 = \{F\}$. The TU-games associated with each strategy profile are given in Table 5. It is easy to check that both TU-games are convex and then they have non-empty cores. The maxmin procedure provides the TU-game given in Table 6. This game is not convex because

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$\psi(X,V)$	2	1	2	5	7	4	9

Table 6 The TU-game provided by the maxmin procedure for Example 6.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$V^{(U,L,F)}$	2	1	2	4	7	4	9
$V^{(D,L,F)}$	1	1	2	5	3	4	9

Table 7 The TU-games V^x in Example 6.

$$\psi(X,V)(13) - \psi(X,V)(1) = 5 > \psi(X,V)(123) - \psi(X,V)(12) = 4$$

but, it has a non-empty core. For every $x \in X$, the TU-game V^x is given in Table 7. Notice that $(3,2,4) \in Core(V^{(U,L,F)}) \cap Core(V^{(D,L,F)})$ and, thus $(3,2,4) \in Core(\psi(X,V))$. Figure 4 shows the relationship among $Core(V^{(U,L,F)})$, $Core(V^{(D,L,F)})$, and $Core(\psi(X,V))$.

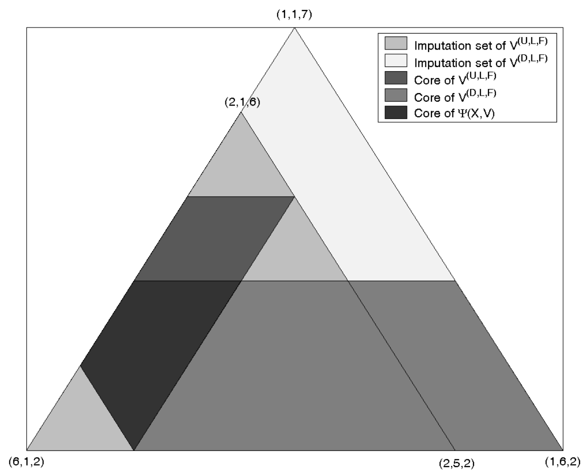

Fig. 4 $Core(V^{(U,L,F)})$, $Core(V^{(D,L,F)})$, and $Core(\psi(X,V))$ in Example 6.

4 Two particular cases

In this section we study two specific cases of TU-games with strategies inspired by the examples described in the Introduction. In Example 2, each TU-game associated to any strategy profile is an airport game. In Example 3, each TU-game associated to any strategy profile belongs to the class of simple games. Airport games and simple games are well-known classes of TU-games which have many applications and have been widely studied in the literature.

4.1 Airport games with strategies

An *airport problem* (Littlechild and Owen 1973) is described as follows. Suppose that $\mathscr{T}$ is the set of types of planes operating in an airport in a particular period. Denote by N_τ the set of movements that are made by planes of type $\tau \in \mathscr{T}$ and by N the set of all movements, i.e. $N = \cup_{\tau \in \mathscr{T}} N_\tau$. Let d_τ be the cost of a runway that is suitable for planes of type τ in the considered period. Without loss of generality, we assume that

$$0 \leq d_1 \leq d_2 \leq \ldots \leq d_{|\mathscr{T}|}.$$

We can associate a cost game to this situation as follows. For every $\emptyset \neq S \subset N$, $c(S)$ is defined as the cost of a runway that can be used by all the movements in S, i.e.

$$c(S) := \max\{d_\tau : S \cap N_\tau \neq \emptyset\}$$

and $c(\emptyset) = 0$. The TU-cost game c is known as an *airport game*.

An *airport game with strategies* is a pair $(X,C) \in SG(N)$, where $C(x)$ is an airport game for all $x \in X$. It means that for all $x \in X$ and $\emptyset \neq S \subset N$,

$$C(x)(S) = \max\{C(x)(j) : \ j \in S\}$$

given $C(x)(j) \geq 0$, for every $j \in N$.

Notice that we are dealing with cost TU-games for each strategy profile. Then, some definitions and properties established for TU-games has to change adequately. Namely, the *core of a cost TU-game* c is the core of the TU-game $-c$ and we say that a cost TU-game c is *concave* if the TU-game $-c$ is convex. Notice also that the maxmin procedure defined for TU-games with strategies becomes the minmax procedure. Then, during this section ψ denotes the minmax procedure.

