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On the fractional Allee logistic equation in the Caputo sense

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ABSTRACT

In the framework of population models, logistic growth and fractional logistic growth has been analyzed. In some situations the so-called Allee effect gives more accurate approximation. In this work, fractional Allee differential equation in the Caputo sense is considered. The solution is obtained by considering formal power series. Numerical computations are presented to compare the truncating series with the classical Allee differential equation.

1. Introduction

Fractional differential equations are useful to describe many physical and biological phenomena such as seepage flow in porous media and in fluid dynamic traffic model [1–4]. Fractional models are adequate to incorporate the memory effect. Some recent fractional models include the epidemic modeling of COVID-19 [5–7] or cholera [8].

The Allee effect may reduce the expansion of a population once that population attains a critical value, a threshold size for the viability of the population [9,10]. In a population model, the growth rate per capita is of the type $u * f(u)$, and it can be logistic growth (with f decreasing function) or weak Allee effect growth (where f is a function initially increasing from a positive value and then decreasing) [11–13].

Logistic growth is well known, and it mainly models the crowding effect of the population. On the other hand, Allee effect describes the phenomenon that for smaller populations, the reproduction and survival of individuals decrease. Weak Allee effect growth function f is always positive, while in the strong Allee effect, there is a critical density below which the growth is negative.

In the study of the role of Allee effect in modeling post resection recurrence of glioblastoma [14], the authors find that indeed in glioblastoma cell cultures the cell proliferation rate is an increasing function of the density at small cell densities suggesting that cooperative behavior of cancer cells is similar to the Allee effect in ecology and can play a critical role in determining the time until tumor recurrence. This is motivated from the experimental results from two glioblastoma cell lines suggesting that in the low density regime the cell growth rate increases with the population density while it decreases at larger densities and it lead to a computational model with Allee effect. As another medical case, in [15] it is considered a mathematical model with Allee effect to describe intra-cellular OCT4 regulation.

In [16], the authors present evidences that the amalgamation fractional order derivative and Allee effect can be used effectively in the study of some dynamics of physical capital and labor force.

Recently, a mathematical model has been developed [17] to analyze the impact of roads on the propagation of invasive plants when its growth is controlled by a function with the Allee effect.

In [18] an Allee effect in prey species and intra-specific competition among predators are incorporated to a prey-dependent tri-trophic food chain model. Mathematically, if the population at time t, $x(t)$, is normalized $0 \le x(t) \le 1$, then

$$x'(t) = x(t)[1 - x(t)][x(t) - a], \quad a > 0, \tag{1.1}$$

with constant solutions $x = 0, 1, a$.

If $x(0) \in [0, 1)$, then

(1) $x'(t) > 0$ for $a < x(0) < 1$, and hence x is increasing, and
(2) $x'(t) < 0$ for $0 < x(0) < a$ and in this case the population decreases.

Eq. (1.1) is a generalization of the logistic differential equation.

We have included some plots of the solutions to (1.1) obtained by using Mathematica [19] (see Fig. 1).

Let us solve (1.1) by considering formal power series. In doing so, let

$$x(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{1.2}$$

Then,

$$x'(t) = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n, \tag{1.3}$$

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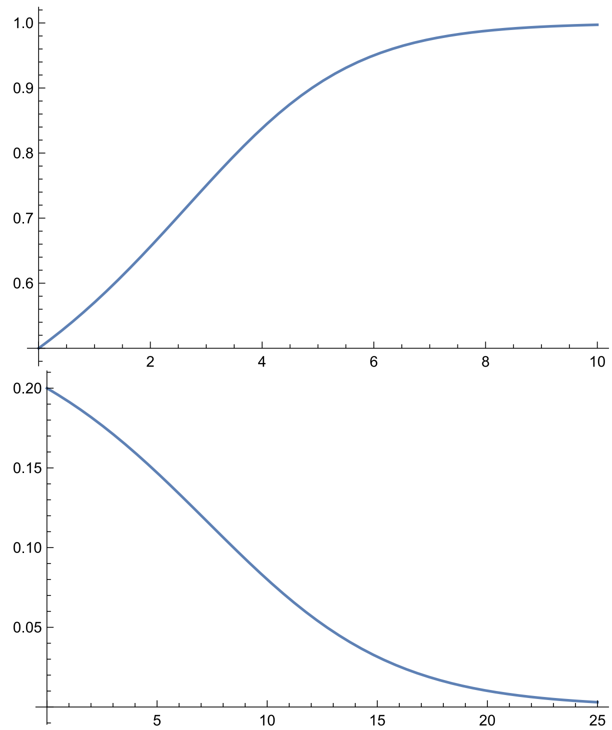

