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# Mathematical analysis and numerical resolution of a heat transfer problem arising in water recirculation

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## Abstract

This work is devoted to the analysis and resolution of a well-posed mathematical model for several processes involved in the artificial circulation of water in a large waterbody. This novel formulation couples the convective heat transfer equation with the modified Navier-Stokes system following a Smagorinsky turbulence model, completed with a suitable set of mixed, nonhomogeneous boundary conditions of diffusive, convective and radiative type. We prove several theoretical results related to existence of solution, and propose a full algorithm for its computation, illustrated with some realistic numerical examples.

*Key words:* Radiation heat transfer, Existence, Uniqueness, Numerical resolution

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## 1 Introduction

Artificial circulation in large waterbodies is a management technique aimed to disrupt stratification of temperature and, consequently, to minimize the development of stagnant zones that may be subject to water quality problems (for instance, low levels of dissolved oxygen or high concentrations of phytoplankton). For its operation, a set of flow pumps take water from the upper

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layers by means of collectors and inject it into the bottom layers, setting up a recirculation pattern that prevents stratification by means of a forced mixing of water. One of the main problems of the temperature stratification is related to algal blooms produced in the upper layers due to high temperature and solar radiation. However, if we circulate water from the bottom layers (where the temperature is lower) to the upper layers, we can mitigate this negative effect. Further details and remarks on several issues related to the optimal design and control of water artificial circulation techniques have been analyzed by the authors in their recent work [19].

Convective heat transfer has been the subject of an intensive mathematical research in last five decades (ranging, for instance, from the pioneering works on the Boussinesq system of Joseph [15] in the 1960s to the present). Among the recent contributions we must mention, for instance, some papers devoted to study related problems in the steady case [4,16], the analysis a time-dependant case, but not including convective phenomena, [20], and some numerical approaches [3,18]. Nevertheless, after an exhaustive search we have not been able to find in the mathematical literature the analysis of the particular problem arising in the setting of our water recirculation model: a coupled problem linking a heat equation with mixed nonlinear boundary conditions to a modified Navier-Stokes equation following the Smagorinsky model of turbulence. Thus, the present work deals with the mathematical analysis and the numerical resolution of this heat transfer problem with specific boundary conditions related to water artificial circulation in a body of water (for instance, a lake or a reservoir). The main difficulties in the study of this problem lie in the nonlinear boundary condition related to the solar irradiation on the surface, the relations between the water temperature in the collectors and the injectors, and the coupling between water temperature and water velocity due to convective effects. We use the Smagorinsky model of turbulence instead of other approaches, like the celebrated  $k - \epsilon$  system, due to the fact that the modified Navier-Stokes equations following the Smagorinsky model of turbulence present very interesting properties from a mathematical viewpoint, in particular the uniqueness of solution and its additional regularity.

The organization of this paper is as follows: First we introduce a well-posed formulation of the physical problem and present a rigorous definition of a solution for the problem. In the central part of the paper we prove the existence of this solution, and in the final part we propose a numerical algorithm for its resolution, showing several computational tests for a realistic example. At the end of the paper we include an appendix with several results for a general heat equation with an advective term and mixed boundary conditions of diffusive, convective and radiative type. These results, as far as we know, are new since we use techniques that will allow us to treat the low regularity of the time derivative of the solution. This lack of regularity represents, together with the nonlinearity of the boundary conditions, the main difficulty.

## 2 Mathematical formulation of the problem

In this section we present in detail the three-dimensional mathematical model under study. So, we consider a convex domain  $\Omega \subset \mathbb{R}^3$  (representing the water-body) whose boundary surface  $\partial\Omega$  can be split into four smooth enough, disjoint sections:  $\Gamma_S$ ,  $\Gamma_C$ ,  $\Gamma_T$  and  $\Gamma_N$ , in such a way that  $\partial\Omega = \Gamma_S \cup \Gamma_C \cup \Gamma_T \cup \Gamma_N$ . Subset  $\Gamma_S$  represents the part of the boundary in contact with air,  $\Gamma_C$  is the part of the boundary where the collectors are located,  $\Gamma_T$  is the part of the boundary where the injectors are located, and  $\Gamma_N$  stands for the rest of the boundary. We will suppose that each collector is linked to an injector by means of a pumped pipeline, and we also assume that there exist  $N_{CT}$  collector/injector pairs  $\{(C^k, T^k)\}_{k=1}^{N_{CT}}$ . Therefore,  $\Gamma_C = \cup_{k=1}^{N_{CT}} C^k$ , and  $\Gamma_T = \cup_{k=1}^{N_{CT}} T^k$ . In Fig. 1 we can see a schematic geometrical configuration of a rectangular domain  $\Omega$  for a particular case of  $N_{CT} = 4$  collector/injector pairs.

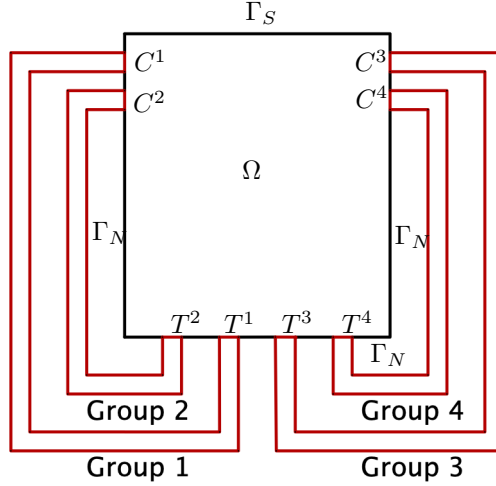


Fig. 1. Geometrical configuration of an example domain  $\Omega$  with  $N_{CT} = 4$  collector/injector pairs, showing the different boundary sections:  $\Gamma_S$ ,  $\Gamma_C = \cup_{k=1}^4 C^k$ ,  $\Gamma_T = \cup_{k=1}^4 T^k$ , and  $\Gamma_N$ .

As above commented, we suppose the boundary  $\partial\Omega$  regular enough to assure the existence of elements  $\varphi^k, \tilde{\varphi}^k \in H^{3/2}(\partial\Omega)$ ,  $k = 1, \dots, N_{CT}$ , satisfying the following assumptions (corresponding to suitable regularizations of the indicator functions of  $T^k$  and  $C^k$ , respectively):

- $\varphi^k(\mathbf{x}), \tilde{\varphi}^k(\mathbf{x}) \geq 0$ , a.e.  $\mathbf{x} \in \partial\Omega$ ,
- $\varphi^k(\mathbf{x}) = 0$ , a.e.  $\mathbf{x} \in \partial\Omega \setminus T^k$ , and  $\int_{T^k} \varphi^k(\mathbf{x}) d\gamma = \mu(T^k)$ ,
- $\tilde{\varphi}^k(\mathbf{x}) = 0$ , a.e.  $\mathbf{x} \in \partial\Omega \setminus C^k$ , and  $\int_{C^k} \tilde{\varphi}^k(\mathbf{x}) d\gamma = \mu(C^k)$ ,

where  $\mu(S)$  represents the area measure of any set  $S \subset \partial\Omega$ .

We denote by  $\theta(\mathbf{x}, t)$  (measured in K) the solution of the following convection-diffusion partial differential equation with nonhomogeneous, nonlinear, mixed boundary conditions:

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta - \nabla \cdot (K \nabla \theta) = 0 \quad \text{in } \Omega \times ]0, T[, \\ \theta = \phi_\theta \quad \text{on } \Gamma_T \times ]0, T[, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_C \times ]0, T[, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = b_1^N (\theta_N - \theta) \quad \text{on } \Gamma_N \times ]0, T[, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = b_1^S (\theta_S - \theta) + b_2^S (T_r^4 - |\theta|^3 \theta) \quad \text{on } \Gamma_S \times ]0, T[, \\ \theta(0) = \theta^0 \quad \text{in } \Omega, \end{array} \right. \quad (1)$$

where Dirichlet boundary condition  $\phi_\theta$  is given by expression:

$$\phi_\theta(\mathbf{x}, t) = \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \int_{-T}^T \rho_\epsilon(t - \epsilon - s) \gamma_\theta^k(s) ds \quad (2)$$

with, for each  $k = 1, \dots, N_{CT}$ ,

$$\gamma_\theta^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} \theta^0 d\gamma & \text{if } s \leq 0, \\ \frac{1}{\mu(C^k)} \int_{C^k} \theta(s) d\gamma & \text{if } s > 0, \end{cases} \quad (3)$$

representing the mean temperature of water in the collector  $C_k$ , and with the weight function  $\rho_\epsilon$  defined by:

$$\rho_\epsilon(t) = \begin{cases} \frac{c}{\epsilon} \exp\left(\frac{t^2}{t^2 - \epsilon^2}\right) & \text{if } |t| < \epsilon, \\ 0 & \text{if } |t| \geq \epsilon, \end{cases} \quad (4)$$

for  $c \in \mathbb{R}$  the positive constant satisfying the unitary condition:

$$\int_{\mathbb{R}} \rho_1(t) dt = 1. \quad (5)$$

In other words, we are assuming that the mean temperature of water at each injector  $T_k$  is a weighted average in time of the mean temperatures of water at its corresponding collector  $C_k$ . In order to obtain the mean temperature at each injector, we convolute the mean temperature at the collector with a smooth function with support in  $(t - 2\epsilon, t)$ . In this way, we have that the temperature in the injector only depends on the mean temperature in the collector in the time interval  $(t - 2\epsilon, t)$ . Parameter  $0 < \epsilon < T$  represents, in a certain sense, the technical characteristics of the pipeline that define the

stay time of water in the pipe. We also suppose that there is not heat transfer through the walls of the pipelines (that is, they are isolated).

Moreover,

- $T > 0$  (s) is the length of the time interval.
- $\mathbf{n}$  is the unit outward normal vector to the boundary  $\partial\Omega$ .
- $K > 0$  ( $\text{m}^2\text{s}^{-1}$ ) is the thermal diffusivity of the fluid:  $K = \frac{\alpha}{\rho c_p}$ , where  $\alpha$  ( $\text{W m}^{-1}\text{K}^{-1}$ ) is the thermal conductivity,  $\rho$  ( $\text{g m}^{-3}$ ) is the density, and  $c_p$  ( $\text{W s g}^{-1}\text{K}^{-1}$ ) is the specific heat capacity of water.
- $b_1^K \geq 0$  ( $\text{m s}^{-1}$ ), for  $K = N, S$ , are the coefficients related to convective heat transfer through the boundaries  $\Gamma_N$  and  $\Gamma_S$ , obtained from the relation  $\rho c_p b_1^K = h^K$ , where  $h^K \geq 0$  ( $\text{W m}^{-2}\text{K}^{-1}$ ) are the convective heat transfer coefficients on each surface. **These coefficients are relevant in the convective heat transfer flux through the frontiers  $\Gamma_S$  and  $\Gamma_N$ ,  $b_1^K(\theta_K - \theta)$ ,  $K = N, S$ .**
- $b_2^S > 0$  ( $\text{m s K}^{-3}$ ) is the coefficient related to radiative heat transfer through the boundary  $\Gamma_S$ , given by  $b_2^S = \frac{\sigma_B \varepsilon}{\rho c_p}$ , where  $\sigma_B$  ( $\text{W m}^{-2}\text{K}^{-4}$ ) is the Stefan-Boltzmann constant and  $\varepsilon$  is the emissivity. **This coefficient is fundamental in the radiative flux through the frontier  $\Gamma_S$  (see, for instance, the classical reference [5] for a complete description of this type of boundary conditions).**
- $\theta^0 \geq 0$  (K) is the initial temperature.
- $\theta_S, \theta_N \geq 0$  (K) are the temperatures related to convection heat transfer on the surfaces  $\Gamma_S$  and  $\Gamma_N$ .
- $T_r \geq 0$  (K) is the radiation temperature on the surface  $\Gamma_S$ , derived from expression  $\sigma_B \varepsilon T_r^4 = (1 - a)R_{sw,net} + R_{lw,down}$ , where  $a$  is the albedo,  $R_{sw,net}$  ( $\text{W m}^{-2}$ ) denotes the net incident shortwave radiation on the surface  $\Gamma_S$ , and  $R_{lw,down}$  ( $\text{W m}^{-2}$ ) denotes the downwelling longwave radiation.