The second example described in the Introduction corresponds to the model of airport games with strategies and illustrates the fact that $\psi(X,C)$ is not an airport game in general. Next, we provide a sufficient condition for the transmission of the non-emptiness of the core in airport games with strategies. This condition is easier to check than the one given in Theorem 4 because we do not need to compute any core.

Theorem 5 *Let $(X,C) \in SG(N)$ be an airport game with strategies. If there is some $i_N \in N$ such that $\psi(X,C)(S) \geq \psi(X,C)(N)$, for every $S \subset N$ with $i_N \in S$, then $\psi(X,C)$ is balanced.*

Proof Recall that the minmax procedure defines the following TU-game:

$$\psi(X,C)(S) = \min_{x_S \in \prod_{i \in S} X_i} \max_{x_{N \setminus S} \in \prod_{i \in N \setminus S} X_i} C(x)(S), \text{ for every } S \subseteq N.$$

Let $i_N \in N$ be such that $\psi(X,C)(S) \geq \psi(X,C)(N)$, for every $S \subset N$ with $i_N \in S$. Take $d_i = \min\{\psi(X,C)(S) \ : \ i \in S\}$, for every $i \in N$. Notice that $d_i \geq 0$, for every $i \in N$ and we have $d_{i_N} = \psi(X,C)(N)$ due to the choice of i_N. Moreover, $d_i \leq d_{i_N}$, for every $i \in N$.

Now, we consider the airport game defined by the ordered collection $(d_1, \ldots, d_{i_N})$. Taking into account that $d_i \leq \psi(X,C)(S)$ for every $i \in S$, then $d(S) = \max\{d_i : \ i \in S\} \leq \psi(X,C)(S)$. Besides, $d(N) = \psi(X,C)(N)$ and d has a non-empty core because it is an airport game. Then, $Core(d) \subseteq Core(\psi(X,C))$ and we obtain that $\psi(X,C)$ has a non-empty core. □

The condition in Theorem 5 holds, for instance, if there is some player i_N with $C(x)(i_N) = \max\{C(x)(j) \ : \ j \in N\}$, for every $x \in X$ (i. e. if i_N is a player with the most costly type of planes for all strategy profiles). Next result proves it.

Corollary 1 *Let $(X,C) \in SG(N)$ be an airport game with strategies such that there is some player i_N with $C(x)(i_N) = \max\{C(x)(j) \ : \ j \in N\}$, for every $x \in X$. Then, $\psi(X,C)$ is balanced.*

Proof Let i_N be such that $C(x)(i_N) = \max\{C(x)(j) \ : \ j \in N\}$, for every $x \in X$. We prove that i_N satisfies the condition of Theorem 5. Notice that there is some $\tilde{x}$ such that

$$\psi(X,C)(N) = \min_{x \in X} C(x)(N) = C(\tilde{x})(N) = \max_{j \in N} C(\tilde{x})(j) = C(\tilde{x})(i_N).$$

We prove that $\psi(X,C)(S) \geq \psi(X,C)(N)$ for every $S \subset N$ with $i_N \in S$. Let $S \subset N$ with $i_N \in S$. Then, there is some $\bar{x} \in X$ such that

$$\begin{aligned} \psi(X,C)(S) &= \min\nolimits_{x_S \in \prod_{i \in S} X_i} \max\nolimits_{x_{N \setminus S} \in \prod_{i \in N \setminus S} X_i} C(x)(S) = C(\bar{x})(S) \\ &= C(\bar{x})(i_N) \geq \min\nolimits_{x \in X} C(x)(N) = C(\tilde{x})(i_N) = \psi(X,C)(N). \end{aligned}$$

Then, applying Theorem 5 we get the result. □

The condition in Theorem 5 is sufficient, but not necessary as Example 7 illustrates.

Example 7 Let us consider $N = \{1,2,3\}$, $X_1 = \{U,D\}$, $X_2 = \{L,R\}$, and $X_3 = \{F\}$. The airport (cost) games associated with each strategy profile are given in Table 8. The minmax procedure provides the TU-game given in Table 9. $\psi(X,C)$ is balanced since, for instance, the allocation $(0,3,5)$ belongs to the core of $\psi(X,C)$. Nevertheless, there is no player satisfying the condition of Theorem 5.

