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On the hyperbolic thermoelasticity with several dissipation mechanisms

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Abstract In this work, we study a two-dimensional problem involving a thermoelastic body with four dissipative mechanisms. The well-known theory proposed by Lord and Shulman is used. The existence and uniqueness of solution is proved by using theory of linear semigroups. Then, introducing some assumptions of the coupling coefficients, we prove that the energy decay is exponential. An extension to the theory provided by Green and Lindsay is briefly presented and to the three-dimensional case is also commented.

Keywords Lord and Shulman theory · Thermoelasticity · Dissipation mechanisms · Energy decay · Existence and uniqueness

1 Introduction

The temporal decay of the solutions to thermomechanical problems has deserved much attention and it has led to many studies. If we restrict ourselves only the thermoelasticity, we can cite the work of Dafermos [4] as a pioneering research in the study of the decay of the solutions. In fact, he proved the asymptotic stability for the one-dimensional case as well as the existence of undamped isothermal solutions. Later, several authors shown [18,26] that, in the one-dimensional case, we can obtain the exponential decay. However, for a dimension greater than one, other authors [12,13] (see also [11]) proved that we cannot expect such behaviour. This is due to the fact that the coupling terms do not translate, in a relevant form, the damping produced by the heat equation to the mechanical part. Therefore, if we want to achieve the exponential decay in a dimension greater than one, we shall introduce more dissipation mechanisms and, at the same time, to propose coupling mechanisms which translate, in an effective way, the thermal dissipation to the mechanical part. Recently, we have performed this kind of studies in the case of the heat conduction of parabolic type for the two-dimensional setting (see [5,6]). The extension to the three-dimensional case was straightforward.

It is convenient to recall that it is well-known the exponential decay of the solutions to different theories of the thermoelasticity in the one-dimensional case [20–22,25].

By other hand, the parabolic theories of the heat conduction have received a lot of criticisms from the physical point of view because, indeed, they allow the instantaneous propagation of the thermal waves, which lead to the violability of the causality principle. For this reason, several authors have been interested to propose

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alternative theories for the heat conduction. The most well-known is the damping hyperbolic heat equation introduced by Cattaneo and Maxwell [3]. From this heat equation, two thermoelastic theories have been derived: the one proposed by Lord and Shulman [17] and the one proposed by Green and Lindsay [8]. We note that both theories have deserved much attention and they have been deeply studied. In particular, the exponential decay is known for the one-dimensional case [1,25].

As we said before, first cases where time decay of thermoelasticity were considered correspond to the parabolic heat conduction. Several authors developed a similar study for hyperbolic thermoelasticity and similar conclusions were obtained and published. However, it is also known that, in the case of a thermoelastic plate, the behavior of the solutions is different depending on the type of heat conduction theory. In the case of the parabolic theories, we know that the solutions generate an analytic semigroup satisfying the exponential decay [14,15], but the behavior in the case of hyperbolic heat conduction is very different. It is known that, for the case of the Green and Lindsay theory, we can obtain the exponential decay, but we lose the analyticity [24]. In the case of the Lord and Shulmann theory, even the exponential decay is not found [23]. Another interesting case can be seen in the study of the strain gradient thermoelasticity. It is known that we can obtain the exponential stability in the case of the isotropic one-dimensional parabolic heat conduction, but the decay is again slow for Green and Lindsay or Lord and Shulmann counterpart [24]. Moreover, in a recent article [7], the authors proved that, for the chiral strain gradient thermoelasticity, we could also find several differences. In this sense, for suitable coupling terms the solutions of the parabolic problem (for the heat conduction) generate an analytic semigroup and then, we find the exponential stability, but we could not expect the exponential decay for the Lord and Shulmann theory. Therefore, we believe that it is of interest, from a thermomechanical point of view, to clarify if the behavior of the solutions changes in a relevant way when hyperbolic heat conduction theories are proposed. In this work, our aim is to make a similar study to the one proposed in the parabolic case [5,6] for a couple of hyperbolic heat conduction theories of thermoelasticity with several dissipative mechanisms. We will see that there are not relevant changes in the solutions when we change the kind of the heat conduction.