Finally,  $\mathbf{v}(\mathbf{x}, t)$  ( $\text{m s}^{-1}$ ) is the water velocity, solution of a modified Navier-Stokes equations following a Smagorinsky model of turbulence:

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \mathbf{v} - \nabla \cdot \Xi(\mathbf{v}) + \nabla p = \alpha_0(\theta - \theta^0)\mathbf{a}_g \quad \text{in } \Omega \times ]0, T[, \\ \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \times ]0, T[, \\ \mathbf{v} = \phi_{\mathbf{g}} \quad \text{on } \partial\Omega \times ]0, T[, \\ \mathbf{v}(0) = \mathbf{v}^0 \quad \text{in } \Omega, \end{array} \right. \quad (6)$$

where  $\mathbf{a}_g = g \mathbf{e}_3$  ( $\text{m s}^{-1}$ ) is the gravity acceleration,  $\alpha_0 = -\frac{1}{\rho} \frac{\partial \rho}{\partial \theta}$  ( $\text{K}^{-1}$ ) is the thermic expansion coefficient. **We must remark here that we are assuming the thermodynamic process to be close to an initial equilibrium state that we denote with the zero subscript, so  $\rho = \rho_0 - \alpha_0 \rho_0(\theta - \theta_0) + k_0 \rho_0(p - p_0) + o(\theta - \theta_0) + o(p - p_0)$ , where  $k_0 = \frac{1}{\rho} \frac{\partial \rho}{\partial p}$  is the isothermal compressibility coefficient.** The details of this approach (known as the Boussinesq model for natural convection) can be consulted, for instance, in Section 10.7 of [6]. Finally,  $\mathbf{v}^0$

is the initial velocity, and boundary field  $\phi_{\mathbf{g}}$  is the element given by:

$$\phi_{\mathbf{g}}(t, \mathbf{x}) = \sum_{k=1}^{N_{CT}} g^k(t) \left[ \frac{\varphi^k(\mathbf{x})}{\mu(T^k)} - \frac{\tilde{\varphi}^k(\mathbf{x})}{\mu(C^k)} \right] \mathbf{n}. \quad (7)$$

with, for each  $k = 1, \dots, N_{CT}$ ,  $g^k(t) \in H^1(0, T)$ , representing the volumetric flow rate by pump  $k$  at each time  $t$  ( $g^k(t) > 0, \forall t \in ]0, T[$ , and  $g^k(0) = 0$ ). The turbulence term  $\Xi(\mathbf{v})$  is given by:

$$\Xi(\mathbf{v}) = \left. \frac{\partial D(e)}{\partial e} \right|_{e=e(\mathbf{v})}, \quad \text{with } e(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad (8)$$

where  $D$  is a potential function (for instance, in the particular case of the classical Navier-Stokes equations,  $D(e) = \nu [e : e]$ , with  $\nu$  ( $\text{m}^2 \text{s}^{-1}$ ) the kinematic viscosity of the water, and, consequently,  $\Xi(\mathbf{v}) = 2\nu e(\mathbf{v})$ ). However, in our case, the Smagorinsky model, the potential function is defined as [17]:

$$D(e) = \nu [e : e] + \frac{2}{3} \nu_{tur} [e : e]^{3/2}, \quad (9)$$

where  $\nu_{tur}$  ( $\text{m}^2$ ) is the turbulent viscosity. [So, for the Smagorinsky case,](#)

$$\begin{aligned} \Xi(\mathbf{v}) &= \left. \frac{\partial D(e)}{\partial e} \right|_{e=e(\mathbf{v})} = 2\nu \epsilon(\mathbf{v}) + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2} \epsilon(\mathbf{v}) \\ &= (2\nu + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2}) \epsilon(\mathbf{v}) = \beta(\epsilon(\mathbf{v})) \epsilon(\mathbf{v}), \end{aligned} \quad (10)$$

with  $\beta(\epsilon(\mathbf{v})) = 2\nu + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2}$

System (6) has been recently studied by the authors in [9], where it is demonstrated the existence and the uniqueness of solution for a recirculation model based in the modified Navier-Stokes equations. In the present work we will use some of the results shown in [9] in order to prove the existence of solution for the coupled problem (1) and (6). The main difficulties of this work relies in the coupling of a heat equation with nonlinear boundary value terms and the modified Navier-Stokes system. The nonlinear terms do not allow us to work with regular solutions for the heat equation, which forces us to use more sophisticated techniques in order to demonstrate existence and uniqueness of solution.

### 3 The concept of solution

We start this section defining the functional spaces used in the definition of solution for the system (1) and (6). So, for the water temperature we consider:

$$\begin{aligned} X_1 &= \{\theta \in H^1(\Omega) : \theta|_{\Gamma_S} \in L^5(\Gamma_S)\}, \\ \widetilde{X}_1 &= \{\theta \in X_1 : \theta|_{\Gamma_T} = 0\}, \end{aligned} \quad (11)$$

and we define the following norm associated to above space  $X_1$ :

$$\|\theta\|_{X_1} = \|\theta\|_{H^1(\Omega)} + \|\theta\|_{L^5(\Gamma_S)}. \quad (12)$$

We have that  $X_1$  is a reflexive separable Banach space (cf. Lemma 3.1 of [8]) and  $\widetilde{X}_1 \subset L^2(\Omega) \subset \widetilde{X}'_1$  is an evolution triple. For the water velocity we consider:

$$\begin{aligned} \mathbf{X}_2 &= \left\{ \mathbf{v} \in [W^{1,3}(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0, \mathbf{v}|_{\Gamma_S \cup \Gamma_N} = \mathbf{0} \right\}, \\ \widetilde{\mathbf{X}}_2 &= \left\{ \mathbf{v} \in [W^{1,3}(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = \mathbf{0} \right\}. \end{aligned} \quad (13)$$

In order to define an appropriate space for the solution of problems (1) and (6), we consider, for a Banach space  $V_1$  and a locally convex space  $V_2$  such that  $V_1 \subset V_2$ , the following Sobolev-Bochner space (cf. Chapter 7 of [21]), for  $1 \leq p, q \leq \infty$ :

$$W^{1,p,q}(0, T; V_1, V_2) = \left\{ u \in L^p(0, T; V_1) : \frac{du}{dt} \in L^q(0, T; V_2) \right\}, \quad (14)$$

where  $\frac{du}{dt}$  denotes the derivative of  $u$  in the sense of distributions. It is well known that, if both  $V_1$  and  $V_2$  are Banach spaces, then  $W^{1,p,q}(0, T; V_1, V_2)$  is also a Banach space endowed with the norm  $\|u\|_{W^{1,p,q}(0,T;V_1,V_2)} = \|u\|_{L^p(0,T;V_1)} + \left\| \frac{du}{dt} \right\|_{L^q(0,T;V_2)}$ .

Then, we define the following spaces that will be used in the mathematical analysis of system (1):

$$\begin{aligned} W_1 &= \{\theta \in W^{1,2,5/4}(0, T; X_1, X'_1) : \\ &\quad \theta|_{\Gamma_S} \in L^5(0, T; L^5(\Gamma_S))\} \cap L^\infty(0, T; L^2(\Omega)), \\ \widetilde{W}_1 &= \{\theta \in W^{1,2,5/4}(0, T; \widetilde{X}_1, \widetilde{X}'_1) : \\ &\quad \theta|_{\Gamma_S} \in L^5(0, T; L^5(\Gamma_S))\} \cap L^\infty(0, T; L^2(\Omega)), \end{aligned} \quad (15)$$



and, for the system (6), we define

$$\begin{aligned}\mathbf{W}_2 &= W^{1,\infty,2}(0, T; \mathbf{X}_2, [L^2(\Omega)]^3) \cap \mathcal{C}([0, T]; \mathbf{X}_2), \\ \widetilde{\mathbf{W}}_2 &= W^{1,\infty,2}(0, T; \widetilde{\mathbf{X}}_2, [L^2(\Omega)]^3) \cap \mathcal{C}([0, T]; \widetilde{\mathbf{X}}_2).\end{aligned}\tag{16}$$

**Hypothesis 1** *We will assume the following hypotheses for coefficients and data of the problem:*

- (a)  $\theta^0 \in X_2$
- (b)  $\theta_S \in L^2(0, T; L^2(\Gamma_S))$
- (c)  $\theta_N \in L^2(0, T; L^2(\Gamma_N))$
- (d)  $T_r \in L^5(0, T; L^5(\Gamma_S))$
- (e)  $\mathbf{v}^0 \in [H_\sigma^2(\Omega)]^3 = \{\mathbf{v} \in [H^2(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = \mathbf{0}\} \subset \widetilde{\mathbf{X}}_2$
- (f)  $g^k \in H^1(0, T)$  with  $g^k(0) = 0, \forall k = 1, \dots, N_{CT}$

**Remark 2** *In order to define in a rigorous way the concept of solution, we will need to extend Dirichlet conditions of  $\theta$  and  $\mathbf{v}$  to the whole domain  $\Omega$ .*

So, for water velocity  $\mathbf{v}$ , thanks to Lemma 2 of [9], for each  $\mathbf{g} \in [H^1(0, T)]^{N_{CT}}$ , there exists an element  $\zeta_{\mathbf{g}} \in W^{1,2,2}(0, T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3)$  such that  $\zeta_{\mathbf{g}}|_{\partial\Omega} = \phi_{\mathbf{g}}$ , with  $\phi_{\mathbf{g}}$  defined by (7). Besides, by Lemma 3 of [9],  $W^{1,2,2}(0, T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3) \subset W^{1,\infty,2}(0, T; [H^2(\Omega)]^3, [H^2(\Omega)]^3) \cap \mathcal{C}([0, T]; [H^2(\Omega)]^3)$  and, then, we can use this element to reformulate the original problem for  $\mathbf{v}$  as an homogeneous Dirichlet boundary condition one.

For water temperature  $\theta$  we can proceed in an analogous way and prove that there exists an extension that allows us to reformulate the problem for  $\theta$  as one with homogeneous boundary conditions.

**Lemma 3** *We have that the following operator is compact*

$$\begin{aligned}R_{\mathbf{h}} : [L^2(0, T)]^{N_{CT}} &\rightarrow W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega)) \\ \mathbf{h} &\rightarrow R_{\mathbf{h}}(\mathbf{h}) = \zeta_{\mathbf{h}},\end{aligned}\tag{17}$$

where:

$$\zeta_{\mathbf{h}}(\mathbf{x}, t) = \sum_{k=1}^{N_{CT}} \beta_0(\varphi^k(\mathbf{x})) \int_{-T}^T \rho_\epsilon(t - \epsilon - s) \gamma_{\mathbf{h}}^k(s) ds,\tag{18}$$

with  $\gamma_{\mathbf{h}}^k(s) \in L^2(-T, T)$ , for  $k = 1, 2, \dots, N_{CT}$ , defined by:

$$\gamma_{\mathbf{h}}^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} \theta^0 d\gamma & \text{if } s \leq 0, \\ h^k(s) & \text{if } s > 0, \end{cases}\tag{19}$$

and  $\beta_0 : u \in H^{3/2}(\partial\Omega) \rightarrow \beta_0(u) \in H^2(\Omega)$  the right inverse of the classical trace operator  $\gamma_0$  (that is,  $(\gamma_0 \circ \beta_0)(u) = u$ .)

We also have that there exists a constant  $C_1$ , that depends continuously on the space-time configuration of our computational domain and  $\theta^0$ , such that:

$$\|\zeta_{\mathbf{h}}\|_{W^{1,2,2}(0,T;H^2(\Omega),H^2(\Omega))} \leq C_1(\theta^0)(1 + \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}}). \quad (20)$$

**PROOF.** Let  $\{\mathbf{h}_n\}_{n \in \mathbb{N}}$  be a bounded sequence in  $[L^2(0, T)]^{N_{CT}}$ . Then, taking subsequences if necessary, we have that  $\mathbf{h}_n \rightharpoonup \mathbf{h}$  weakly in  $[L^2(0, T)]^{N_{CT}}$ . We also have that  $\gamma_{\mathbf{h}_n} \rightharpoonup \gamma_{\mathbf{h}}$  weakly in  $[L^2(-T, T)]^{N_{CT}}$  and, if we denote by

$$b_{\mathbf{h}}^k(t) = \int_{-T}^T \rho_\epsilon(t - \epsilon - s) \gamma_{\mathbf{h}}^k(s) ds, \quad (21)$$

we obtain that  $b_{\mathbf{h}_n}^k(t) \rightarrow b_{\mathbf{h}}^k(t)$  pointwise a.e.  $t \in [0, T]$ ,  $\forall k = 1, \dots, N_{CT}$ . Thus, the sequence  $\{b_{\mathbf{h}_n}^k\}_{n \in \mathbb{N}}$  is bounded by a function in  $[L^2(0, T)]^{N_{CT}}$ , so we have the strong convergence in  $[L^2(0, T)]^{N_{CT}}$ . We can repeat the same argument with the time derivative of  $\{b_{\mathbf{h}_n}^k\}_{n \in \mathbb{N}}$ , obtaining that  $b_{\mathbf{h}_n}^k \rightarrow b_{\mathbf{h}}^k$  strongly in  $H^1(0, T)$ , thus  $\zeta_{\mathbf{h}_n} \rightarrow \zeta_{\mathbf{h}}$  strongly in  $W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$ . Finally, by the properties of the operator  $\beta_0$  we have that

$$\|\beta_0(\varphi^k)\|_{H^2(\Omega)} \leq C \|\varphi^k\|_{H^{3/2}(\partial\Omega)}, \quad k = 1, \dots, N_{CT}. \quad (22)$$

So, thanks to the regularity of function  $\rho_\epsilon$ , it is clear that  $b_{\mathbf{h}}^k \in H^1(0, T)$ , and

$$\|b_{\mathbf{h}}^k\|_{H^1(0,T)} \leq C \|\gamma_{\mathbf{h}}^k\|_{L^2(-T,T)}. \quad (23)$$

In the other hand,

$$\begin{aligned} \|\gamma_{\mathbf{h}}^k\|_{L^2(-T,T)}^2 &= \int_{-T}^T (\gamma_{\mathbf{h}}^k(s))^2 ds \\ &= \frac{1}{\mu(C^k)^2} \int_{-T}^0 \left( \int_{C^k} \theta^0 d\gamma \right)^2 + \int_0^T (h^k(s))^2 ds \\ &\leq \frac{T}{\mu(C^k)} \|\theta^0\|_{L^2(\partial\Omega)}^2 + \|h^k\|_{L^2(0,T)}^2. \end{aligned} \quad (24)$$

Thereby, we have the following inequality:

$$\begin{aligned} \|\zeta_{\mathbf{h}}\|_{W^{1,2,2}(0,T;H^2(\Omega),H^2(\Omega))} &\leq C \left( \|\theta^0\|_{L^2(\partial\Omega)} + \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}} \right) \sum_{k=1}^{N_{CT}} \mu(T^k) \\ &\leq C_1(\theta^0) \left( 1 + \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}} \right), \end{aligned}$$

where  $C(\theta^0)$  is a positive constant than depends continuously on the spatial-time configuration of our computational domain and on the initial tempera-

ture. Moreover, it is worthwhile remarking here that we can make this constant as small as we want by considering data appropriately.  $\blacksquare$

Finally, the following technical lemma will be necessary in order to guaranty that the sum of an element of  $W_1$  plus an element of  $W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$  makes sense.