We have concave games for every strategy profile in an airport game with strategies, (X,C). Despite this fact, the resulting TU-game $\psi(X,C)$ is not concave in general, as we can see in Example 7 by taking coalitions $\{1,2,3\}$, $\{1,2\}$, $\{2,3\}$ and $\{2\}$.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
C(U,L,F)	1	8	2	8	2	8	8
C(U,R,F)	2	9	5	9	5	9	9
C(D,L,F)	5	10	7	10	7	10	10
C(D,R,F)	6	7	9	7	9	9	9

Table 8 The airport games associated with each strategy profile in Example 7.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$\psi(X,C)$	2	9	9	7	5	9	8

Table 9 The TU-game provided by the minmax procedure for Example 7.

4.2 Simple games with strategies

A *simple game with strategies* is a pair $(X,V) \in SG(N)$ where $V(x)$ is a simple game for all $x \in X$. It means that for all $x \in X$ and $S \subset N$,

i) $V(x)(S) \in \{0,1\}$, for every $S \subset N$,

ii) $V(x)(N) = 1$, and

iii) $V(x)(S) \leq V(x)(T)$, for every $S \subset T \subset N$.

Condition *iii*) indicates that for every $x \in X$, $V(x)$ is a monotone game. As a consequence of Theorem 3, the TU-game built using the maxmin procedure, $\psi(X,V)$, is monotone. Besides, it is clear that $\psi(X,V)(N) = 1$. Then, the maxmin procedure provides a simple game too. The third situation described in the Introduction is a particular case of a simple game with strategies.

In general, the core of a simple game w is empty unless there is some veto player (i.e., a player $i \in N$ with $w(N \setminus \{i\}) = 0$); besides, any allocation in the core of a simple game splits $w(N)$ among veto players. We characterize the balancedness of the simple game resulting from the maxmin procedure applied to a simple game with strategies in the following result.

Theorem 6 *Let $(X,V) \in SG(N)$ be a simple game with strategies. Then, the core of $\psi(X,V)$ is non-empty if and only if there is some $i \in N$ such that, for every $x_{N\setminus\{i\}}$, player i is a veto player in $V(x_{N\setminus\{i\}},\bar{x}_i)$ for some $\bar{x}_i \in X_i$.*

Proof First we assume that $Core(\psi(X,V))$ is non-empty. Then, there is a veto player i in $\psi(X,V)$ because this TU-game is a simple game. Using Theorem 4, we have $Core(\psi(X,V)) = \cap_{x\in X} Core(V^x)$ with

$$V^x(S) = \min_{\tilde{x}_{N\setminus S}\in\prod_{j\in N\setminus S} X_j} V(x_S,\tilde{x}_{N\setminus S})(S),$$

for every $S \subsetneq N$ and $V^x(N) = 1$, for every $x \in X$. Since $\emptyset \neq Core(\psi(X,V)) = \cap_{x\in X} Core(V^x)$, we have $Core(V^x) \neq \emptyset$, for every $x \in X$. Besides, V^x is a simple game, for every $x \in X$. Then, due to the choice of i, we have that i is a veto player in V^x, for every $x \in X$. This means that for every $x_{N\setminus\{i\}}$ there is some $\bar{x}_i \in X_i$ such that $V(x_{N\setminus\{i\}},\bar{x}_i)(N\setminus\{i\}) = 0$. This proves the condition.

Now we assume that there is some $i \in N$ such that, for every $x_{N\setminus\{i\}}$, player i is a veto player in $V(x_{N\setminus\{i\}},\bar{x}_i)$ for some $\bar{x}_i \in X_i$. Then, $Core(V^x) \neq \emptyset$, for every $x \in X$. Moreover, the allocation $a \in \mathbb{R}^n$ with $a_i = 1$ and $a_j = 0$ for every $j \in N\setminus\{i\}$ belongs to $Core(V^x)$, for every $x \in X$. Then, $\cap_{x\in X} Core(V^x) \neq \emptyset$ and using Theorem 4, we obtain that $Core(\psi(X,V)) \neq \emptyset$.