The paper is structured as follows. In Sect. 2, we describe the two-dimensional problem involving a thermoelastic material of Lord–Shulman type with four dissipation mechanisms. The basic assumptions are also recalled. Then, in Sect. 3 we write our problem as a Cauchy problem in an adequate Hilbert space and we prove the existence of solutions. The exponential decay of the solutions is shown in the next section whenever a certain assumption (which implies that the coupling terms are not isotropic) is hold. Section 5 is devoted to the Green and Lindsay theory. We make a short sketch of the results and the basic arguments used in the previous sections to obtain similar results as the ones of the Lord and Shulman theory. Finally, we conclude the work with some further comments regarding the difficulty to weaken the required conditions to obtain the exponential decay.

2 Basic equations of the thermoelastic model

In this work, we consider a two-dimensional domain B with a boundary ∂B assumed smooth enough to guarantee the use of the divergence theorem.

We will study in B the problem determined by the following system of partial differential equations:

$$\begin{aligned} \rho \ddot{u}_i &= \mu \Delta u_i + (\lambda + \mu) u_{j,ji} + A^l_{ik} \theta_{l,k}, \\ m_{lp} \dot{\hat{\theta}}_p &= K_{lp} \Delta \theta_p + A^l_{ik} \dot{\hat{u}}_{i,k} + \xi^l_{pq} (\theta_p - \theta_q), \end{aligned} \tag{1}$$

where $i, j = 1, 2$ and $p, q, l = 1, \ldots, 4$, and we have used the notation:

$$\hat{f} = f + d \dot{f} \quad \text{for a given} \quad d > 0. \tag{2}$$

It is worth noting that this system corresponds to the thermoelasticity of Lord–Shulman with four dissipative mechanisms.

As usual, here we denote by u_i the displacement vector, θ_1 and θ_2 are two different temperatures, θ_3 and θ_4 determine two dissipative mechanisms (which can be associated to the mass diffusion), ρ is the mass density, λ and μ are the Lamé coefficients, K_{lp} is the matrix of thermal conductivity (or the mass dissipation), m_{lp} is the matrix of thermal capacity (or the mass dissipation), ξ^l_{pq} are the coefficients related to the relative temperatures (concentrations) and A^l_{pq} are the coupling coefficients.

Due to the type of problem we propose, we assume that $\xi_{12}^2 = -\xi_{12}^1 = l_1$, $\xi_{34}^2 = -\xi_{34}^1 = \xi_{12}^4 = -\xi_{12}^3 = l_2$, $\xi_{34}^4 = -\xi_{34}^3 = l_3$ and the remaining $\xi_{pq}^l = 0$ (see [10]).

It is usual to find processes where two temperatures θ_1, θ_2 act at the same time as we can see in the classical works of Ieşan [10] or Gurtin and de La Penha [9]. They are usually understood in the context of mixtures. At the same time, the mass diffusion is a dissipative mechanism which is considered and studied for a long time ago [19]. Of course, we can also consider it in a similar way of the two temperatures in order to obtain two mechanisms of mass diffusion (for θ_3, θ_4).

We can note that, using notation (2), system (1) can be written as:

$$\begin{aligned} &\rho\ddot{\hat{u}}_i = \mu\Delta\hat{u}_i + (\lambda+\mu)\hat{u}_{j,ji} + A_{ik}^l(\theta_{l,k} + d\dot{\theta}_{l,k}), \\ &m_{lp}(\dot{\theta}_p + d\ddot{\theta}_p) = K_{lp}\Delta\theta_p + A_{ik}^l\dot{\hat{u}}_{i,k} + \xi_{pq}^l(\theta_p - \theta_q). \end{aligned} \tag{3}$$

From now on, in order to simplify the writing, we will omit the hat over all the variables of system (3).