**Lemma 4** *We have that the following inclusion is compact:*

$$W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega)) \subset\subset W_1. \quad (25)$$

Now, we define the concept of solution for coupled system (1) and (6) in terms of homogeneous Dirichlet systems, so we establish the following notations:

- $\xi = \theta - \zeta_{\mathbf{h}} \in \widetilde{W}_1$ , with  $\zeta_{\mathbf{h}} \in W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$  the extension obtained from Lemma 3, where:

$$h^k(s) = \frac{1}{\mu(C^k)} \int_{C^k} \theta(s) d\gamma, \quad k = 1, 2, \dots, N_{CT}. \quad (26)$$

- $\mathbf{z} = \mathbf{v} - \zeta_{\mathbf{g}} \in \widetilde{\mathbf{W}}_2$ , with  $\zeta_{\mathbf{g}} \in W^{1,2,2}(0, T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3)$  the extension of the trace given in Lemma 2 of [9].

Thus, using above notations, we can reformulate the state system (1) and (6) in the following way:

$$\left\{ \begin{array}{l} \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi - \nabla \cdot (K \nabla \xi) \\ \quad = -\frac{\partial \zeta_{\mathbf{h}}}{\partial t} - \mathbf{v} \cdot \nabla \zeta_{\mathbf{h}} + \nabla \cdot (K \nabla \zeta_{\mathbf{h}}) \quad \text{in } \Omega \times (0, T), \\ \xi = 0 \quad \text{on } T^k \times (0, T), \quad \text{for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \xi}{\partial \mathbf{n}} = -K \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} \quad \text{on } C^k \times (0, T), \quad \text{for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \xi}{\partial \mathbf{n}} = b_1^N \left( \theta_N - \zeta_{\mathbf{h}} - \frac{K}{b_1^N} \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} - \xi \right) \quad \text{on } \Gamma_N \times (0, T), \\ K \frac{\partial \xi}{\partial \mathbf{n}} = b_1^S \left( \theta_S - \zeta_{\mathbf{h}} - \frac{K}{b_1^S} \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} - \xi \right) \\ \quad + b_2^S \left( T_r^4 - |\xi + \zeta_{\mathbf{h}}|^3 (\xi + \zeta_{\mathbf{h}}) \right) \quad \text{on } \Gamma_S \times (0, T), \\ \xi(0) = \theta^0 - \zeta_{\mathbf{h}}(0) \quad \text{in } \Omega. \end{array} \right. \quad (27)$$

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{z}}{\partial t} + \nabla(\zeta_{\mathbf{g}} + \mathbf{z})\mathbf{z} + \nabla \mathbf{z} \zeta_{\mathbf{g}} \\ -\operatorname{div} \left( 2\nu \epsilon(\mathbf{z}) + 2\nu_{tur} \int_{\Omega} \left[ \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) \right]^{1/2} \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) \right) \\ + \nabla p = \alpha_0(\theta - \theta^0) \mathbf{a}_g - \frac{\partial \zeta_{\mathbf{g}}}{\partial t} - \nabla \zeta_{\mathbf{g}} \zeta_{\mathbf{g}} \\ + 2\nu \nabla \cdot \epsilon(\zeta_{\mathbf{g}}) \quad \text{in } \Omega \times (0, T), \\ \mathbf{z} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T), \\ \mathbf{z}(0) = \mathbf{v}^0 \quad \text{in } \Omega. \end{array} \right. \quad (28)$$

It is worthwhile remarking here that above system shows homogeneous Dirichlet boundary conditions and, consequently, we will be able to define the concept of solution of the original state systems (1) and (6) in terms of the modified state systems (27) and (28). It should be also noted that, in the case of equation (1), the coupling terms in the Dirichlet boundary conditions are now transferred to the partial differential equation in system (27).

**Definition 5** *A pair  $(\theta, \mathbf{v}) \in W_1 \times \mathbf{W}_2$  is said to be a solution of problem (1) and (6) if there exist elements  $(\xi, \mathbf{z}) \in \widetilde{W}_1 \times \widetilde{\mathbf{W}}_2$  such that:*

- (1)  $\mathbf{v} = \zeta_{\mathbf{g}} + \mathbf{z}$ , with  $\zeta_{\mathbf{g}} \in W^{1,2;2}(0, T; [H^2(\Omega)]^3, [H^2(\Omega)]^3)$  the reconstruction of the trace given in Lemma 2 of [9], and  $\theta = \zeta_{\mathbf{h}} + \xi$ , with  $\zeta_{\mathbf{h}} \in W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$  the extension obtained in Lemma 3, where:

$$h^k(s) = \frac{1}{\mu(C^k)} \int_{C^k} \theta(s) d\gamma, \quad k = 1, 2, \dots, N_{CT}. \quad (29)$$

- (2)  $\mathbf{z}(0) = \mathbf{v}^0$ , and  $\xi(0) = \theta^0 - \zeta_{\mathbf{h}}(0)$ , a.e.  $\mathbf{x} \in \Omega$ .  
(3)  $(\xi, \mathbf{z})$  verifies the following variational formulation:

$$\begin{aligned} & \int_{\Omega} \frac{\partial \xi}{\partial t} \eta d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot \nabla \xi \eta d\mathbf{x} + K \int_{\Omega} \nabla \xi \cdot \nabla \eta d\mathbf{x} + b_1^N \int_{\Gamma_N} \xi \eta d\gamma \\ & + b_1^S \int_{\Gamma_S} \xi \eta d\gamma + b_2^S \int_{\Gamma_S} |\xi + \zeta_{\mathbf{h}}|^3 (\xi + \zeta_{\mathbf{h}}) \eta d\gamma = \int_{\Omega} H_{\mathbf{h}} \eta d\mathbf{x} \\ & + \int_{\Gamma_C} g_{\mathbf{h}}^C \eta d\gamma + b_1^N \int_{\Gamma_N} g_{\mathbf{h}}^N \eta d\gamma + b_1^S \int_{\Gamma_S} g_{\mathbf{h}}^S \eta d\gamma + b_2^S \int_{\Gamma_S} T_r^4 \eta d\gamma, \\ & \quad \text{a.e. } t \in ]0, T[, \quad \forall \eta \in \widetilde{X}_1. \end{aligned} \quad (30)$$

$$\begin{aligned} & \int_{\Omega} \frac{\partial \mathbf{z}}{\partial t} \cdot \boldsymbol{\eta} d\mathbf{x} + \int_{\Omega} \nabla(\zeta_{\mathbf{g}} + \mathbf{z})\mathbf{z} \cdot \boldsymbol{\eta} d\mathbf{x} + \int_{\Omega} \nabla \mathbf{z} \zeta_{\mathbf{g}} \cdot \boldsymbol{\eta} d\mathbf{x} \\ & + 2\nu \int_{\Omega} \epsilon(\mathbf{z}) : \epsilon(\boldsymbol{\eta}) d\mathbf{x} \\ & + 2\nu_{tur} \int_{\Omega} \left[ \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) \right]^{1/2} \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\eta}) d\mathbf{x} \\ & = \int_{\Omega} \mathbf{H}_{\mathbf{g}} \cdot \boldsymbol{\eta} d\mathbf{x}, \quad \text{a.e. } t \in ]0, T[, \quad \forall \boldsymbol{\eta} \in \widetilde{\mathbf{X}}_2, \end{aligned} \quad (31)$$

where

$$\begin{aligned}
H_{\mathbf{h}} &= \frac{\partial \zeta_{\mathbf{h}}}{\partial t} - \mathbf{v} \cdot \nabla \zeta_{\mathbf{h}} + \nabla \cdot (K \nabla \zeta_{\mathbf{h}}) \in L^2(0, T; L^2(\Omega)), \\
g_{\mathbf{h}}^C &= -K \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} \in L^2(0, T; L^2(\Gamma_C)), \\
g_{\mathbf{h}}^N &= \theta_N - \zeta_{\mathbf{h}} - \frac{K}{b_1^N} \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} \in L^2(0, T; L^2(\Gamma_N)), \\
g_{\mathbf{h}}^S &= \theta_S - \zeta_{\mathbf{h}} - \frac{K}{b_1^S} \frac{\partial \zeta_{\mathbf{h}}}{\partial \mathbf{n}} \in L^2(0, T; L^2(\Gamma_S)), \\
\mathbf{H}_{\mathbf{g}} &= \alpha_0(\theta - \theta^0) \mathbf{a}_g - \frac{\partial \zeta_{\mathbf{g}}}{\partial t} - \nabla \zeta_{\mathbf{g}} \zeta_{\mathbf{g}} \in L^2(0, T; [L^2(\Omega)]^3).
\end{aligned} \tag{32}$$

#### 4 Existence of solution

We will prove now that, under certain hypotheses over coefficients and data, there exists a unique solution for the system (1) and (6) in the sense of Definition 5. The procedure used here for demonstrating the existence of solution is based in the Schauder fixed point Theorem (cf. section 9.5 of [7]) and is similar to one employed by the authors, for instance, in [10]. **The main difficulties in the present case lie in the coupling of the Dirichlet conditions for the water temperature, and in the nonlinear radiation terms. To overcome these difficulties we will need to define a compact extension for the nonhomogeneous Dirichlet conditions and to prove novel results for the heat equation with radiation boundary conditions (see Appendix A).**

So, we consider the following operator:

$$\begin{aligned}
\mathbf{M} : L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}} &\rightarrow L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}} \\
(\theta^*, \mathbf{h}^*) &\rightarrow \mathbf{M}(\theta^*, \mathbf{h}^*) = (\theta, \mathbf{h})
\end{aligned} \tag{33}$$

where:

- $\mathbf{v} \in \mathbf{W}_2$  is such that  $\mathbf{z} = \mathbf{v} - \zeta_{\mathbf{g}} \in \widetilde{\mathbf{W}}_2$  is the solution of the following problem:

$$\left\{ \begin{array}{l}
\frac{\partial \mathbf{z}}{\partial t} + \nabla(\zeta_{\mathbf{g}} + \mathbf{z})\mathbf{z} + \nabla \mathbf{z} \zeta_{\mathbf{g}} \\
- \nabla \cdot \left( 2\nu \epsilon(\mathbf{z}) + 2\nu_{tur} \left[ \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) \right]^{1/2} \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) \right) \\
+ \nabla p = \alpha_0(\theta^* - \theta^0) \mathbf{a}_g - \frac{\partial \zeta_{\mathbf{g}}}{\partial t} - \nabla \zeta_{\mathbf{g}} \zeta_{\mathbf{g}} + 2\nu \nabla \cdot \epsilon(\zeta_{\mathbf{g}}) \text{ in } \Omega \times ]0, T[, \\
\mathbf{z} = \mathbf{0} \text{ on } \Gamma \times ]0, T[, \\
\mathbf{z}(0) = \mathbf{v}^0 - \zeta_{\mathbf{g}}(0) \text{ in } \Omega,
\end{array} \right. \tag{34}$$

with  $\zeta_{\mathbf{g}} \in W^{1,2,2}(0, T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3)$  the extension obtained from the Lemma 2 of [9].

- $\theta \in W_1$  is such that  $\xi = \theta - \zeta_{\mathbf{h}^*} \in \widetilde{W}_1$  is the solution of:

$$\left\{ \begin{array}{l} \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi - \nabla \cdot (K \nabla \xi) = -\frac{\partial \zeta_{\mathbf{h}^*}}{\partial t} - \mathbf{v} \cdot \nabla \zeta_{\mathbf{h}^*} \\ \quad + \nabla \cdot (K \nabla \zeta_{\mathbf{h}^*}) \quad \text{in } \Omega \times ]0, T[, \\ \xi = 0 \quad \text{on } T^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \xi}{\partial \mathbf{n}} = -K \frac{\partial \zeta_{\mathbf{h}^*}}{\partial \mathbf{n}} \quad \text{on } C^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \xi}{\partial \mathbf{n}} = b_1^N (\theta_N - \zeta_{\mathbf{h}^*} - \frac{K}{b_1^N} \frac{\partial \zeta_{\mathbf{h}^*}}{\partial \mathbf{n}} - \xi) \quad \text{on } \Gamma_N \times ]0, T[, \\ K \frac{\partial \xi}{\partial \mathbf{n}} = b_1^S (\theta_S - \zeta_{\mathbf{h}^*} - \frac{K}{b_1^S} \frac{\partial \zeta_{\mathbf{h}^*}}{\partial \mathbf{n}} - \xi) \\ \quad + b_2^S (T_r^4 - |\xi + \zeta_{\mathbf{h}^*}|^3 (\xi + \zeta_{\mathbf{h}^*})) \quad \text{on } \Gamma_S \times ]0, T[, \\ \xi(0) = \theta(0) - \zeta_{\mathbf{h}^*}(0) \quad \text{in } \Omega, \end{array} \right. \quad (35)$$

with  $\zeta_{\mathbf{h}^*} \in W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$  defined as in Lemma 3.