□

5 Concluding remarks

In this paper we study situations where the players cooperate and, previously, can choose some strategy that changes the values of cooperation. In order to allocate the benefits of cooperation using this strategic power, we apply the maxmin procedure to obtain a new

S	{1}	{2}	{1,2}
V(A,E)	4	3	10
V(B,E)	2	5	9
V(C,E)	2	4	7
V(D,E)	6	1	7

Table 10 The TU-games associated with each strategy profile in Concluding remarks section.

TU-game. This analysis generalizes the models that appear in Carpente et al. (2005) in the sense that we consider a non-additive characteristic function for each strategy profile. We characterize the procedure using adaptations of some well-known and intuitive properties. Actually, we present an axiomatic characterization that is an extension (goes beyond the additive characteristic functions) and an improvement (without a monotonicity property) of the one given by Carpente et al. (2005). Additionally, we study under which conditions some properties of the initial TU-games are transmitted to the TU-game obtained as a result of applying the maxmin procedure. Finally, we provide some sufficient conditions to obtain a balanced TU-game in case of airport games with strategies and characterize the non-emptiness of the core in case of simple games with strategies.

Other procedures to transform a TU-game with strategies into a TU-game could be introduced and compared with the maxmin procedure. One possibility is to consider the following optimistic version of the maxmin procedure. The *maxmax procedure* is the map $\bar{\psi} : SG \to G$ given by $\bar{\psi}(X,V)(S) = \max_{x \in X} V(x)(S)$ for all $(X,V) \in SG(N)$ and all $S \subset N$. Neither the superaditivity nor the balancedness property are transmitted by the maxmax procedure as we see in the following example. Let us take $N = \{1,2\}$, $X_1 = \{A,B,C,D\}$ and $X_2 = \{E\}$. The TU-games associated with each strategy profile are given in Table 10. It is easy to check that $V(x)$ is superadditive and balanced for every $x \in X$. Nevertheless, the maxmax procedure provides the TU-game: $v(1)=6$, $v(2)=5$, and $v(12)=10$, which is neither superadditive nor balanced.

Our model of TU-games with strategies is related with the joint games introduced in Assa et al. (2016). Given a family of K TU-games in $G(N)$ $v_1,\dots,v_K$ the corresponding joint game $v_1\bullet\cdots\bullet v_K$ is defined by:

$$v_1\bullet\cdots\bullet v_K(S)=\max\{\sum_{k=1}^{K}v_k(S_k)\ :\ S_i\cap S_j=\emptyset, i\neq j, S=\bigcup_{k=1}^{K}S_k\},$$

for every $S\subset N$. Notice that we can associate a TU-game with strategies (X,V) to $v_1\bullet\cdots\bullet v_K$ as follows. We take $X_i=\{v_1,\dots,v_K\}$ for every $i\in N$, $X=\prod_{i\in N}X_i$, and $V:X\to G(N)$ given by $V(x)(S)=\sum_{k=1}^{K}v_k(S_k)$, where $S_k=\{i\in S\ :\ x_i=v_k\}$ for every $k=1,\dots,K$, $S\subset N$, and $x\in X$. In order to obtain the TU-game $v_1\bullet\cdots\bullet v_K$ we only need to apply the maxmax procedure to (X,V).

Acknowledgements The authors would like to thank two anonymous referees for their helpful suggestions to improve this article. This work has been supported by the Ministerio de Economía y Competitividad through grants PGC2018-097965-B-100, MTM2017-87197-C3-1-P, MTM2017-87197-C3-2-P, MTM2014-53395-C3-1-P, MTM2014-53395-C3-3-P, MTM2014-54199-P, and by the Xunta de Galicia through the European Regional Development Fund (Grupos de Referencia Competitiva ED431C-2016-015 and ED431C-2016-040, and Centro Singular de Investigación de Galicia ED431G/01).

Conflict of interest

The authors declare that they have no conflict of interest.

References

Alonso-Meijide, J. M., Carreras, F., Costa, J. & García-Jurado, I. (2015). The proportional partitional Shapley value. *Discrete Applied Mathematics*, 187, 1–11.

Assa, H., Elliston, S. & Lehrer, E. (2016). Joint games and compatibility. *Economic Theory*, 61(1), 91–113.

Aumann, R. J. & Dreze, J. H. (1974). Cooperative games with coalition structures. *International Journal of Game Theory*, 3, 217–237.