Fig. 1. Numerical solutions to (1.1) with $a = 1/4$. For $x(0) = 1/2$ (up) and for $x(0) = 1/5$ (down).

$$x^2(t) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j a_{n-j}\right) t^n, \tag{1.4}$$

$$x^3(t) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{j} a_k a_{j-k}\right) a_{n-j}\right) t^n. \tag{1.5}$$

Since

$$x(1-x)(x-a) = -x^3 + ax^2 + x^2 - ax = -x^3 + (a+1)x^2 - ax = P(x),$$

if we substitute the formal power series expansion into (1.1) the following recurrence relation holds for the coefficients

$$a_{n+1} = \frac{1}{n+1}\left[-a\,a_n + (a+1)\sum_{j=0}^{n} a_j a_{n-j} - \sum_{j=0}^{n}\left(\sum_{k=0}^{j} a_k a_{j-k}\right) a_{n-j}\right], \tag{1.6}$$

valid for $n \geq 0$ with the initial condition $a_0 = x(0)$.

We have included in Fig. 2 some plots of the solution of the differential Eq. (1.1), by comparing the numerical solution provided by Mathematica [19] with the solution expressed in terms of truncating polynomials.

2. Fractional Allee logistic equation in the Caputo sense

Let

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\xi)}{(t-\xi)^{\alpha}}\,d\xi, \quad t \in [0,b],$$

be the Caputo fractional derivative of order α of an absolutely continuous function $f : [0,b] \mapsto \mathbf{R}$, where $\Gamma(z)$ stands for the gamma function [20]. As it is well-known, we have [3]

$$D^{\alpha} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}, \qquad \nu > -1, \quad \alpha > 0. \tag{2.1}$$

Let us first introduce the fractional Allee logistic equation in the Caputo sense, that is,