- $\mathbf{h} \in [L^2(0, T)]^{N_{CT}}$  is such that:

$$h^k(s) = \frac{1}{\mu(C^k)} \int_{C^k} \theta(s) d\gamma, \text{ for } k = 1, 2, \dots, N_{CT}. \quad (36)$$

The following technical results are necessary to prove that the operator  $M$  defined in (33) is well defined. The first one corresponds to the existence of solution for problem (34), and the second one is related to the existence of solution for problem (35).

**Theorem 6** *Within the framework established in Hypothesis 1, given elements  $\zeta_{\mathbf{g}} \in W^{1,2,2}(0, T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3)$  and  $\theta^* \in L^2(0, T; L^2(\Omega))$ , there exists an element  $\mathbf{v} \in \mathbf{W}_2$  such that  $\mathbf{z} = \mathbf{v} - \zeta_{\mathbf{g}} \in \widetilde{\mathbf{W}}_2$  is the unique solution of problem (34) in the following sense:*

$$\begin{aligned} & \int_{\Omega} \frac{\partial \mathbf{z}}{\partial t} \cdot \boldsymbol{\eta} d\mathbf{x} + \int_{\Omega} \nabla(\zeta_{\mathbf{g}} + \mathbf{z}) \mathbf{z} \cdot \boldsymbol{\eta} d\mathbf{x} \\ & \quad + \int_{\Omega} \nabla \mathbf{z} \zeta_{\mathbf{g}} \cdot \boldsymbol{\eta} d\mathbf{x} + 2\nu \int_{\Omega} \epsilon(\mathbf{z}) : \epsilon(\boldsymbol{\eta}) d\mathbf{x} \\ & + 2\nu_{tur} \int_{\Omega} [\epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\zeta_{\mathbf{g}} + \mathbf{z})]^{1/2} \epsilon(\zeta_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\eta}) d\mathbf{x} \\ & = \int_{\Omega} \mathbf{H}_{\mathbf{g}} \cdot \boldsymbol{\eta} d\mathbf{x}, \quad \text{a.e. } t \in ]0, T[, \quad \forall \boldsymbol{\eta} \in \widetilde{\mathbf{X}}_2, \end{aligned} \quad (37)$$

with  $\mathbf{z}(0) = \mathbf{v}^0$ , a.e.  $\mathbf{x} \in \Omega$ , where:

$$\mathbf{H}_{\mathbf{g}} = \alpha_0(\theta^* - \theta^0) \mathbf{a}_g - \frac{\partial \zeta_{\mathbf{g}}}{\partial t} - \nabla \zeta_{\mathbf{g}} \zeta_{\mathbf{g}} \in L^2(0, T; [L^2(\Omega)]^3). \quad (38)$$

Besides, we have the following estimates:

$$\begin{aligned} & \|\mathbf{z}\|_{L^\infty(0,T;[L^2(\Omega)]^3)} + \|\mathbf{z}\|_{L^2(0,T;[W^{1,2}(\Omega)]^3)} + \|\mathbf{z}\|_{L^3(0,T;[W^{1,3}(\Omega)]^3)} \\ & \leq C_2(\mathbf{v}^0, \theta^0, \mathbf{g}) \left[ 1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))} \right] \end{aligned} \quad (39)$$

$$\begin{aligned} & \left\| \frac{\partial \mathbf{z}}{\partial t} \right\|_{L^2(0,T;[L^2(\Omega)]^3)} + \|\epsilon(\mathbf{z})\|_{L^\infty(0,T;[L^3(\Omega)]^{3 \times 3})} + \|\epsilon(\mathbf{z})\|_{L^\infty(0,T;[L^2(\Omega)]^{3 \times 3})} \\ & \leq C_3(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp\left(1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))}^2\right) \left[ 1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))} \right]. \end{aligned} \quad (40)$$

where  $C_2$  and  $C_3$  are positive constants that depend continuously with respect to the space-time configuration of our computational domain,  $\mathbf{v}^0$ ,  $\theta^0$  and  $\mathbf{g}$ .

**PROOF.** It is a direct consequence of Theorem 8 in [9]. In this theorem the authors prove the wellposedness (existence, uniqueness and regularity of solution) for a modified Navier-Stokes system with non-homogeneous Dirichlet boundary conditions, by building a continuous extension of the Dirichlet condition and a Galerking approximation of the corresponding problem with homogeneous boundary conditions.  $\blacksquare$

**Theorem 7** *Within the framework established in Hypothesis 1, given elements  $\zeta_{\mathbf{h}^*} \in W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$  and  $\mathbf{v} \in L^{10/3}(0, T; [L_\sigma^3(\Omega)]^3)$ , there exists an element  $\theta \in W_1$  such that  $\xi = \theta - \zeta_{\mathbf{h}^*} \in \widetilde{W}_1$  is the unique solution of problem (35) in the following sense:*

$$\begin{aligned} & \int_\Omega \frac{\partial \xi}{\partial t} \eta \, d\mathbf{x} + \int_\Omega \mathbf{v} \cdot \nabla \xi \eta \, d\mathbf{x} + K \int_\Omega \nabla \xi \cdot \nabla \eta \, d\mathbf{x} + b_1^N \int_{\Gamma_N} \xi \eta \, d\gamma \\ & + b_1^S \int_{\Gamma_S} \xi \eta \, d\gamma + b_2^S \int_{\Gamma_S} |\xi + \zeta_{\mathbf{h}^*}|^3 (\xi + \zeta_{\mathbf{h}^*}) \eta \, d\gamma = \int_\Omega H_{\mathbf{h}^*} \eta \, d\mathbf{x} \\ & + \int_{\Gamma_C} g_{\mathbf{h}^*}^C \eta \, d\gamma + b_1^N \int_{\Gamma_N} g_{\mathbf{h}^*}^N \eta \, d\gamma + b_1^S \int_{\Gamma_S} g_{\mathbf{h}^*}^S \eta \, d\gamma \\ & + b_2^S \int_{\Gamma_S} T_r^A \eta \, d\gamma, \quad a.e. \, t \in ]0, T[, \quad \forall \eta \in \widetilde{X}_1, \end{aligned} \quad (41)$$

with  $\xi(0) = \theta^0 - \zeta_{\mathbf{h}^*}(0)$ , a.e.  $\mathbf{x} \in \Omega$ .

Besides, we have the following estimates:

$$\begin{aligned} & \|\xi\|_{L^\infty(0,T;L^2(\Omega))} + \|\xi\|_{L^2(0,T;\widetilde{X}_1)} + \|\xi\|_{L^5(0,T;L^5(\Gamma_S))} \\ & \leq C_4(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}} \right], \end{aligned} \quad (42)$$

$$\begin{aligned} & \left\| \frac{d\xi}{dt} \right\|_{L^{5/4}(0,T;\widetilde{X}_1)} \leq C_5(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \right. \\ & \left. \|\mathbf{v}\|_{L^{10/3}(0,T;[L^3(\Omega)]^3)}^2 + \|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}}^2 \right], \end{aligned} \quad (43)$$

where  $C_4$  and  $C_5$  are positive constants that depend continuously with respect to the space-time configuration of our computational domain,  $\theta^0$ ,  $\theta_N$ ,  $\theta_S$  and  $T_r$ .

**PROOF.** It is a straightforward consequence of Theorem 14 that we will prove in Appendix A. In our case, we have to choose there  $\Gamma_R = \Gamma_S$ ,  $\Gamma_L = \Gamma_T$  and  $\Gamma_A = \Gamma_C \cup \Gamma_N$ .  $\blacksquare$

**Lemma 8** *The operator  $M$  defined in (33) is well defined and compact.*

**PROOF.** Thanks to above Theorems 6 and 7, it is straightforward that the operator  $M$  is well defined. Let us check now its compactness.

So, given a bounded sequence  $\{(\theta_n^*, \mathbf{h}_n^*)\}_{n \in \mathbb{N}} \subset L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}}$ , we have, using estimates (39) and (40), that the corresponding sequence  $\{\mathbf{z}_n\}_{n \in \mathbb{N}} \subset \widetilde{\mathbf{W}}_2$  of solutions for the problem (37) is bounded in  $\widetilde{\mathbf{W}}_2$ . Then, we have, taking subsequences if necessary, that:

- $\theta_n^* \rightharpoonup \theta$  in  $L^2(0, T; L^2(\Omega))$ ,
- $\mathbf{h}_n^* \rightharpoonup \mathbf{h}$  in  $[L^2(0, T)]^{N_{CT}}$ ,
- $\mathbf{z}_n \rightarrow \mathbf{z}$  strongly in  $L^p(0, T; [L^q(\Omega)]^3)$ , for all  $1 < p < \infty$ , and  $2 \leq q < \infty$ ,
- $\mathbf{z}_n \rightharpoonup \mathbf{z}$  weakly in  $L^3(0, T; \mathbf{X})$ ,
- $\frac{d\mathbf{z}_n}{dt} \rightharpoonup \frac{d\mathbf{z}}{dt}$  weakly in  $L^2(0, T; [L^2(\Omega)]^3)$ ,
- $\nabla \mathbf{z}_n \rightharpoonup^* \nabla \mathbf{z}$  weakly-\* in  $L^\infty(0, T; [L^3(\Omega)]^3)$ ,
- $\beta(\epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}_n)) \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}_n) \rightharpoonup \beta(\epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})) \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})$  weakly in  $L^{3/2}(0, T; \widetilde{\mathbf{X}}')$ ,

where  $\mathbf{z} \in \widetilde{\mathbf{W}}_2$  is the solution of (37) associated to  $\theta$ . The last convergence is a consequence of the monotony of operator (see [9] for more details)

$$\begin{aligned} A : \widetilde{\mathbf{X}}_2 &\rightarrow \widetilde{\mathbf{X}}'_2 \\ \mathbf{z} &\rightarrow A(\mathbf{z}), \end{aligned} \tag{44}$$

where, for any  $\boldsymbol{\xi} \in \widetilde{\mathbf{X}}_2$ ,

$$\langle A(\mathbf{z}), \boldsymbol{\xi} \rangle = \int_{\Omega} \beta(\epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})) \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\xi}) \, d\mathbf{x}. \tag{45}$$

Now, using the results proved in Lemma 3, we have that the corresponding sequence  $\{\zeta_{\mathbf{h}_n^*}\}$  converges to  $\boldsymbol{\zeta}_{\mathbf{h}}$  strongly in  $W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$ , and, consequently,

- $H_{\mathbf{h}_n^*} \rightarrow H_{\mathbf{h}}$  in  $L^2(0, T; L^2(\Omega))$ ,
- $g_{\mathbf{h}_n^*}^C \rightarrow g_{\mathbf{h}}^C$  in  $L^2(0, T; L^2(\Gamma_C))$ ,



- $g_{\mathbf{h}_n^*}^N \rightarrow g_{\mathbf{h}}^N$  in  $L^2(0, T; L^2(\Gamma_N))$ ,
- $g_{\mathbf{h}_n^*}^S \rightarrow g_{\mathbf{h}}^S$  in  $L^2(0, T; L^2(\Gamma_S))$ .

Finally, thanks to estimates (42) and (43), the corresponding sequence  $\{\xi_n\}_{n \in \mathbb{N}} \subset \widetilde{W}_1$  is bounded. Thus, taking subsequences if necessary, we have thanks to a straightforward adaptation of Lemma 19 in Appendix A, that

- $\xi_n \rightharpoonup \xi$  in  $L^2(0, T; \widetilde{X})$ ,
- $\xi_n \rightharpoonup^* \xi$  in  $L^\infty(0, T; L^2(\Omega))$ ,
- $\xi_n \rightarrow \xi$  in  $L^{10/3-\epsilon}(0, T; L^{10/3-\epsilon}(\Omega))$ ,
- $\xi_n \rightarrow \xi$  in  $L^2(0, T; L^2(\Gamma_C))$ ,
- $\xi_n \rightarrow \xi$  in  $L^5(0, T; L^5(\Gamma_R))$ .