We will also assume the boundary conditions:

$$u_i(\boldsymbol{x},t) = \theta_{l,j}(\boldsymbol{x},t)n_j(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \partial B, \tag{4}$$

where $i, j = 1, 2$, $l = 1, \ldots, 4$ and $n_j(\boldsymbol{x})$ represents the outward unit normal vector to ∂B at point x. Moreover, we also prescribe some initial conditions:

$$\begin{aligned} &u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}) \quad \boldsymbol{x} \in B, \\ &\theta_j(\boldsymbol{x},0) = \theta_j^0(\boldsymbol{x}), \quad \dot{\theta}_j(\boldsymbol{x},0) = \vartheta_j^0(\underline{\mathrm{x}}) \quad \boldsymbol{x} \in B, \end{aligned} \tag{5}$$

where $i = 1, 2$ and $j = 1, \ldots, 4$.

We will study problem (3)-(5) under the following assumptions:

(i) $\rho > 0$, $\mu > 0$ and $\lambda + \mu > 0$.
(ii) The matrices m_{ij} and K_{ij} are symmetric; that is, $m_{ij} = m_{ji}$ and $K_{ij} = K_{ji}$ for $i, j = 1, \ldots, 4$.
(iii) The matrices m_{ij}, K_{ij} and l_i are positive definite; that is, there exist three positive constants C_1, C_2 and C_3 such that

$$\begin{aligned} &m_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_i, \\ &K_{ij}\xi_i\xi_j \geq C_2\xi_i\xi_i, \\ &l_1\xi_1^2 + 2l_2\xi_1\xi_2 + l_3\xi_2^2 \geq C_3(\xi_1^2 + \xi_2^2). \end{aligned}$$

The meaning of the condition on the mass density is clear, meanwhile the assumptions on the Lamé constants guarantee that the elastic part is positive definite (and then, the stability is deduced). Assumptions (ii) are natural to work with several dissipation mechanisms, although the symmetry of the term K_{ij} is not a consequence of the thermomechanical axioms. Condition (iii) is related with the property of the heat (mass diffusion) conductor and it leads to the dissipation of the variables θ_i.

It is relevant to point out that the solutions to problem (3)–(5) satisfy the equality:

$$E(t) + 2\int_0^t D(s)\,ds = E(0), \tag{6}$$

where

$$\begin{aligned} E(t) = \int_B \Big(&\rho\dot{u}_i\dot{u}_i + 2\mu e_{ij}e_{ij} + \lambda e_{ii}e_{jj} + m_{lk}(\theta_l + d\dot{\theta}_l)(\theta_k + d\dot{\theta}_k) + dK_{lk}\nabla\theta_l\nabla\theta_k \\ &+d\left[l_1(\theta_1-\theta_2)^2 + 2l_2(\theta_1-\theta_2)(\theta_3-\theta_4) + l_3(\theta_3-\theta_4)^2\right]\Big)\,da, \end{aligned}$$

and

$$\begin{aligned} D(t) = \int_B \Big(&K_{lp}\nabla\theta_l\nabla\theta_p + l_1(\theta_1-\theta_2)^2 + 2l_2(\theta_1-\theta_2)(\theta_3-\theta_4) \\ &+l_3(\theta_3-\theta_4)^2\Big)\,da. \end{aligned}$$

3 Existence of solutions

In this section, we will obtain an existence and uniqueness result to problem (3)–(5), under the assumptions (i)-(iii), in the following Hilbert space:

$$\mathcal{H} = [H_0^1(B)]^2 \times [L^2(B)]^2 \times [H_*^1(B)]^4 \times [L_*^2(B)]^4,$$

where $H_0^1(B)$ and $L^2(B)$ are the classical Sobolev spaces, and

$$L_*^2(B) = \left\{ f \in L^2(B);\ \int_B f\, da = 0 \right\}, \quad H_*^1(B) = H^1(B) \cap L_*^2(B).$$

In this space, we will consider the inner product with the norm defined by

$$\|(u, v, \theta, \vartheta)\|^2 = \int_B \Big(\rho v_i \overline{v_i} + 2\mu e_{ij} \overline{e_{ij}} + \lambda e_{ii} \overline{e_{jj}} + m_{lk}(\theta_l + d\vartheta_l)\overline{(\theta_k + d\vartheta_k)} \\ + d K_{lk} \nabla\theta_l \overline{\nabla\theta_k} + d\xi^l_{pq}(\theta_p - \theta_q)\overline{\theta_l} \Big)\, da.$$

As usual, a bar over a variable denotes the conjugated complex. It is also important to note that this norm is equivalent to the usual one in the Hilbert space $\mathcal{H}$ by using assumptions (i)–(iii).