$$D^{\alpha} x(t) = x(t)[1-x(t)][a-x(t)] = P(x(t)). \tag{2.2}$$

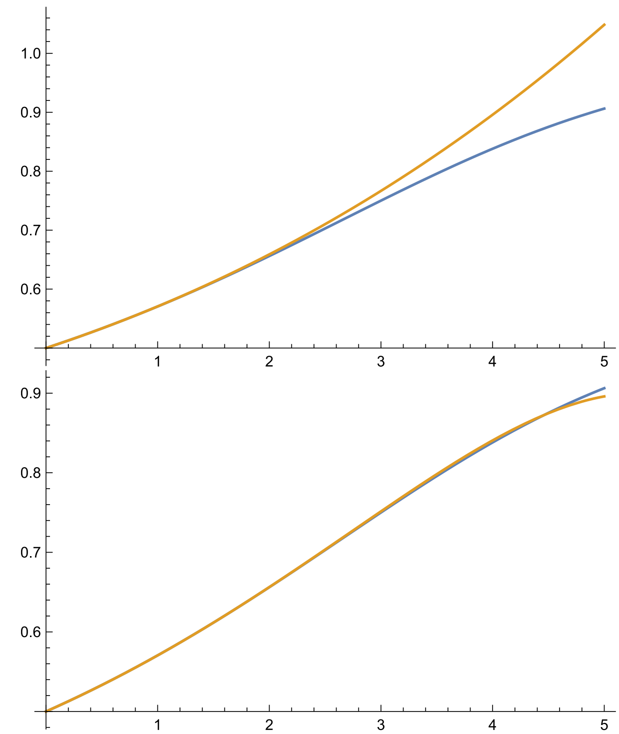

Fig. 2. Comparison for $t \in [0,5]$ between the numerical solutions to (1.1) obtained by using Mathematica command NSolve (in blue color) and truncating polynomials from (1.2) in orange color. We have considered $n = 3$ (up) and $n = 5$ (down). In both examples we have considered $a = 1/4$ as well as the initial condition $x(0) = 1/2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The fractional logistic equation has been considered recently by many authors (see e.g. [21–29] and references therein).

Let us consider the formal power series expansion in powers of t^{α} as in [22]

$$x(t) = \sum_{n=0}^{\infty} b_n(\alpha)(t^{\alpha})^n, \tag{2.3}$$

such that formally

$$D^{\alpha} x(t) = \sum_{n=0}^{\infty} b_{n+1}(\alpha)\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}(t^{\alpha})^n. \tag{2.4}$$

By proceeding in a similar way as in the classical situation, we obtain the following recurrence relation for the coefficients $b_n(\alpha)$, denoted only by b_n to simplify the notation, in (2.3)

$$b_{n+1} = \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\left[-a\,b_n + (a+1)\sum_{j=0}^{n} b_j b_{n-j} - \sum_{j=0}^{n}\left(\sum_{k=0}^{j} b_k b_{j-k}\right) b_{n-j}\right], \tag{2.5}$$

valid for $n \geq 0$ with an initial condition $b_0 = x(0)$.

For $b_0 = 1/2$ we can now compute recursively the truncated polynomials obtained from the formal series (2.3), namely

$$p_1(t;\alpha) = \frac{1}{16}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 8\right),$$

$$p_2(t;\alpha) = \frac{1}{64}\left(\frac{4t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left(t^2\right)^{\alpha}}{\Gamma(2\alpha+1)} + 32\right),$$

$$p_3(t;\alpha) = \frac{1}{1024}\left(\frac{4\left(t^3\right)^{\alpha}}{\Gamma(3\alpha+1)} + \frac{16\left(t^2\right)^{\alpha}}{\Gamma(2\alpha+1)} + \frac{64t^{\alpha}\Gamma(\alpha+1) - \frac{\left(t^3\right)^{\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}}{\Gamma(\alpha+1)^2} + 512\right),$$

which in the limit as $\alpha \to 1$ converge to those obtained from (1.2).

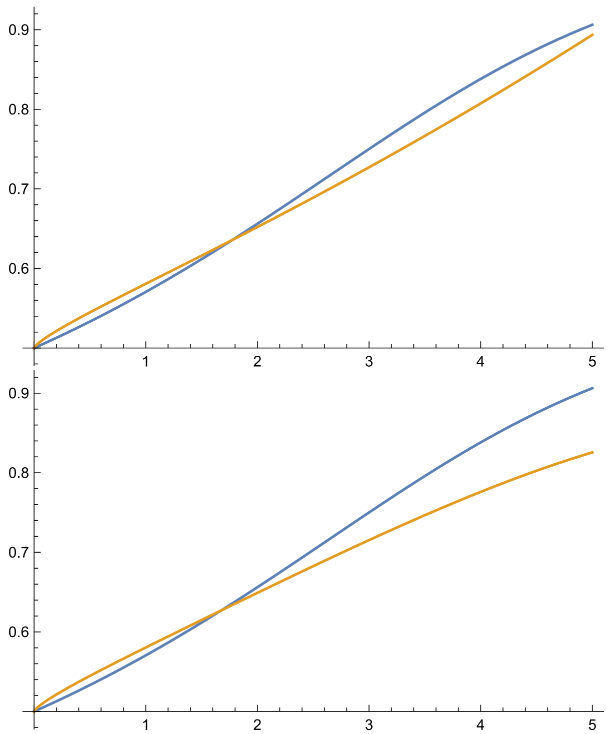

Fig. 3. Comparison for $t \in [0,5]$ between the numerical solutions to (1.1) obtained by using Mathematica command NSolve (in blue color) and truncating polynomials from (2.3) in orange color. We have considered $n = 3$ (up) and $n = 5$ (down). In both examples we have considered $\alpha = 0.75$, $a = 1/4$ as well as $x(0) = 1/2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