Using the same techniques that we present in Appendix A for the demonstration of Theorem 14, we can pass to the limit in the variational formulation of  $\xi_n$  and prove that  $\xi \in \widetilde{W}_2$  is the solution of (7) associated to  $\theta$  and  $\mathbf{h}$ . Thus, we have that

- $\theta_n = \xi_n + \zeta_{\mathbf{h}_n^*} \rightarrow \theta = \xi + \zeta_{\mathbf{h}}$  in  $L^2(0, T; L^2(\Omega))$ ,
- $\theta_n = \xi_n + \zeta_{\mathbf{h}_n^*} \rightarrow \theta = \xi + \zeta_{\mathbf{h}}$  in  $L^2(0, T; L^2(\partial\Omega))$ ,

and, consequently,  $M(\theta_n^*, \mathbf{h}_n^*) \rightarrow M(\theta, \mathbf{h})$  in  $L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}}$ , which concludes the proof.  $\blacksquare$

**Theorem 9** *Given positive constants  $\widehat{C}^1$  and  $\widehat{C}^2$ , there exist coefficients and data small enough such that the operator  $M$  defined in (33) has a fixed point in the space  $\{(\theta, \mathbf{h}) \in L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}} : \|\theta\|_{L^2(0, T; L^2(\Omega))} \leq \widehat{C}^1, \|\mathbf{h}\|_{[L^2(0, T)]^{N_{CT}}} \leq \widehat{C}^2\}$ . Moreover, the corresponding  $(\theta, \mathbf{v}) \in W_1 \times \mathbf{W}_2$  is a solution for the system (1) and (6) in the sense of Definition 5.*

**PROOF.** The existence is a direct consequence of the Schauder fixed point Theorem. Given an element  $(\theta^*, \mathbf{h}^*) \in L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}}$ , we have, thanks to (20), (39), (40), (42) and (43), the following estimates for  $(\theta, \mathbf{h}) = M(\theta^*, \mathbf{h}^*)$ :

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{W}_2} &\leq C_6(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \|\theta^*\|_{L^2(0, T; L^2(\Omega))}^2) \left[1 + \|\theta^*\|_{L^2(0, T; L^2(\Omega))}\right], \\
\|\theta\|_{W_1} &\leq C_7(\theta^0, \theta_N, \theta_S, T_r) \left[1 + \|\mathbf{v}\|_{\mathbf{W}_2}^2 + \|\mathbf{h}^*\|_{[L^2(0, T)]^{N_{CT}}}^2\right], \\
\|\mathbf{h}\|_{[L^2(0, T)]^{N_{CT}}} &\leq C_8(\theta^0, \theta_N, \theta_S, T_r) \left[1 + \|\mathbf{v}\|_{\mathbf{W}_2}^2 + \|\mathbf{h}^*\|_{[L^2(0, T)]^{N_{CT}}}^2\right],
\end{aligned} \tag{46}$$

where  $C_6, C_7, C_8$  are positive constants that depend continuously on the coefficients and data. If we take the first inequality to the second and third ones,

we obtain that:

$$\begin{aligned}\|\theta\|_{W_1} &\leq C_7(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}}^2 \right. \\ &\quad \left. + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))}^2) [1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))}^2] \right], \\ \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}} &\leq C_8(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}}^2 \right. \\ &\quad \left. + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))}^2) [1 + \|\theta^*\|_{L^2(0,T;L^2(\Omega))}^2] \right].\end{aligned}$$

So, if we suppose that  $\|\theta^*\|_{L^2(0,T;L^2(\Omega))} \leq \widehat{C}_1$  and  $\|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}} \leq \widehat{C}_2$ , we have that

$$\begin{aligned}\|\theta\|_{W_1} &\leq C_7(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \widehat{C}_2^2 + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \widehat{C}_1^2) [1 + \widehat{C}_1^2] \right], \\ \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}} &\leq C_8(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \widehat{C}_2^2 + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \widehat{C}_1^2) [1 + \widehat{C}_1^2] \right].\end{aligned}$$

Thus, we are led to solve the following inequality:

$$\begin{aligned}C_7(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \widehat{C}_2^2 + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \widehat{C}_1^2) [1 + \widehat{C}_1^2] \right] &\leq \widehat{C}_1, \\ C_8(\theta^0, \theta_N, \theta_S, T_r) \left[ 1 + \widehat{C}_2^2 + C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \exp(1 + \widehat{C}_1^2) [1 + \widehat{C}_1^2] \right] &\leq \widehat{C}_2.\end{aligned}\tag{47}$$

However, it is obvious that, given  $\widehat{C}_1$  and  $\widehat{C}_2$ , we can consider small enough data  $\mathbf{v}^0, \mathbf{g}, \theta^0, \theta_N, \theta_S$  and  $T_r$ , such that

$$C_9(\mathbf{v}^0, \theta^0, \mathbf{g}) \leq \frac{1}{\exp(1 + \widehat{C}_1^2) [1 + \widehat{C}_1^2]},\tag{48}$$

$$C_7(\theta^0, \theta_N, \theta_S, T_r) \leq \frac{\widehat{C}_1}{2 + \widehat{C}_2},\tag{49}$$

$$C_8(\theta^0, \theta_N, \theta_S, T_r) \leq \frac{\widehat{C}_2}{2 + \widehat{C}_2}.\tag{50}$$

Then, choosing suitable coefficients and data that verify (48)-(50), we have that  $M$  maps elements of the set  $\{(\theta^*, \mathbf{h}^*) \in L^2(0, T; L^2(\Omega)) \times [L^2(0, T)]^{N_{CT}} : \|\theta^*\|_{L^2(0,T;L^2(\Omega))} \leq \widehat{C}_1, \|\mathbf{h}^*\|_{[L^2(0,T)]^{N_{CT}}} \leq \widehat{C}_2\}$  into itself. Thus, thanks to Schauder fixed point Theorem, there exists a fixed point  $(\theta, \mathbf{h})$  of operator  $M$ , such that the corresponding  $(\theta, \mathbf{v})$  is a solution of the coupled system (1) and (6) in the sense of Definition 5.  $\blacksquare$

## 5 Numerical resolution

Once proved in above section that the coupled system (1) and (6) admits a solution, we will introduce here a full numerical algorithm in order to compute it, and show several computational test for a realistic example.

We must recall here that our main aim is related to understanding which is the best strategy for reducing the water temperature in the upper layers. In order to achieve this objective, and for the sake of completeness, we will consider an algorithm able to deal with more general states than those we have presented in previous mathematical analysis of the problem. In particular, in the numerical resolution proposed here we will also take into account the possibility that  $g^k(t)$  takes non-positive values. To be exact, if  $g^k(t) > 0$  we will say that the pump  $k$  is turbinating (water enters by the collector  $C^k$  and is turbinated by the corresponding pipeline to the injector  $T^k$ ), and if  $g^k(t) < 0$  we will say that the pump  $k$  is pumping (water enters by injector  $T^k$  and is pumped to the collector  $C^k$ ). As it is evident, the situation  $g^k(t) = 0$  corresponds to the case in which the pump  $k$  is off. In addition, we will also suppose that the parameter  $\epsilon$  used in the definition (2) tends to zero, that is, the mean temperature in the injectors is equal to the mean temperature in the collectors. **It is essential emphasizing here that it is also possible to perform a similar mathematical analysis for the general case (with the obvious embarrassing notations), under the only assumption of the existence of a partition of the time interval verifying that the groups do not change their state within any element of the partition (as can be seen in following section devoted to the numerical examples).**

### 5.1 Space-time discretization

For the discretization of the problem, let us consider a regular partition  $0 = t_0 < t_1 < \dots < t_N = T$  of the time interval  $[0, T]$  such that  $t_{n+1} - t_n = \Delta t = \frac{1}{\alpha}$ ,  $\forall n = 0, \dots, N - 1$ , and a family of meshes  $\tau_h$  for the domain  $\Omega$  with characteristic size  $h$ . Associated to this family of meshes, we also consider three compatible finite element spaces  $Z_h$ ,  $\mathbf{W}_h$  and  $M_h$  corresponding, respectively, to the water temperature, velocity and the pressure of water. From the computational viewpoint, for the generation of the mesh associated to the domain and for the numerical resolution of the system, we propose the use of FreeFem++ [14]. Finally, we have employed an Uzawa algorithm [12] for computing the solution of the Stokes problems that appears after the discretization, and a fixed point algorithm for solving the nonlinearities.

So, we consider the following space-time discretization for system (1) and (6):

- (1) *Dirichlet condition for the water velocity:* **We consider the following ap-**

proximation for a function  $\mathbf{g} = (g^1, \dots, g^{N_{CT}}) \in [H^1(0, T)]^{N_{CT}}$ ,

$$g^k(x) = \sum_{n=0}^N g^{k,n} e^n(x), \quad (51)$$

where, for all  $n = 0, \dots, N$ ,  $e^n \in \mathcal{C}([0, T])$  is such that  $e^n|_{[t^{n-1}, t_n]} \in \mathcal{P}_1([t^{n-1}, t_n])$ ,  $n = 1, \dots, N$  and  $e^n(t^k) = \delta_{k,n}$ ,  $k, n = 0, \dots, N$ . It is well known that the linear closure of the functions of the basis  $\{e^n\}_{n=0}^N$  is a vector subspace of  $H^1(0, T)$ , so if we suppose that  $g^{k,0} = 0$ ,  $k = 1, \dots, N_{CT}$ , we can consider the following coordinate vector in the basis of the corresponding subspace of  $[H^1(0, T)]^{N_{CT}}$ :

$$\mathbf{g} = \underbrace{(g^{1,1}, g^{2,1}, \dots, g^{N_{CT},1})}_{\mathbf{g}^1}, \dots, \underbrace{(g^{1,N}, g^{2,N}, \dots, g^{N_{CT},N})}_{\mathbf{g}^N} \in \mathbb{R}^{N \times N_{CT}}, \quad (52)$$

with  $g^{k,n} \in [-M, M]$ ,  $k = 1, \dots, N_{CT}$ ,  $n = 1, \dots, N$ , with  $M > 0$  a technical bound related to mechanical characteristics of pumps.

- (2) *Coupling of temperature in collectors and injectors*: We denote by  $\theta^n \in Z_h$  the water temperature at time step  $n = 0, \dots, N$ . Then, we can consider the following approximation in the case of  $g^{k,n} > 0$ ,  $k = 1, \dots, N_{CT}$ ,  $n = 1, \dots, N$ , for (3) by functions  $\gamma_\theta^k$ ,  $k = 1, \dots, N_{CT}$ , :

$$\begin{aligned} \gamma_\theta^k(t) &= \frac{1}{\mu(C^k)} \left[ \chi_{(-\infty, t_0)} \int_{C^k} \theta^0 d\gamma \right. \\ &\left. + \sum_{n=1}^N \chi_{[t_{n-1}, t_n]} \int_{C^k} \theta^{n-1} d\gamma + \chi_{[t_N, \infty)} \int_{C^k} \theta^N d\gamma \right] \end{aligned}$$

Moreover, if we assume the value  $\epsilon = \frac{\Delta t}{2}$  in the definition (4) of function  $\rho_\epsilon$  we have that the support of  $\rho_{\Delta t/2}(t^n - \frac{\Delta t}{2} - s)$  is contained in  $(t^n - \Delta t, t^n) = (t^{n-1}, t^n)$ , for all  $n = 1, \dots, N$ , and then:

$$\begin{aligned} \phi_\theta^n(\mathbf{x}) &= \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \int_{-T}^T \rho_{\Delta t/2}(t^n - \frac{\Delta t}{2} - s) \gamma_\theta^k(s) ds \\ &= \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \int_{t_{n-1}}^{t_n} \rho_{\Delta t/2}(t^n - \frac{\Delta t}{2} - s) \left[ \frac{1}{\mu(C^k)} \int_{C^k} \theta^{n-1} d\gamma \right] ds \\ &= \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \frac{1}{\mu(C^k)} \int_{C^k} \theta^{n-1} d\gamma. \end{aligned}$$

Finally, we approximate each element  $\varphi^k$  by the indicator function of the injector  $T^k$ ,  $k = 1, \dots, N_{CT}$ , and each element  $\tilde{\varphi}^k$  by the indicator function of the collector  $C^k$ ,  $k = 1, \dots, N_{CT}$ . Thus, the temperature in each injector at time step  $t_n$  is the mean temperature in the corresponding collector at time step  $t_{n-1}$ . The previous approximation is still valid in the general case with obvious modifications. For instance, if we suppose that

$g^{k,n} < 0$ ,  $k = 1, \dots, N_{CT}$ ,  $n = 1, \dots, N$ , then, we consider the Dirichlet condition in the collectors:

$$\phi_\theta^n(\mathbf{x}) = \sum_{k=1}^{N_{CT}} \tilde{\varphi}^k(\mathbf{x}) \frac{1}{\mu(T^k)} \int_{T^k} \theta^{n-1} d\gamma.$$

(3) *Water temperature:* Given  $\theta^0 \in Z_h$ ,  $\theta^1 \in Z_h$  is the solution of:

$$\begin{aligned} \alpha \int_{\Omega} \theta^1 \eta d\mathbf{x} + K \int_{\Omega} \nabla \theta^1 \cdot \nabla \eta d\mathbf{x} + b_1^N \int_{\Gamma_N} \theta^1 \eta d\gamma + b_1^S \int_{\Gamma_S} \theta^1 \eta d\gamma \\ + b_2^S \int_{\Gamma_S} |\theta^1|^3 \theta^1 \eta d\gamma = \alpha \int_{\Omega} (\theta^0 \circ X^0) \eta d\mathbf{x} + b_1^N \int_{\Gamma_N} \theta_1^1 \eta d\gamma \\ + b_1^S \int_{\Gamma_S} \theta_S^1 \eta d\gamma + b_2^S \int_{\Gamma_S} (T_r^1)^4 \eta d\gamma, \quad \forall \eta \in Z_h, \end{aligned} \quad (53)$$

where the discrete characteristic  $X^0(\mathbf{x}) = \mathbf{x} - \Delta t \mathbf{v}^0(\mathbf{x})$ .