We can write our problem in the following form:

$$\frac{d}{dt} U(t) = \mathcal{A} U(t), \quad U(0) = (u^0, v^0, \theta^0, \vartheta^0), \tag{7}$$

where the matrix differential operator $\mathcal{A}$ is defined as

$$\mathcal{A} \begin{pmatrix} u_i \\ v_i \\ \theta_d \\ \vartheta_d \end{pmatrix} = \begin{pmatrix} v_i \\ \rho^{-1}(\mu \Delta u_i + (\lambda + \mu) u_{j,ji} + A^l_{ik}(\theta_{l,k} + d\vartheta_{l,k})) \\ \vartheta_d \\ d^{-1} n_{dl}(K_{lp} \Delta\theta_p + A^l_{ik} v_{i,k} + \xi^l_{pq}(\theta_p - \theta_q)) - d^{-1}\vartheta_d \end{pmatrix}.$$

Here, (n_{dl}) is the inverse matrix of (m_{lp}). It is convenient to remark that the domain of the operator $\mathcal{A}$ are the elements of the Hilbert space $\mathcal{H}$ such that $v_i \in H_0^1(B)$, $\vartheta_l \in H_*^1(B)$, $\mu\Delta u_i + (\lambda + \mu) u_{j,ji} \in L^2(B)$ and $\theta_l \in H^2(B) \cap L_*^2(B)$.

Remark 1 Although, as usual we assume the second condition of (4) at the domain of the operator, we actually state the suitable ones. That should be more given in the form:

$$K_{lp}\theta_{p,j} n_j = 0 \quad \text{on} \quad \partial B.$$

For the same reason, it is implicitly assumed that $\theta_{p,j} n_j = 0$ on the boundary.

Lemma 1 *For each element U of the domain of the operator $\mathcal{A}$ we find that*

$$Re < \mathcal{A}U, U > \leq 0. \tag{8}$$

Proof If we apply the divergence theorem and we use the boundary conditions (4) we obtain

$$Re < \mathcal{A}U, U >= -\int_B \Big(K_{lm} \nabla\theta_l \overline{\nabla\theta_m} + l_1(\theta_1 - \theta_2)\overline{(\theta_1 - \theta_2)} \\ + l_3(\theta_3 - \theta_4)\overline{(\theta_3 - \theta_4)} + l_2[(\theta_1 - \theta_2)\overline{(\theta_3 - \theta_4)} + (\theta_3 - \theta_4)\overline{(\theta_1 - \theta_2)}] \Big)\, da.$$

From the assumptions (iii) we can see that condition (8) is satisfied. □

Lemma 2 *Zero belongs to the resolvent of the operator $\mathcal{A}$.*

Proof Let us consider $F = (f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4)$ an element of the Hilbert space. We must prove the existence of an element U in the domain of $\mathcal{A}$ such that

$$\mathcal{A}U = F.$$

We can observe that it implies that

$$\begin{array}{l} v_i = f_i, \\ \mu\Delta u_i + (\lambda + \mu)u_{j,ji} + A^l_{ik}(\theta_{l,k} + d\vartheta_{l,k}) = \rho f_{2+i}, \\ \vartheta_p = g_p, \\ K_{lp}\Delta\theta_p + A^l_{ik}v_{i,k} + \xi^l_{pq}(\theta_p - \theta_q) - m_{lp}\vartheta_p = dm_{lp}h_p. \end{array}$$

Here, $i, j, k = 1, 2$ and $p, q, l = 1, \ldots, 4$.