Balog, D., Bátyi, T. L., Csóka, P. & Pintér, M. (2017). Properties and comparison of risk capital allocation methods. *European Journal of Operational Research*, 259(2), 614–625.

Bilbao, J. M. (2000). *Cooperative games on combinatorial structures*, *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*, vol. 26. Kluwer Academic Publishers, Boston, MA.

Brandenburger, A. & Stuart, H. (2007). Biform games. *Management Science*, 53, 537–549.

Carpente, L., Casas-Méndez, B., García-Jurado, I. & van den Nouweland, A. (2005). Values for strategic games in which players cooperate. *International Journal of Game Theory*, 33(3), 397–419.

Çetiner, D. & Kimms, A. (2013). Assessing fairness of selfish revenue sharing mechanisms for airline alliances. *Omega*, 41, 641–652.

Ertemel, S. & Kumar, R. (2018). Proportional rules for state contingent claims. *International Journal of Game Theory*, 47(1), 229–246.

Fernández, J. R., Gallego, I., Jiménez-Losada, A. & Ordóñez, M. (2016). Cooperation among agents with a proximity relation. *European Journal of Operational Research*, 250(2), 555–565.

Fiestras-Janeiro, M. G., García-Jurado, I., Meca, A. & Mosquera, M. A. (2015). Cooperation on capacitated inventory situations with fixed holding costs. *European Journal of Operational Research*, 241(3), 719–726.

González-Díaz, J., García-Jurado, I. & Fiestras-Janeiro, M. G. (2010). *An introductory course on mathematical game theory*, *Graduate Studies in Mathematics*, vol. 115. American Mathematical Society.

Goyal, T. & Kaushal, S. (2017). An intelligent scheduling scheme for real-time traffic management using Cooperative Game Theory and AHP-TOPSIS methods for next generation telecommunication networks. *Expert Systems with Applications*, 86, 125–134.

Habis, H. & Herings, P. J.-J. (2011). Transferable utility games with uncertainty. *Journal of Economic Theory*, 146(5), 2126–2139.

Kalai, A. & Kalai, E. (2013). Cooperation in strategic games revisited. *The Quarterly Journal of Economics*, 128(2), 917–966.

Kimms, A. & Kozeletskyi, I. (2016). Core-based cost allocation in the cooperative traveling salesman problem. *European Journal of Operational Research*, 248(3), 910–916.

Li, Y., Xie, J., Wang, M. & Liang, L. (2016). Super efficiency evaluation using a common platform on a cooperative game. *European Journal of Operational Research*, 255(3), 884–892.

Littlechild, S. C. & Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science*, 20(3), 370–372.

Lozano, S., Hinojosa, M. A. & Mármol, A. M. (2015). Set-valued DEA production games. *Omega*, 52, 92–100.

Mirás-Calvo, M. A. & Sánchez-Rodríguez, E. (2008). *Juegos cooperativos con utilidad transferible usando MATLAB : TUGlab*. Servizo de Publicacións da Universidade de Vigo.

Myerson, R. B. (1977). Graphs and Cooperation in Games. *Mathematics of Operations Research*, 2(3), 225–229.

Naber, S. K., de Ree, D. A., Spliet, R. & van den Heuvel, W. (2015). Allocating CO2 emission to customers on a distribution route. *Omega*, 54, 191–199.

von Neumann, J. & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Owen, G. (1977). Values of games with a priori unions. In R. Henn & O. Moeschlin (eds.), *Mathematical Economics and Game Theory*, (pp. 76–88), Lecture Notes in Economics and Mathematical Systems, Springer Berlin Heidelberg.

Platkowski, T. (2016). Evolutionary coalitional games. *Dynamic Games and Applications*, 6(3), 396–408.

Ui, T. (2000). A Shapley value representation of potential games. *Games and Economic Behavior*, 31(1), 121–135.

Appendix

Independence of the properties of Theorem 1.

Next we define some procedures in order to show the independence of the properties of Theorem 1.

Individual objectivity. Take the map $\psi_0 : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by:

$$\psi_0(X,V)(S) = 0$$

for all $S \subset N$.