Then, for each  $n = 1, \dots, N$ ,  $\theta^{n+1} \in Z_h$ , with

$$\frac{\partial \theta^{n+1}}{\partial \mathbf{n}} \Big|_{C^k} = 0, \quad \theta_{|T^k}^{n+1} = \frac{1}{\mu(C^k)} \int_{C^k} \theta^{n+1} d\gamma \quad \text{if } g^{k,n} > 0, \quad (54)$$

$$\theta_{|C^k}^{n+1} = \frac{1}{\mu(T^k)} \int_{T^k} \theta^{n+1} d\gamma, \quad \frac{\partial \theta^{n+1}}{\partial \mathbf{n}} \Big|_{T^k} = 0 \quad \text{if } g^{k,n} < 0, \quad (55)$$

$$\frac{\partial \theta^{n+1}}{\partial \mathbf{n}} \Big|_{C^k} = 0, \quad \frac{\partial \theta^{n+1}}{\partial \mathbf{n}} \Big|_{T^k} = 0 \quad \text{if } g^{k,n} = 0, \quad (56)$$

for all  $k = 1, \dots, N_{CT}$ , is the solution of:

$$\begin{aligned} \alpha \int_{\Omega} \theta^{n+1} \eta d\mathbf{x} + K \int_{\Omega} \nabla \theta^{n+1} \cdot \nabla \eta d\mathbf{x} + b_1^N \int_{\Gamma_N} \theta^{n+1} \eta d\gamma \\ + b_1^S \int_{\Gamma_S} \theta^{n+1} \eta d\gamma + b_2^S \int_{\Gamma_S} |\theta^{n+1}|^3 \theta^{n+1} \eta d\gamma = \alpha \int_{\Omega} (\theta^n \circ X^n) \eta d\mathbf{x} \\ + b_1^N \int_{\Gamma_N} \theta_N^{n+1} \eta d\gamma + b_1^S \int_{\Gamma_S} \theta_S^{n+1} \eta d\gamma + b_2^S \int_{\Gamma_S} (T_r^{n+1})^4 \eta d\gamma, \quad \forall \eta \in Z_h, \end{aligned} \quad (57)$$

where the discrete characteristic  $X^n(\mathbf{x}) = \mathbf{x} - \Delta t \mathbf{v}^n(\mathbf{x})$ , for  $n = 1, \dots, N$ , and where the functional space  $Z_h^n$  is given by:

$$\left\{ \eta \in Z_h : \eta \Big|_{\left( \cup_{k=1}^{N_{CT}} \frac{1}{2}(1 - \text{sign}(g^{k,n})) C^k \right) \Big|_{\text{sign}(g^{k,n})} \cup \left( \cup_{k=1}^{N_{CT}} \frac{1}{2}(1 + \text{sign}(g^{k,n})) T^k \right) \Big|_{\text{sign}(g^{k,n})}} = 0 \right\}, \quad (58)$$

with  $\text{sign}(y)$  denoting the sign function:

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y < 0, \\ 0 & \text{if } y = 0. \end{cases}$$

For instance, in the case of  $g^{k,n} > 0$  for all  $k = 1, \dots, N_{CT}$ ,  $n = 1, \dots, N$ , then  $(\cup_{k=1}^{N_{CT}} \frac{1}{2}(1 - \text{sign}(g^{k,n}))C^k) |\text{sign}(g^{k,n})| \cup (\cup_{k=1}^{N_{CT}} \frac{1}{2}(1 + \text{sign}(g^{k,n}))T^k) = \cup_{k=1}^{N_{CT}} T^k$ , so

$$Z_h^n = \{\eta \in Z_h : \eta|_{\cup_{k=1}^{N_{CT}} T^k} = 0\}, \quad n = 1, \dots, N. \quad (59)$$

That is, we are considering the Dirichlet condition on the injectors. In the opposite case,  $g^{k,n} < 0$  for all  $k = 1, \dots, N_{CT}$ ,  $n = 1, \dots, N$ , we have that

$$Z_h^n = \{\eta \in Z_h : \eta|_{\cup_{k=1}^{N_{CT}} C^k} = 0\}, \quad n = 1, \dots, N, \quad (60)$$

and we are considering the Dirichlet condition on the collectors. In the general case, we can have alternating Dirichlet conditions on the collectors and injectors, so the definition for the space  $Z_h^n$ ,  $n = 1, \dots, N$ , given in (58) covers all the possibilities.

- (4) *Water velocity and pressure:* Given  $\mathbf{v}^0 \in \mathbf{V}_h$ , for each  $n = 0, 1, \dots, N-1$ , the pair velocity/pressure  $(\mathbf{v}^{n+1}, p^{n+1}) \in \mathbf{W}_h \times M_h$ , with:

$$\mathbf{v}|_{T^k}^{n+1} = -\frac{g^{k,n+1}}{\mu(T^k)} \mathbf{n}, \quad \mathbf{v}|_{C^k}^{n+1} = \frac{g^{k,n+1}}{\mu(C^k)} \mathbf{n}, \quad \forall k = 1, \dots, N_{CT}, \quad (61)$$

is the solution of:

$$\begin{aligned} & \alpha \int_{\Omega} \mathbf{v}^{n+1} \cdot \mathbf{z} \, d\mathbf{x} + 2\nu \int_{\Omega} \epsilon(\mathbf{v}^{n+1}) : \epsilon(\mathbf{z}) \, d\mathbf{x} \\ & + 2\nu_{tur} \int_{\Omega} [\epsilon(\mathbf{v}^{n+1}) : \epsilon(\mathbf{v}^{n+1})]^{1/2} \epsilon(\mathbf{v}^{n+1}) : \epsilon(\mathbf{z}) \, d\mathbf{x} \\ & - \int_{\Omega} p^{n+1} \nabla \cdot \mathbf{z} \, d\mathbf{x} - \int_{\Omega} \nabla \cdot \mathbf{v}^{n+1} q \, d\mathbf{x} = \alpha \int_{\Omega} (\mathbf{v}^n \circ X^n) \cdot \mathbf{z} \, d\mathbf{x} \\ & + \int_{\Omega} \alpha_0 (\theta^{n+1} - \theta^0) \mathbf{a}_g \cdot \mathbf{z} \, d\mathbf{x}, \quad \forall \mathbf{z} \in \mathbf{V}_h, \quad \forall q \in M_h, \end{aligned} \quad (62)$$

where the functional space  $\mathbf{V}_h = \{\mathbf{z} \in \mathbf{W}_h : \mathbf{z}|_{\cup_{k=1}^{N_{CT}} (T^k \cup C^k)} = \mathbf{0}\}$ .

**Remark 10** *It is worthwhile noting here that in above scheme we have to compute one additional time step for the water temperature. This shift is motivated by the dependence diagram shown in Fig. 2. We observe that, due to the time discretization proposed here, the Dirichlet boundary condition for the hydrodynamic model begins to have influence from the second step of time for the water temperature. In the first time step  $n = 1$ , water circulation does not affect the water temperature, so for this first time step we consider  $\Gamma_N = \partial\Omega \setminus \Gamma_S$ . However, for  $n \geq 2$ , we impose at each of the  $N_{CT}$  collector/injector pairs one of the boundary conditions (54), (55) or (56), depending on the sign of the Dirichlet condition  $g^{k,n-1}$  in previous time step.*

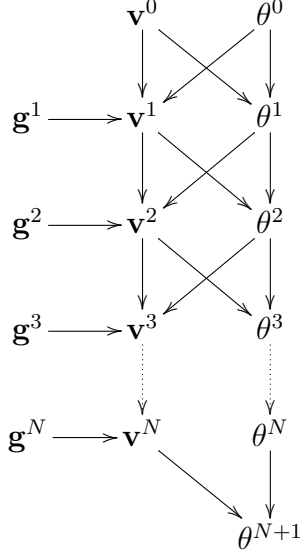


Fig. 2. Dependence scheme for the discretized variables.

## 5.2 Numerical results

This final subsection is devoted to present some numerical results that we have obtained using realistic coefficients and data. Nevertheless, for the sake of clarity and comprehensibility, we will show results for a simplified 2D domain. For this purpose, we have considered a rectangular domain  $\Omega = [0, 16] \times [0, 19]$  (measured in meters), corresponding a reservoir, in which we have distributed  $N_{CT} = 4$  collector/injector pairs with a symmetrical configuration similar to that shown in Fig. 1. For the time discretization we have chosen a time step of  $\Delta t = 1800$  seconds with  $N = 96$  time steps (which represents a time period of 2 days), and for the space discretization we have used a regular mesh formed by triangles of characteristic size  $h = 0.5$  meters (corresponding to 1599 vertices). Finally, the finite element spaces employed for space discretizations have been the Taylor-Hood element  $\mathcal{P}_2/\mathcal{P}_1$  for the hydrodynamic model, and the Lagrange  $\mathcal{P}_2$  element for the water temperature. In Fig. 3 we can observe the evolution of **the radiation temperature**:

$$T_r = \left( \frac{1}{\varepsilon \sigma_B} [(1 - a)R_{sw,net} + R_{lw,down}] \right)^{1/4} \quad (63)$$

along the whole period of 2 days ( $17.28 \cdot 10^4$  seconds), considering  $a = 0.1$ ,  $R_{sw,net} = 1000 \text{ W m}^{-2}$ ,  $R_{lw,down} = 350 \text{ W m}^{-2}$  (typical values in mediterranean countries during the summer) and multiplied by a sinusoidal function in order to simulate the effects of day and night. The parameters used for the numerical resolution of the coupled system can be seen in Table 1.

In order to analyze the influence of water artificial circulation in the thermal behavior of top 1.5 meters from water upper layer we have solved the problem

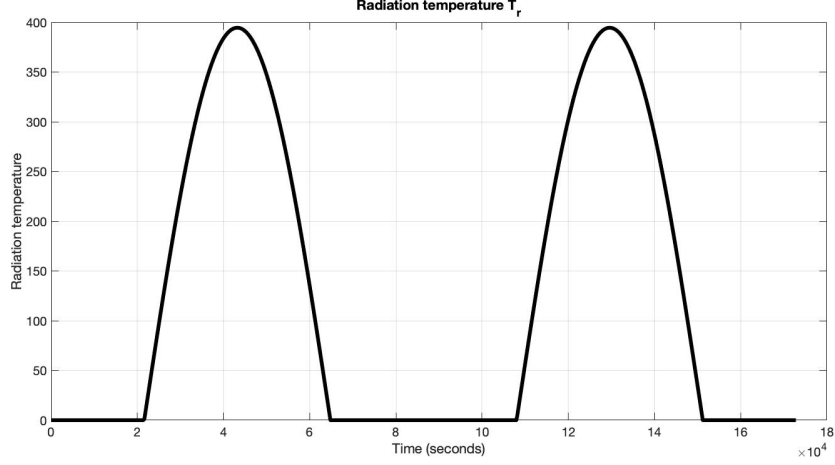


Fig. 3. Profile of the radiation temperature  $T_r$  for the whole time interval of  $T = 17.28 \cdot 10^4$  seconds (2 days).

Parameters	Values	Units
$\nu$	$1.3 \cdot 10^{-3}$	$\text{m}^2 \text{s}^{-1}$
$\nu_{tur}$	$5.0 \cdot 10^{-2}$	$\text{m}^2$
$K$	$1.4 \cdot 10^{-5}$	$\text{m}^2 \text{s}^{-1}$
$h^N, h^S$	$3.0 \cdot 10^2$	$\text{W m}^{-2} \text{K}^{-1}$
$\rho$	$9.9 \cdot 10^2$	$\text{g m}^{-3}$
$c_p$	4.2	$\text{W s g}^{-1} \text{K}^{-1}$
$\theta^0$	283.0	K
$\theta_S$	286.0	K
$\theta_N$	283.0	K
$\alpha_0$	$8.7 \cdot 10^{-7}$	$\text{K}^{-1}$

Table 1  
Physical parameters for the numerical example.

in five different scenarios:

- (1) NNNN: In this configuration we take  $g^{k,n} = 0$ , for all  $k = 1, \dots, N_{CT}$  and  $n = 1, \dots, N$  (reference configuration with all the groups off).
- (2) TTTT: In this configuration we take  $g^{k,n} = 2.0 \cdot 10^{-3}$ , for all  $k = 1, \dots, N_{CT}$  and  $n = 1, \dots, N$  (all the groups are turbinating).
- (3) PPPP: In this configuration we take  $g^{k,n} = -2.0 \cdot 10^{-3}$ , for all  $k = 1, \dots, N_{CT}$  and  $n = 1, \dots, N$  (all the groups are pumping).
- (4) TPTP: In this configuration we take  $g^{1,n} = g^{3,n} = 2.0 \cdot 10^{-3}$  and  $g^{2,n} = g^{4,n} = -2.0 \cdot 10^{-3}$ , for all  $k = 1, \dots, N_{CT}$  and  $n = 1, \dots, N$  (groups 1 and 3 are turbinating, and groups 2 and 4 are pumping).



- (5) PTPT: In this configuration we take  $g^{1,n} = g^{3,n} = -2.0 \cdot 10^{-3}$  and  $g^{2,n} = g^{4,n} = 2.0 \cdot 10^{-3}$ , for all  $k = 1, \dots, N_{CT}$  and the groups 2 and 4 are turbinating).

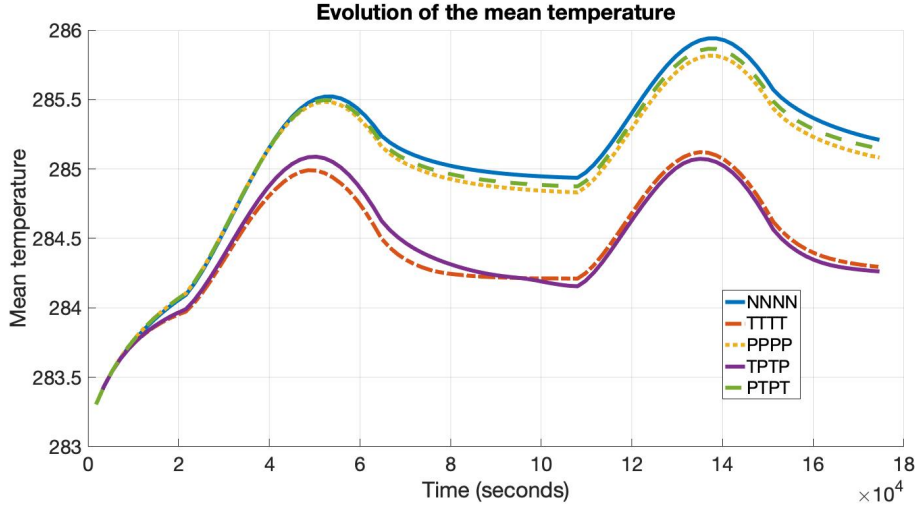


Fig. 4. Evolution of the mean temperature in the top 1.5 meters upper layer of the water domain for the five scenarios under study.