Clearly, we can obtain v_i and ϑ_p and therefore, it leads to the following linear system:

$$\begin{array}{l} \mu\Delta u_i + (\lambda + \mu)u_{j,ji} + A^l_{ik}\theta_{l,k} = \rho f_{2+i} - dA^l_{ik}g_{l,k}, \\ K_{lp}\Delta\theta_p + \xi^l_{pq}(\theta_p - \theta_q) = -A^l_{ik}f_{i,k} + m_{lp}g_p + dm_{lp}h_p. \end{array} \tag{9}$$

Keeping in mind that the right-hand side of the second system of equations of the previous system is in $L^2(B)$ and the required assumptions on K_{lp} and ξ^l_{pq}, we can conclude the existence of a solution $\theta_p \in H^1(B)$. Thus, introducing this solution in the first equation of system (9) we can find the solution for u_i. Moreover, we can obtain the existence of a positive constant C, independent of F, such that

$$\|U\| \leq C\|F\|.$$

□

If we apply the corollary of Lumer–Phillips to the Hille–Yosida theorem, we have shown the following.

Theorem 1 *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions in the space $\mathcal{H}$.*

Remark 2 Since the operator $\mathcal{A}$ generates a C^0-semigroup of contractions, we see that, if $(\hat{u}(0), \hat{v}(0), \theta(0), \vartheta(0))$ belongs to the domain of the operator, we find that

$$(\hat{u}(t), \hat{v}(t), \theta(t), \vartheta(t)) \in C^0([0, t]; \mathcal{D}(\mathcal{A})) \cap C^1([0, t]; \mathcal{H}).$$

Therefore, we obtain the regularity for the variables θ and ϑ. With respect to the displacement field $u(t)$, we note that

$$u(t) + d\dot{u}(t) \in C^1([0, t]; [H^1_0(B)]^2),$$

and so,

$$\frac{d}{dt}\left(e^{d^{-1}t}u(t)\right) \in C^1([0, t]; [H^1_0(B)]^2).$$

Thus, we have

$$u(t) \in C^2([0, t]; [H^1_0(B)]^2).$$

In a similar way, we can see that $v(t) \in C^1([0, t]; [H^1_0(B)]^2)$.

However, we point out that we must also impose strict conditions on the initial data.

4 Exponential decay for a particular choice of the coupling coefficients

In this section, we aim to prove the exponential decay of the solutions obtained in the previous section whenever the coupling matrix $\left(A_{ij}^l\right)$ given by

$$\begin{pmatrix} A_{11}^1 & A_{12}^1 & A_{21}^1 & A_{22}^1 \\ A_{11}^2 & A_{12}^2 & A_{21}^2 & A_{22}^2 \\ A_{11}^3 & A_{12}^3 & A_{21}^3 & A_{22}^3 \\ A_{11}^4 & A_{12}^4 & A_{21}^4 & A_{22}^4 \end{pmatrix}$$

has rank 4. It is stated in the following.

Theorem 2 *If the coupling matrix $\left(A_{ij}^l\right)$ has rank 4, then the solutions to problem (7) decay in an exponential form.*

Proof In order to prove this result, we will use the characterization of the semigroup of contractions exponentially stable. From [16] we know that this is equivalent to show that the imaginary axis is within its resolvent and that the condition

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\| < \infty \tag{10}$$

holds.

First, we will prove the condition about the resolvent. It will be done by contradiction. If such condition is not satisfied, then there exist a sequence of real numbers $\beta_n \to \beta \neq 0$ and a sequence of vectors in the domain of the operator, with unit norm, such that

$$\|(i\beta_n I - \mathcal{A})U_n\| \to 0. \tag{11}$$

This convergence is equivalent to the following convergences:

$$i\beta_n u_{jn} - v_{jn} \to 0 \quad \text{in} \quad H^1(B), \tag{12}$$

$$i\beta_n \rho v_{jn} - (\mu\Delta u_{jn} + (\lambda+\mu)u_{kn,kj} + A_{jk}^l\theta_{ln,k} + dA_{jk}^l\vartheta_{ln,k}) \to 0 \quad \text{in} \quad L^2(B), \tag{13}$$

$$i\beta_n\theta_{qn} - \vartheta_{qn} \to 0 \quad \text{in} \quad H^1(B), \tag{14}$$

$$i\beta_n d m_{qp}\vartheta_{pn} - (K_{qp}\Delta\theta_{pn} + A_{ik}^q v_{in,k} + \xi_{rs}^q(\theta_{rn} - \theta_{sn}) - m_{qp}\vartheta_{pn}) \to 0 \quad \text{in} \quad L^2(B). \tag{15}$$