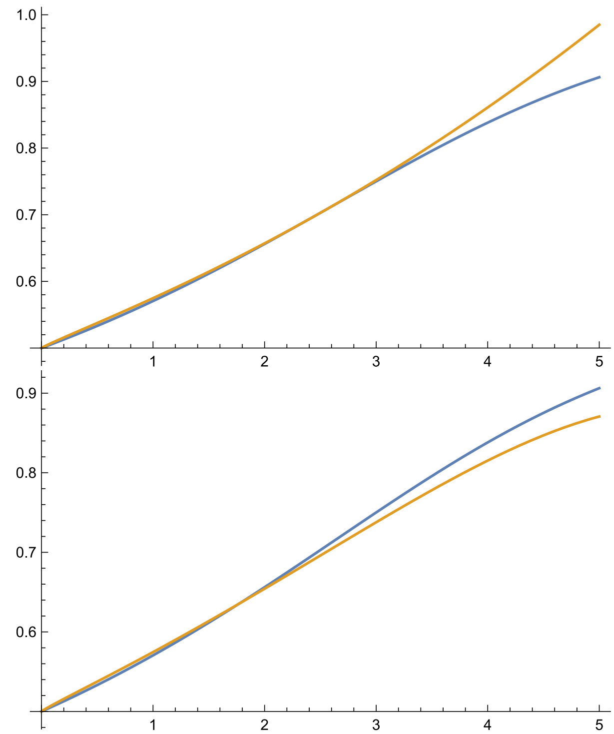

Fig. 4. Comparison for $t \in [0,5]$ between the numerical solutions to (1.1) obtained by using Mathematica command NSolve (in blue color) and truncating polynomials from (2.3) in orange color. We have considered $n = 3$ (up) and $n = 5$ (down). In both examples we have considered $\alpha = 0.9$, $a = 1/4$ as well as $x(0) = 1/2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Conclusions

In population models the logistic differential equation and the fractional logistic differential equation(s) have been considered in the literature. Moreover, in some situations the so-called Allee effect might provide more accurate approximation. In this work, we have considered fractional Allee differential equation in the Caputo sense for the first time, to the best of our knowledge. The solution has been obtained by considering formal power series, and numerical approximations have been computed by truncating the series (see Figs. 3 and 4).

Declaration of competing interest

The authors declare no conflict of interest.

Data availability

No data was used for the research described in the article.