The procedure ψ_0 satisfies all the properties but individual objectivity. Namely, it is clear that it does not satisfy individual objectivity. Now we will show that it satisfies the other properties:

- *Irrelevance of weakly dominated strategies*: Let $(X,V) \in SG(N)$, $i \in N$, and $S \subset N$ with $i \in S$. If strategy $x_i \in X_i$ is weakly dominated in S, then

$$\psi_0(X,V)(S) = 0 = \psi_0(X^{-x_i},V)(S).$$

- *Irrelevance of weakly dominated threats*: Let $(X,V) \in SG(N)$, $j \in N$, and $S \subset N \setminus \{j\}$. If strategy $x_j \in X_j$ is a weakly dominated threat to coalition S, then

$$\psi_0(X,V)(S) = 0 = \psi_0(X^{-x_j},V)(S).$$

- *Merge invariance*: Let $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$, then, for all $T \subset N \setminus S$,

$$\psi_0(X,V)(T) = 0 = \psi_0(X^S,V^S)(T)$$

and

$$\psi_0(X,V)(T \cup S) = 0 = \psi_0(X^S,V^S)(T \cup \{[S]\}).$$

Irrelevance of weakly dominated strategies. Take the *minmin procedure* $\underline{\psi} : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by:

$$\underline{\psi}(X,V)(S) = \min_{x \in X} V(x)(S)$$

for all $S \subset N$.

The minmin procedure satisfies all the properties but irrelevance of weakly dominated strategies. It is clear that it does not satisfy irrelevance of weakly dominated strategies. Namely, take $(X,V) \in SG(N)$, $i \in N$, $S \subset N$ with $i \in S$, and let $x_i \in X_i$ be such that $V(\bar{x}_{-i}, x_i')(S) > V(\bar{x}_{-i}, x_i)(S)$ for all $\bar{x}_{-i} \in \prod_{j \in N \setminus \{i\}} X_j$ and $x_i' \in X_i$ with $x_i \neq x_i'$. Then, x_i is weakly dominated in S but

$$\underline{\psi}(X,V)(S) = \min_{x \in X} V(x)(S) < \min_{x \in X^{-x_i}} V(x)(S) = \underline{\psi}(X^{-x_i}, V)(S).$$

Now we will show that it satisfies the other properties:

- *Individual objectivity*: Let $(X,V) \in SG(N)$ and a player $i \in N$ be such that $V(x)(i) = c$, for all $x \in X$, then

$$\underline{\psi}(X,V)(i) = \min_{x \in X} c = c$$

- *Irrelevance of weakly dominated threats*: Let $(X,V) \in SG(N)$, $j \in N$, and $S \subset N \setminus \{j\}$. If strategy $x_j \in X_j$ is a weakly dominated threat to coalition S, then

$$\underline{\psi}(X,V)(S) = \min_{x \in X} V(x)(S) = \min_{x \in X^{-x_j}} V(x)(S) = \underline{\psi}(X^{-x_j}, V)(S).$$

- *Merge invariance*: Let $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$, then, for all $T \subset N \setminus S$,

$$\underline{\psi}(X,V)(T) = \min_{x \in X} V(x)(T) = \min_{x^S \in X^S} V^S(x^S)(T) = \underline{\psi}(X^S, V^S)(T)$$

and

$$\underline{\psi}(X,V)(T \cup S) = \min_{x \in X} V(x)(T \cup S) = \min_{x^S \in X^S} V^S(x^S)(T \cup \{[S]\}) = \underline{\psi}(X^S, V^S)(T \cup \{[S]\}).$$

Irrelevance of weakly dominated threats. Take the *maxmax procedure* $\overline{\psi} : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by:

$$\overline{\psi}(X,V)(S) = \max_{x \in X} V(x)(S)$$

for all $S \subset N$.

The maxmax procedure satisfies all the properties but irrelevance of weakly dominated threats. It is clear that it does not satisfy irrelevance of weakly dominated threats. Namely, take $(X,V) \in SG(N)$, $j \in N$, $S \subset N \setminus \{j\}$, and let $x_j \in X_j$ be such that $V(\bar{x}_{-j}, x'_j)(S) < V(\bar{x}_{-j}, x_j)(S)$ for all $\bar{x}_{-j} \in \prod_{k \in N \setminus \{j\}} X_k$ and $x'_j \in X_j$ with $x_j \neq x'_j$. Then, x_j is a weakly dominated threat to coalition S but

$$\overline{\psi}(X,V)(S) = \max_{x \in X} V(x)(S) > \max_{x \in X^{-x_j}} V(x)(S) = \overline{\psi}(X^{-x_j}, V)(S).$$