From the dissipation inequality we find that

$$\nabla\theta_{ln} \to 0,$$

and so, it follows that $\beta_n^{-1}\nabla\vartheta_{ln} \to 0$. If we multiply convergence (15) by $\beta_n^{-1}\vartheta_{qn}$ we find that $\vartheta_{pn} \to 0$. If we divide now convergence (15) by β_n^{-1} we have

$$\beta_n^{-1}K_{qp}\Delta\theta_{pn} + iA_{ik}^q u_{in,k} \to 0 \quad \text{in} \quad L^2(B)$$

for $q = 1, \ldots, 4$. If we multiply now q-th convergence by $A_{ik}^q u_{in,k}$ and we take into account that

$$< \beta_n^{-1}\Delta\theta_{pn}, A_{ik}^q u_{in,k} >= - < \nabla\theta_{pn}, \beta_n^{-1}A_{ik}^q\nabla u_{in,k} >\to 0,$$

because $\beta_n^{-1}u_{in}$ is bounded in $H^2(B)$ since the elastic part is positive definite, we conclude that

$$A_{ik}^q u_{in,k} \to 0 \quad \text{in} \quad L^2(B) \text{ for } q = 1, \ldots, 4.$$

Since we have assumed that the matrix of coefficients has rank four, we can conclude that $u_{in} \to 0$ in $H^1(B)$. If we multiply now convergence (13) by u_{in} we also conclude that $v_{in} \to 0$, and we arrive to a contradiction because we assumed that $i\beta \neq 0$ did not belong to the resolvent.

In order to show the asymptotic condition (10), we can follow a similar argument. Hence, we assume that this condition does not hold. Therefore, there will exist a sequence of real numbers $\beta_n \to \infty$ and a sequence of unit norm vectors, belonging to the domain of the operator $\mathcal{A}$, such that convergence (11) is fulfilled. Now, we can repeat the same argument used previously, since the key point was to assume that β_n did not tend to zero, and we arrive again to a contradiction. Therefore, the asymptotic condition is satisfied and the theorem is proved. □

5 An application to the Green-Lindsay theory

We can also consider the case when the dissipative mechanisms are given by the Green and Linday theory. The system of equations is now the following:

$$\begin{array}{l}\rho\ddot{u}_i = \mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A^l_{ik}(\theta_{l,k} + \alpha\dot{\theta}_{l,k}),\\ m_{lp}\ddot{\theta}_p + d_{lp}\dot{\theta}_p = K_{lp}\Delta\theta_p + A^l_{ik}\dot{u}_{i,k} + \xi^l_{pq}(\theta_p - \theta_q).\end{array} \tag{16}$$

For this system, we impose again the boundary and initial conditions given in (4) and (5), respectively. We also assume conditions (i)–(iii) but we will also need that the matrix

$$(d_{lp}\alpha - m_{lp})$$

is positive definite; that is,

(iv) there exists a positive constant C_4 such that $(d_{lp}\alpha - m_{lp})\xi_l\xi_p \geq C_4\xi_i\xi_i$.

We note that condition (iv) is compatible with the usual assumptions within the Green-Lindsay theory.

We can observe that the relation (6) is satisfied when

$$\begin{aligned}E(t) = \int_B \Big(&\rho\dot{u}_i\dot{u}_i + 2\mu e_{ij}e_{ij} + \lambda e_{ii}e_{jj} + \alpha^{-1}m_{lp}(\theta_l + \alpha\dot{\theta}_l)(\theta_p + \alpha\dot{\theta}_p)\\ &+\alpha\left[l_1(\theta_1-\theta_2)^2 + 2l_2(\theta_1-\theta_2)(\theta_3-\theta_4) + l_3(\theta_3-\theta_4)^2\right]\\ &+(d_{lp} - \alpha^{-1}m_{lp})\theta_l\theta_p + \alpha K_{lp}\nabla\theta_l\nabla\theta_p\Big)\,da,\end{aligned}$$

and

$$\begin{aligned}D(t) = \int_B \Big(&l_1(\theta_1-\theta_2)^2 + 2l_2(\theta_1-\theta_2)(\theta_3-\theta_4) + l_3(\theta_3-\theta_4)^2\\ &+K_{lp}\nabla\theta_l\nabla\theta_p + (d_{lp}\alpha - m_{lp})\dot{\theta}_l\dot{\theta}_p\Big)\,da.\end{aligned}$$