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References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of the Fractional Differential Equations, North-Holland Mathematics Studies, 2006.
[2] K.B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, 1974.
[3] S. Samko, A.A. Kilbas, O. Marichev, Fractional Integrals and Derivatives, Taylor & Francis, 1993.
[4] V.S. Erturk, E. Godwe, D. Baleanu, P. Kumar, J.H. Asad, A. Jajarmi, Novel fractional-order Lagrangian to describe motion of beam on nanowire, Acta Phys. Pol. A 140 (3) (2021) 265–272.
[5] F. Ndaïrou, I. Area, J.J. Nieto, C.J. Silva, D.F.M. Torres, Fractional model of COVID-19 applied to Galicia, Spain and Portugal, Chaos Solitons Fractals 144 (2021) 110652.
[6] D. Baleanu, M.H. Abadi, A. Jajarmi, K.Z. Vahid, J.J. Nieto, A new comparative study on the general fractional model of COVID-19 with isolation and quarantine effects, Alex. Eng. J. 61 (2022) 4779–4791.
[7] J.J. Nieto, Fractional Euler numbers and generalized proportional fractional logistic differential equation, Fract. Calc. Appl. Anal. 25 (2022) 876–886.
[8] D. Baleanu, F.A. Ghassabzade, J.J. Nieto, A. Jajarmi, On a new and generalized fractional model for a real Cholera outbreak, Alex. Eng. J. 61 (2022) 9175–9186.
[9] W.C. Allee, E. Bowen, Studies in animal aggregations: mass protection against colloidal silver among goldfishes, J. Exp. Zool. 61 (1932) 185–207, http://dx.doi.org/10.1002/jez.1400610202.
[10] F. Courchamp, J. Berec, J. Gascoigne, Allee Effects in Ecology and Conservation, Oxford University Press, New York, 2008.
[11] W.C. Allee, Animal aggregations: A study in general sociology, University of Chicago Press, 1931.
[12] Y. Su, J. Wei, J. Shi, Hopf bifurcations in a reaction–diffusion population model with delay effect, J. Differential Equations 247 (2009) 1156–1184, http://dx.doi.org/10.1016/j.jde.2009.04.017.
[13] P.A. Stephens, W.J. Sutherland, R.P. Freckleton, What is the Allee effect? Oikos 87 (1999) 185–190.
[14] Z. Neufeld, W. von Witt, D. Lakatos, J. Wang, B. Hegedus, A. A. Czirok, The role of Allee effect in modelling post resection recurrence of glioblastoma, PLoS Comput. Biol. 13 (2017) e1005818, http://dx.doi.org/10.1371/journal.pcbi.1005818.

[15] L-E-. Wadkin, S. Orozco-Fuentes, I. Neganova, M. Lako, N.G. Parker, A. Shukurov, A mathematical modelling framework for the regulation of intra-cellular OCT4 in human pluripotent stem cells, PLoS One 16 (2021) e0254991, http://dx.doi.org/10.1371/journal.pone.0254991.

[16] N.A. Khan, O.A. Razzaq, F. Riaz, A. Ahmadian, N. Senu, Dynamics of fractional order nonlinear system: A realistic perception with neutrosophic fuzzy number and Allee effect, J. Adv. Res. 32 (2021) 109–118.

[17] B. Deeley, N. Petrovskaya, Propagation of invasive plant species in the presence of a road, J. Thoret. Biol. 548 (2022) 111196, http://dx.doi.org/10.1016/j.jtbi.2022.111196.

[18] L.N. Guin, G. Mandal, M. Mondal, S. Chakravarty, A chaotic tri-trophic food chain model supplemented by Allee effect, Int. J. Dyn. Control 11 (2023) 528–554.

[19] Wolfram Research, Inc., Mathematica, Version 13.1, Champaign, IL, 2022.

[20] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions, 9th printing, Dover, New York, 1972.

[21] I. Area, J. Losada, J.J. Nieto, A note on the fractional logistic equation, Physica A 444 (2016) 182–187.

[22] I. Area, J.J. Nieto, Power series solution of the fractional logistic equation, Physica A 573 (2021) 125947.

[23] I. Area, J.J. Nieto, Fractional-order logistic differential equation with Mittag-Leffler-type kernel, Fract. Fract. 5 (2021) 273.

[24] M. D'Ovidio, P. Loreti, S. Sarv Ahrabi, Modified fractional logistic equation, Physica A 505 (2018) 818–824.

[25] A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. El-Saka, On the fractional-order logistic equations, Appl. Math. Lett. 20 (2007) 817–823.

[26] L.N. Kaharuddin, C. Phang, S.S. Jamaian, Solution to the fractional logistic equation by modified Eulerian numbers, Eur. Phys. J. Plus 135 (2) (2020) 229.

[27] J. Nieto, Solution of a fractional logistic ordinary differential equation, Appl. Math. Lett. 123 (2022) 107568.

[28] M. Ortigueira, G. Bengochea, A new look at the fractionalization of the logistic equation, Physica A 467 (2017) 554–561.

[29] B.J. West, Exact solution to fractional logistic equation, Physica A 429 (2015) 103–108.