Now we will show that it satisfies the other properties:

- *Individual objectivity*: Let $(X,V) \in SG(N)$ and a player $i \in N$ be such that $V(x)(i) = c$, for all $x \in X$, then

$$\overline{\psi}(X,V)(i) = \max_{x \in X} c = c$$

- *Irrelevance of weakly dominated strategies*: Let $(X,V) \in SG(N)$, $i \in N$, and $S \subset N$ with $i \in S$. If strategy $x_i \in X_i$ is weakly dominated in S, then

$$\overline{\psi}(X,V)(S) = \max_{x \in X} V(x)(S) = \max_{x \in X^{-x_j}} V(x)(S) = \overline{\psi}(X^{-x_j}, V)(S).$$

- *Merge invariance*: Let $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$, then, for all $T \subset N \setminus S$,

$$\overline{\psi}(X,V)(T) = \max_{x \in X} V(x)(T) = \max_{x^S \in X^S} V^S(x^S)(T) = \overline{\psi}(X^S, V^S)(T)$$

  and

$$\overline{\psi}(X,V)(T \cup S) = \max_{x \in X} V(x)(T \cup S) = \max_{x^S \in X^S} V^S(x^S)(T \cup \{[S]\}) = \overline{\psi}(X^S, V^S)(T \cup \{[S]\}).$$

Merge invariance. Take the map $\psi_1 : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by:

$$\psi_1(X,V)(S) = \psi(X,V)(S) + (|S| - 1)$$

for all $\emptyset \neq S \subset N$.

The procedure ψ_1 satisfies all the properties but merge invariance. Namely, it is clear that it does not satisfy merge invariance since if we take $(X,V) \in SG(N)$ and $\emptyset \neq S \subset N$ with $|S| > 1$, then for all $T \subset N \setminus S$,

$$\begin{aligned}\psi_1(X,V)(T \cup S) &= \psi(X,V)(T \cup S) + (|T| + |S| - 1) = \psi(X^S,V^S)(T \cup \{[S]\}) + (|T| + |S| - 1)\\ &> \psi(X^S,V^S)(T \cup \{[S]\}) + (|T| + |\{[S]\}| - 1) = \psi_1(X^S,V^S)(T \cup \{[S]\})\end{aligned}$$

where the inequality follows from $|S| > 1 = |\{[S]\}|$.

Now we will show that it satisfies the other properties:

- *Individual objectivity*: Let $(X,V) \in SG(N)$ and a player $i \in N$ be such that $V(x)(i) = c$, for all $x \in X$, then

  $$\psi_1(X,V)(i) = \psi(X,V)(i) = c$$

  since ψ satisfies individual objectivity.
- *Irrelevance of weakly dominated strategies*: Let $(X,V) \in SG(N)$, $i \in N$, and $S \subset N$ with $i \in S$. If strategy $x_i \in X_i$ is weakly dominated in S, then

  $$\psi_1(X,V)(S) = \psi(X,V)(S) + (|S| - 1) = \psi(X^{-x_i},V)(S) + (|S| - 1) = \psi_1(X^{-x_i},V)(S)$$

  since ψ satisfies irrelevance of weakly dominated strategies.
- *Irrelevance of weakly dominated threats*: Let $(X,V) \in SG(N)$, $j \in N$, and $S \subset N \setminus \{j\}$. If strategy $x_j \in X_j$ is a weakly dominated threat to coalition S, then

  $$\psi_1(X,V)(S) = \psi(X,V)(S) + (|S| - 1) = \psi(X^{-x_j},V)(S) + (|S| - 1) = \psi_1(X^{-x_j},V)(S)$$

  since ψ satisfies irrelevance of weakly dominated threats.

Next table shows a summary of the procedures and properties.

	ψ	ψ_0	$\underline{\psi}$	$\overline{\psi}$	ψ_1
Individual objectivity	Yes	No	Yes	Yes	Yes
Irrelevance of weakly dominated strategies	Yes	Yes	No	Yes	Yes
Irrelevance of weakly dominated threats	Yes	Yes	Yes	No	Yes
Merge invariance	Yes	Yes	Yes	Yes	No