We can consider this problem in the Hilbert space given in Sect. 3 and to define an inner product with the norm:

$$\begin{aligned}\|(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{\vartheta})\|^2 = \int_B \Big(&\rho v_i\overline{v_i} + 2\mu e_{ij}\overline{e_{ij}} + \lambda e_{ii}\overline{e_{jj}}\\ &+\alpha\xi^l_{pq}(\theta_p - \theta_q)\overline{\theta_l} + \alpha^{-1}m_{lp}(\theta_l + \alpha\vartheta_l)\overline{(\theta_p + \alpha\vartheta_p)}\\ &+(d_{lp} - \alpha^{-1}m_{lp})\theta_l\overline{\theta_p} + \alpha K_{lp}\nabla\theta_l\overline{\nabla\theta_p}\Big)\,da.\end{aligned}$$

We can write the problem in the form (7), where

$$\mathcal{A}\begin{pmatrix}u_i\\ v_i\\ \theta_d\\ \vartheta_d\end{pmatrix} = \begin{pmatrix}v_i\\ \rho^{-1}(\mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A^l_{ik}(\theta_{l,k} + \alpha\vartheta_{l,k}))\\ \vartheta_d\\ n_{dl}(K_{lp}\Delta\theta_p + A^l_{ik}v_{i,k} + \xi^l_{pq}(\theta_p-\theta_q) - d_{lp}\vartheta_p)\end{pmatrix}.$$

It is clear that the domain of the operator $\mathcal{A}$ is made of the elements of the Hilbert space $\mathcal{H}$ such that $v_i \in H^1_0(B)$, $\vartheta_l \in H^1_*(B)$, $\mu\Delta u_i + (\lambda+\mu)u_{j,ji} \in L^2(B)$ and $\theta_l \in H^2(B) \cap L^2_*(B)$. Therefore, it is dense in $\mathcal{H}$.

We can see that

$$\begin{aligned}Re < \mathcal{A}U, U >= -\int_B \Big(&K_{lp}\nabla\theta_l\overline{\nabla\theta_p} + (d_{lp}\alpha - m_{lp})\vartheta_l\overline{\vartheta_p} + l_1(\theta_1-\theta_2)\overline{(\theta_1-\theta_2)}\\ &+l_3(\theta_3-\theta_4)\overline{(\theta_3-\theta_4)} + l_2\left[(\theta_1-\theta_2)\overline{(\theta_3-\theta_4)} + (\theta_3-\theta_4)\overline{(\theta_1-\theta_2)}\right]\Big)\,da\end{aligned}$$

for every $U \in \mathcal{D}(\mathcal{A})$. Therefore, $\mathcal{A}$ is a dissipative operator.

By other hand, if $F = (f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4)$ and we try to solve the system:

$$\begin{array}{l}v_i = f_i,\\ \mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A^l_{ik}(\theta_{l,k} + \alpha\vartheta_{l,k}) = \rho f_{2+i},\\ \vartheta_p = g_p,\\ K_{lp}\Delta\theta_p + A^l_{ik}v_{i,k} + \xi^l_{pq}(\theta_p - \theta_q) - d_{lp}\vartheta_p = m_{lp}h_p.\end{array}$$

This system leads to the following coupled equations:

$$\mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A^l_{ik}\theta_{l,k} = \rho f_{2+i} - \alpha A^l_{ik} g_{l,k},$$
$$K_{lp}\Delta\theta_p + \xi^l_{pq}(\theta_p - \theta_q) = m_{lp}h_p - A^l_{ik} f_{i,k} + d_{lp}g_p,$$

which can be solved similarly as in the proof of Lemma 2. Again, we can conclude an existence and uniqueness result for the solutions as in Theorem 1.

Finally, we can show the exponential decay of the solutions. Following the same arguments used in the proof of Theorem 2 we find that

$$i\beta_n u_{jn} - \vartheta_{jn} \to 0 \quad \text{in} \quad H^1(B),$$
$$i\beta_n \rho v_{jn} - (\mu\Delta u_{jn} + (\lambda+\mu)u_{kn,kj} + A^l_{jk}(\theta_{ln,k} + \alpha\vartheta_{ln,k}) \to 0 \quad \text{in} \quad L^2(B),$$
$$i\beta_n\theta_{qn} - \vartheta_{qn} \to 0 \quad \text{in} \quad H^1(B),$$
$$i\beta_n m_{lp}\vartheta_{pn} - (K_{qp}\Delta\theta_{pn} + A^q_{ik}v_{in,k} + \xi^l_{pq}(\theta_p - \theta_q) - d_{lp}\vartheta_p) \to 0 \quad \text{in} \quad L^2(B).$$

In this case, the energy inequality leads to

$$\nabla\theta_{pn} \to 0, \quad \vartheta_{pn} \to 0.$$

From now on, we can follow the same procedure as in the proof of Theorem 2 and therefore, we can also obtain the exponential decay of the solutions whenever the coupling matrix has rank 4, that we state in the following.

Theorem 3 *Under conditions (i)-(iv), the problem determined by the Cauchy problem (7) associated to the system (16) has a unique solution. If we also assume that the rank of the matrix (A^k_{ij}) is four, then the solutions decay in an exponential way.*

6 Further comments

It is relevant to note that Theorems 2 and 3 have been shown assuming that the coupling matrix had maximum rank. Clearly, this requirement was a key point in the proof. However, in order to satisfy this condition it is needed that its terms are not isotropic. In fact, in the isotropic case we can follow the example provided by Dafermos [4] and we can obtain undamped isothermal solutions. This example could be generalized to domains satisfying the following condition (see [2]):

CONDITION D. There exists a nonzero vector field $\phi \in [H^1(B)]^2$ such that

$$\phi_{i,jj} + \gamma^2\phi_i = 0 \text{ for a given } \gamma \neq 0, \quad \phi_{i,i} = 0.$$

In the case $\lambda + \mu = 0$ we can consider the system:

$$\rho\ddot{u}_1 = \mu\Delta u_1 + A_{11}\theta_{1,1} + A_{22}\theta_{2,2},$$
$$\rho\ddot{u}_2 = \mu\Delta u_2 - A_{11}\theta_{1,1} - A_{22}\theta_{2,2},$$
$$m_1(\dot\theta_1 + d\ddot\theta_1) = K_1\Delta\theta_1 - l_1(\theta_1-\theta_2) - l_2(\theta_3-\theta_4) + A_{11}\dot u_{1,1} - A_{11}\dot u_{2,1},$$
$$m_2(\dot\theta_2 + d\ddot\theta_2) = K_2\Delta\theta_2 + l_1(\theta_1-\theta_2) + l_2(\theta_3-\theta_4) + A_{22}\dot u_{1,2} - A_{22}\dot u_{2,2},$$
$$m_3(\dot\theta_3 + d\ddot\theta_3) = K_3\Delta\theta_3 - l_2(\theta_1-\theta_2) - l_3(\theta_3-\theta_4),$$
$$m_4(\dot\theta_4 + d\ddot\theta_4) = K_4\Delta\theta_4 + l_2(\theta_1-\theta_2) + l_3(\theta_3-\theta_4).$$

If we impose the initial conditions:

$$u_1(x,0) = u_2(x,0) = u^0(x), \quad \dot u_1(x,0) = \dot u_2(x,0) = v^0(x),$$
$$\theta_i(x,0) = \vartheta_i(x,0) = 0 \quad i = 1,\ldots,4,$$

we also obtain undamped isothermal solutions. In this case, the matrix of coupling coefficients has rank 2.

Finally, we point out that we can extend the results of the previous sections to the case where the elasticity tensor is positive definite, or to the three-dimensional case but, in this case, we shall consider nine dissipative mechanisms and the coupling matrix must have maximum rank. Moreover, it is worth remarking that similar examples can be also considered for the Green-Lindsay theories.

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Declarations

Conflict of interest The authors declare there is no conflict of interest.

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