In Fig. 4 we present the evolution of the mean temperature in the top 1.5 meters upper layer along the whole time interval corresponding to two days. We can clearly distinguish here that the best configurations correspond, in a very evident manner, to the second scenario (TTTT) and to the fourth one (TPTP). Moreover, we can notice how third and fifth scenarios (PPPP and PTPT, respectively) do not improve in a significant way the reference configuration (NNNN).

Finally, we show in Fig. 5 water temperatures and velocities at last time step for NNNN and TTTT configurations and, in Fig. 6, the behavior of water at same time step for configurations TPTP and PTPT. (In all of the cases, velocities have been multiplied by an amplifying factor to make their graphic representations more perceptible). As we can easily notice, the best strategies correspond to evacuating the excess of temperature in the upper layers to the bottom layers instead of refrigerating the upper layers with cold water from the bottom ones.

Although we only present here one realistic example of application of our approach to understand the behaviour of water velocity and temperature in the upper section of the domain, we have developed many other numerical experiences for different choices of parameters and data (that will not be presented here for the sake of conciseness). However, from these computational tests we can derive two important consequences: For the modified Navier-Stokes equations, the second member -corresponding to the thermic term- shows less influence in the numerical results than the Smagorinsky turbulence

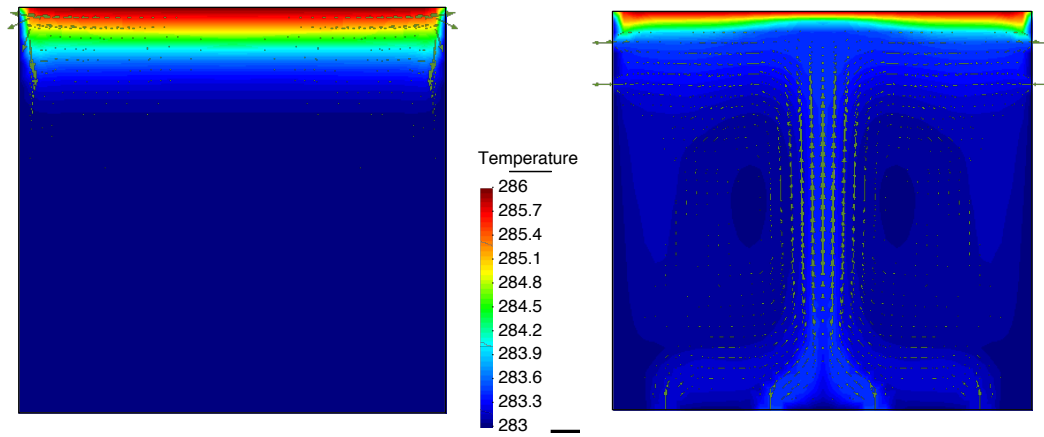


Fig. 5. Water temperatures and velocities (multiplied by factors 1500/500) in the last time step for the worst configuration NNNN (left) and the best configuration TTTT (right).

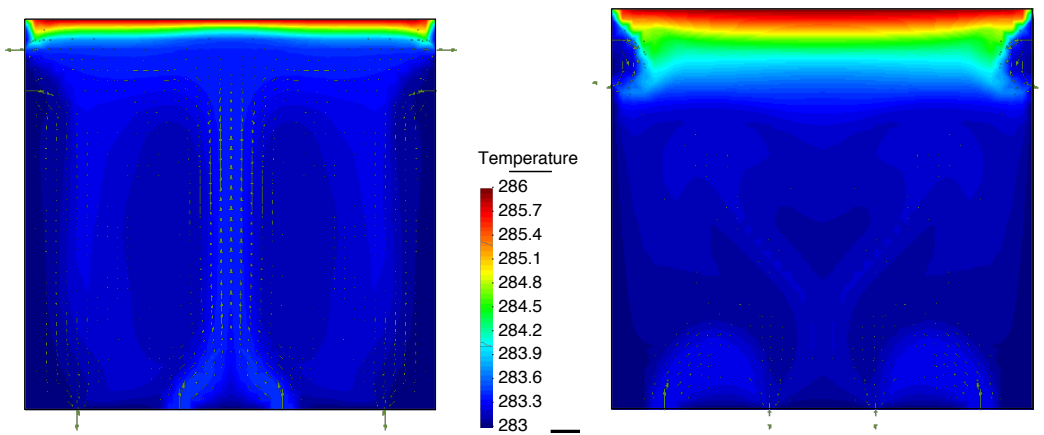


Fig. 6. Water temperatures and velocities (multiplied by a factor 500) in the last time step for the “good” configuration TPTP (left) and the “bad” configuration PTPT (right).

term. For the convective heat equation, the radiation term in the nonlinear boundary condition affects in a significative way the final results. Finally, for a better resolution of the numerical examples, it would be possible to use, instead of an uniform mesh like the one employed in previous example, a finer mesh in the neighbourhoods of collectors and injectors. Nevertheless, this approach would mean a significant increase in the computational time, already quite high in the current case (especially in the part referring to the resolution of the hydrodynamic problem).

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## A A radiation heat transfer problem with nonhomogeneous mixed boundary conditions

In this appendix we mathematically analyze a heat equation with an advective term and mixed boundary conditions of diffusive, convective and radiative type. The essential difficulty for demonstrating the existence and uniqueness of solution lies, in one hand, in the presence of nonlinear boundary conditions of radiative type and, in the other hand, the lack of regularity of the time derivative of the solutions. This lack of regularity does not allow us to take the solution itself as a test function in the variational formulation, and we are led to use more refined techniques (similar to those employed, for instance, in [1]).

So, we suppose that we have a convex domain  $\Omega \subset \mathbb{R}^3$ , whose boundary can be split into three disjoint, smooth enough parts:  $\Gamma_L$ ,  $\Gamma_A$  and  $\Gamma_R$ , with  $\partial\Omega = \Gamma_L \cup \Gamma_A \cup \Gamma_R$ . We denote by  $\theta$  the solution of the following initial-boundary value problem:

$$\left\{ \begin{array}{l} \frac{\partial\theta}{\partial t} + \mathbf{v} \cdot \nabla\theta - \nabla \cdot (K\nabla\theta) = f \quad \text{in } \Omega \times ]0, T[, \\ \theta = \theta_L \quad \text{on } \Gamma_L \times ]0, T[, \\ K\nabla\theta \cdot \mathbf{n} = b_1^A(\theta_A - \theta) \quad \text{on } \Gamma_A \times ]0, T[, \\ K\nabla\theta \cdot \mathbf{n} = b_1^R(\theta_R - \theta) + b_2^R(\varphi^4 - |\theta|^3\theta) \quad \text{on } \Gamma_R \times ]0, T[, \\ \theta(0) = \theta^0 \quad \text{in } \Omega, \end{array} \right. \quad (\text{A.1})$$

where  $K > 0$  ( $\text{m}^2 \text{s}^{-1}$ ) is the thermal diffusivity,  $b_1^H \geq 0$  ( $\text{m s}^{-1}$ ), for  $H = A, R$ , are the coefficients related to convective heat transfer through the boundaries  $\Gamma_A$  and  $\Gamma_R$ , and  $b_2^R > 0$  ( $\text{m s K}^{-3}$ ) is the coefficient related to radiative heat transfer through the boundary  $\Gamma_R$ ,  $\theta^0 \geq 0$  (K) is the initial temperature,  $\theta_L \geq 0$  (K) is Dirichlet temperature on  $\Gamma_L$ ,  $\varphi \geq 0$  (K) is the radiation temperature on  $\Gamma_R$ , and  $\theta_A, \theta_R \geq 0$  (K) are the temperatures related to convection heat transfer in surfaces  $\Gamma_A$  and  $\Gamma_R$ .

**Remark 11** *In this work we will suppose that  $\Gamma_L$ ,  $\Gamma_A$  and  $\Gamma_R$  are nonempty, but all the results can be easily extended to the case where  $\Gamma_A$  and/or  $\Gamma_R$  are empty sets. The only drawback is when  $\Gamma_L = \emptyset$  because in this case we*

cannot use Poincare type inequalities, and we should apply another type of techniques for obtaining energy estimates in the Galerkin approximation. A related problem with  $\Gamma_L = \emptyset$  was studied, for instance, in [20].

We consider the following spaces

$$\begin{aligned} X &= \{\theta \in H^1(\Omega) : \theta|_{\Gamma_R} \in L^5(\Gamma_R)\}, \\ \widetilde{X} &= \{\theta \in X : \theta|_{\Gamma_L} = 0\}, \end{aligned} \quad (\text{A.2})$$

and

$$\begin{aligned} W &= \{\theta \in W^{1,2,5/4}(0, T; X, X') : \\ &\quad \theta|_{\Gamma_R} \in L^5(0, T; L^5(\Gamma_R))\} \cap L^\infty(0, T; L^2(\Omega)), \\ \widetilde{W} &= \{\theta \in W^{1,2,5/4}(0, T; \widetilde{X}, \widetilde{X}') : \\ &\quad \theta|_{\Gamma_R} \in L^5(0, T; L^5(\Gamma_R))\} \cap L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (\text{A.3})$$

**Hypothesis 12** *We will assume the following hypotheses for the coefficients and data:*

- (1)  $\theta^0 \in L^2(\Omega)$ .
- (2)  $f \in L^2(0, T; L^2(\Omega))$ .
- (3)  $\mathbf{v} \in L^{10/3}(0, T; [L_\sigma^3(\Omega)]^3)$ .
- (4)  $\theta_A \in L^2(0, T; L^2(\Gamma_A))$ .
- (5)  $\theta_R \in L^2(0, T; L^2(\Gamma_R))$ .
- (6)  $\varphi \in L^5(0, T; L^5(\Gamma_R))$ .
- (7)  $\theta_L(\mathbf{x}, t) = g(t)\theta_D(\mathbf{x})$ , a.e.  $(\mathbf{x}, t) \in \Gamma_L \times ]0, T[$ , with  $g \in H^1(0, T)$  and  $\theta_D = \widehat{\theta_D}|_{\Gamma_L}$ , where  $\widehat{\theta_D} \in H^{3/2}(\partial\Omega)$ .

**Definition 13** *Within the framework established in Hypothesis 12, we say that an element  $\theta \in W$  is a solution of the system (A.1) if there exists  $\xi \in \widetilde{W}$  such that:*

- $\theta = \zeta_D + \xi$ , with  $\zeta_D = g\beta_0(\widehat{\theta_D}) \in W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega))$ , where  $\beta_0$  is the right inverse of trace operator  $\gamma_0 : H^2(\Omega) \rightarrow H^{3/2}(\partial\Omega)$ .
- $\xi(0) = \theta^0 - \zeta_D(0)$ , a.e.  $\mathbf{x} \in \Omega$ .
- $\xi \in \widetilde{W}$  is the solution of the following variational formulation:

$$\begin{aligned} &\int_\Omega \frac{\partial \xi}{\partial t} \eta \, d\mathbf{x} + \int_\Omega \mathbf{v} \cdot \nabla \xi \eta \, d\mathbf{x} + K \int_\Omega \nabla \xi \cdot \nabla \eta \, d\mathbf{x} \\ &+ b_1^A \int_{\Gamma_A} \xi \eta \, d\gamma + b_1^R \int_{\Gamma_R} \xi \eta \, d\gamma + b_2^R \int_{\Gamma_R} |\xi + \zeta_D|^3 (\xi + \zeta_D) \eta \, d\gamma \\ &= \int_\Omega H_D \eta \, d\mathbf{x} + b_1^A \int_{\Gamma_A} g_D^A \eta \, d\gamma \\ &+ b_1^R \int_{\Gamma_R} g_D^R \eta \, d\gamma + b_2^R \int_{\Gamma_R} \varphi^4 \eta \, d\gamma, \quad \text{a.e. } t \in ]0, T[, \quad \forall \eta \in \widetilde{X}, \end{aligned} \quad (\text{A.4})$$

where some of previous integrals must be understood as duality pairs, and

$$\begin{aligned}
H_D &= -\mathbf{v} \cdot \nabla \zeta_D + \frac{\partial \zeta_D}{\partial t} + \nabla \cdot (K \nabla \zeta_D) \in L^2(0, T; L^2(\Omega)), \\
g_D^A &= \theta_A - \zeta_D - \frac{D}{b_1^A} \nabla \zeta_D \cdot \mathbf{n} \in L^2(0, T; L^2(\Gamma_A)), \\
g_D^R &= \theta_R - \zeta_D - \frac{D}{b_1^R} \nabla \zeta_D \cdot \mathbf{n} \in L^2(0, T; L^2(\Gamma_R)).
\end{aligned} \tag{A.5}$$

We have the following result that we will prove in the following subsections:

**Theorem 14** *Within the framework established in Hypothesis 12, there exists a unique solution  $\xi \in \widetilde{W}$  of equation (A.4) in the sense of Definition 13. Moreover, there exists a constant  $C > 0$ , such that this solution satisfies the following inequalities:*

$$\begin{aligned}
& \|\xi\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|\xi\|_{L^2(0, T; H^1(\Omega))}^2 + \|\xi\|_{L^5(0, T; L^5(\Gamma_S))}^5 \\
& \leq C \left[ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 + \|H_D\|_{L^2(0, T; L^2(\Omega))}^2 + \|g_D^A\|_{L^2(0, T; L^2(\Gamma_A))}^2 \right. \\
& \quad \left. + \|g_D^R\|_{L^2(0, T; L^2(\Gamma_R))}^2 + \|\zeta_D\|_{L^5(0, T; L^5(\Gamma_R))}^5 + \|\varphi\|_{L^5(0, T; L^5(\Gamma_R))}^5 \right],
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
& \left\| \frac{d\xi}{dt} \right\|_{L^{5/4}(0, T; \widetilde{X}')} \leq C \left[ \|H_D\|_{L^2(0, T; L^2(\Omega))}^2 + \|g_D^A\|_{L^2(0, T; L^2(\Gamma_A))}^2 \right. \\
& + \|g_D^R\|_{L^2(0, T; L^2(\Gamma_R))}^2 + \|\varphi\|_{L^5(0, T; L^5(\Gamma_R))}^4 + \|\mathbf{v}\|_{L^{10/3}(0, T; [L^3(\Omega)]^3)} \|\xi\|_{L^2(0, T; \widetilde{X})} \\
& \quad \left. + \|\xi\|_{L^2(0, T; \widetilde{X})} + \|\xi\|_{L^5(0, T; L^5(\Gamma_R))}^4 + \|\zeta_D\|_{L^5(0, T; L^5(\Gamma_R))}^4 \right].
\end{aligned} \tag{A.7}$$

In order to better understand the proof of the previous result, we will divide it in five parts: in the first part we will prove some technical results related to the space where we look for the solution. In the second part we will obtain the Galerkin approximation of problem (A.1). In the third part we will analyze the differential equation obtained from the Galerkin discretization. In the fourth part we will derive the convergence of the Galerkin approximation in a suitable space. Finally, in fifth part we will prove Theorem 14.

### A.1 Part 1: Some technical results

We have the following lemma that we will use in following subsections:

**Lemma 15** *Let  $Z \subset \widetilde{X}$  a Banach space such that  $L^5(\Gamma_R) \subset Z'$ . The following inclusion is compact:*

$$\begin{aligned} & \{\theta \in W^{1,2,5/4}(0, T; \widetilde{X}, Z') : \\ & \theta|_{\Gamma_R} \in L^5(0, T; L^5(\Gamma_R))\} \subset\subset L^5(0, T; L^5(\Gamma_R)) \end{aligned} \quad (\text{A.8})$$

**PROOF.** The proof is a direct consequence of Aubin and Lions Lemma (see, for instance, Lemma 7.7 of [21]), the compactness of  $\widetilde{X}$  in  $L^5(\Gamma_R)$ , and the dominated convergence theorem in Banach spaces.  $\blacksquare$

**Remark 16** *We have that  $W^{1,2,2}(0, T; H^2(\Omega), H^2(\Omega)) \subset \mathcal{C}([0, T]; H^2(\Omega))$  (in fact, it is well known that if  $p, q \geq 1$  and  $V_1 \subset V_2$  continuously, then  $W^{1,p,q}(0, T; V_1, V_2) \subset \mathcal{C}([0, T]; V_2)$  continuously), and that  $\mathcal{C}([0, T]; H^2(\Omega)) \subset W$ . Then, the sum  $\theta_D + \xi$  makes sense in the space  $W$ . Moreover,  $\mathbf{v} \cdot \nabla \zeta_D \in L^{10/3}(0, T; L^2(\Omega))$ .*

*We can also consider as a Dirichlet condition the restriction to  $\Gamma_L$  of one element of the space  $W^{1,2,2}(0, T; H^{s-1/2}(\partial\Omega), H^{s-5/2}(\partial\Omega))$ , with  $s \geq 2$ . In this case, we can obtain an extension in the space  $W^{1,2,2}(0, T; H^s(\Omega), H^{s-2}(\Omega))$  (cf. Theorem 3.2 of [11]) and, if we want to ensure that  $W^{1,2,2}(0, T; H^s(\Omega), H^{s-2}(\Omega)) \subset W$ , we can take, for instance,  $s \geq 3$ .*

## A.2 Part 2: Galerkin approximation

In this part we will construct a sequence of approximations that will converge to a solution of problem (A.1). So, let  $\{\omega_n\}_{n \in \mathbb{N}} \subset \widetilde{X}$  be a dense subset of independent vectors of  $\widetilde{X}$ , which we can assume orthonormal in  $L^2(\Omega)$ , such that the projection

$$P_N(v) = \sum_{k=1}^N \left( \int_{\Omega} \omega_k v \, d\mathbf{x} \right) \omega_k \quad (\text{A.9})$$

is selfadjoint and  $\|P_N|_Z\|_{\mathcal{L}(Z, Z)} \leq 1$ ,  $\forall N \in \mathbb{N}$ , where  $Z$  is a Banach space, as given in Lemma 15. Then, for  $N \in \mathbb{N}$ , we denote by:

$$\xi_N = \sum_{n=1}^N \xi_n^N(t) \omega_n, \quad (\text{A.10})$$

where the coefficients  $\xi_n^N(t)$ ,  $n = 1, \dots, N$ , are such that  $\xi_N$  is the solution of the following differential equation:

$$\begin{aligned} & \int_{\Omega} \frac{\partial \xi_N}{\partial t} \omega_k \, d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot \nabla \xi_N \omega_k \, d\mathbf{x} + K \int_{\Omega} \nabla \xi_N \cdot \nabla \omega_k \, d\mathbf{x} + b_1^A \int_{\Gamma_A} \xi_N \omega_k \, d\gamma \\ & + b_1^R \int_{\Gamma_R} \xi_N \omega_k \, d\gamma + b_2^R \int_{\Gamma_R} |\xi_N + \zeta_D|^3 (\xi_N + \zeta_D) \omega_k \, d\gamma = \int_{\Omega} H_D \omega_k \, d\mathbf{x} \\ & + b_1^A \int_{\Gamma_A} g_D^A \omega_k \, d\gamma + b_1^R \int_{\Gamma_R} g_D^R \omega_k \, d\gamma + b_2^R \int_{\Gamma_R} \varphi^4 \omega_k \, d\gamma, \quad \forall k = 1, \dots, N, \end{aligned} \quad (\text{A.11})$$

which can be rewritten in the following standard formulation:

$$\begin{cases} \frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}(t), t), & a.e. t \in ]0, T[, \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \quad (\text{A.12})$$

where:

$$\mathbf{y}(t) = (\xi_1^N(t), \xi_2^N(t), \dots, \xi_N^N(t))^T, \quad (\text{A.13})$$

$$\mathbf{y}_0 = ((\xi_0^N, \omega_1), (\xi_0^N, \omega_2), \dots, (\xi_0^N, \omega_N))^T, \quad (\text{A.14})$$

$$\mathbf{F}(\mathbf{y}, t) = \begin{pmatrix} \langle f(t), \omega_1 \rangle - a(t; \mathbf{y} \cdot \boldsymbol{\omega}, \omega_1) \\ \langle f(t), \omega_2 \rangle - a(t; \mathbf{y} \cdot \boldsymbol{\omega}, \omega_2) \\ \vdots \\ \langle f(t), \omega_N \rangle - a(t; \mathbf{y} \cdot \boldsymbol{\omega}, \omega_N) \end{pmatrix}, \quad (\text{A.15})$$

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_N)^T, \quad (\text{A.16})$$

$$\begin{aligned} \langle f(t), \omega_k \rangle &= \int_{\Omega} H_D(t) \omega_k d\mathbf{x} + b_1^A \int_{\Gamma_A} g_D^A(t) \omega_k d\gamma \\ &+ b_1^R \int_{\Gamma_R} g_D^R(t) \omega_k d\gamma + b_2^R \int_{\Gamma_R} \varphi^4(t) \omega_k d\gamma, \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} a(t; \mathbf{y} \cdot \boldsymbol{\omega}, \omega_k) &= \int_{\Omega} \mathbf{v}(t) \cdot \nabla(\mathbf{y} \cdot \boldsymbol{\omega}) \omega_k d\mathbf{x} \\ &+ K \int_{\Omega} \nabla(\mathbf{y} \cdot \boldsymbol{\omega}) \cdot \nabla \omega_k d\mathbf{x} \\ &+ b_1^A \int_{\Gamma_A} (\mathbf{y} \cdot \boldsymbol{\omega}) \omega_k d\gamma + b_1^R \int_{\Gamma_R} (\mathbf{y} \cdot \boldsymbol{\omega}) \omega_k d\gamma \\ &+ b_2^R \int_{\Gamma_R} |(\mathbf{y} \cdot \boldsymbol{\omega}) + \zeta_D(t)|^3 ((\mathbf{y} \cdot \boldsymbol{\omega}) + \zeta_D(t)) \omega_k d\gamma, \end{aligned} \quad (\text{A.18})$$

where  $k = 1, \dots, N$ , and  $\xi_0^N = P_N(\theta^0 - \zeta_D(0))$  is the projection of  $\theta^0 - \zeta_D(0)$  onto  $\widetilde{X}_N = \langle \{\omega_1, \dots, \omega_N\} \rangle$ .

Thus, we say that an element  $\xi_N \in W^{1,5/4}(0, T; \widetilde{X}_N)$  is a solution of system (A.11) if it satisfies the ordinary differential equation problem (A.12). We must recall here that  $\|\xi_0^N\|_{L^2(\Omega)} \leq \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}$ ,  $\forall N \in \mathbb{N}$ , and that, if  $\xi_N$  is a solution of problem (A.11), then  $\mathbf{y} \in \mathcal{C}([0, T]; \mathbb{R}^N)$ .

**Lemma 17** *Within the framework established in Hypothesis 12, there exists a constant  $C > 0$  independent of  $N$  such that:*

$$\begin{aligned} &\|\xi_N\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|\xi_N\|_{L^2(0, T; H^1(\Omega))}^2 + \|\xi_N\|_{L^5(0, T; L^5(\Gamma_S))}^5 \\ &\leq C \left[ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 + \|H_D\|_{L^2(0, T; L^2(\Omega))}^2 + \|g_D^A\|_{L^2(0, T; L^2(\Gamma_A))}^2 \right. \\ &\quad \left. + \|g_D^R\|_{L^2(0, T; L^2(\Gamma_R))}^2 + \|\zeta_D\|_{L^5(0, T; L^5(\Gamma_R))}^5 + \|\varphi\|_{L^5(0, T; L^5(\Gamma_R))}^5 \right], \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned}
& \left\| \frac{d\xi_N}{dt} \right\|_{L^{5/4}(0,T;Z')} \leq C \left[ \|H_D\|_{L^2(0,T;L^2(\Omega))} + \|g_D^A\|_{L^2(0,T;L^2(\Gamma_A))} \right. \\
& + \|g_D^R\|_{L^2(0,T;L^2(\Gamma_R))} + \|\varphi\|_{L^5(0,T;L^5(\Gamma_R))}^4 + \|\mathbf{v}\|_{L^{10/3}(0,T;[L^3(\Omega)]^3)} \|\xi_N\|_{L^2(0,T;\tilde{X})} \\
& \left. + \|\xi_N\|_{L^2(0,T;\tilde{X})} + \|\xi_N\|_{L^5(0,T;L^5(\Gamma_R))}^4 + \|\zeta_D\|_{L^5(0,T;L^5(\Gamma_R))}^4 \right]. \tag{A.20}
\end{aligned}$$

**PROOF.** Multiplying (A.11) by  $\xi_k^N(t)$ , summing in  $k$  and adding to both sides the term

$$b_2^R \int_{\Gamma_R} |\xi_N(s) + \zeta_D(s)|^3 (\xi_N(s) + \zeta_D(s)) \zeta_D(s) \, d\mathbf{x}, \tag{A.21}$$

we have:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\xi_N(s)\|_{L^2(\Omega)}^2 + K \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2 + b_2^R \|\xi_N(s) + \zeta_D(s)\|_{L^5(\Gamma_R)}^5 \\
& \leq \|H_D(s)\|_{L^2(\Omega)} \|\xi_N(s)\|_{L^2(\Omega)} + b_1^A \|g_D^A(s)\|_{L^2(\Gamma_A)} \|\xi_N(s)\|_{L^2(\Gamma_A)} \\
& + b_1^R \|g_D^R(s)\|_{L^2(\Gamma_R)} \|\xi_N(s)\|_{L^2(\Gamma_R)} + b_2^R \|\varphi(s)\|_{L^5(\Gamma_R)}^4 \|\xi_N(s)\|_{L^5(\Gamma_R)} \\
& + b_2^R \|\xi_N(s) + \zeta_D(s)\|_{L^5(\Gamma_R)}^4 \|\zeta_D(s)\|_{L^5(\Gamma_R)}. \tag{A.22}
\end{aligned}$$

As a consequence of the continuity of trace operator and of the inequalities of Young and Poincaré, we obtain:

$$\begin{aligned}
\|H_D(s)\|_{L^2(\Omega)} \|\xi_N(s)\|_{L^2(\Omega)} & \leq \frac{C_1}{\epsilon_1} \|H_D(s)\|_{L^2(\Omega)}^2 + \epsilon_1 \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2, \\
\|g_D^A(s)\|_{L^2(\Gamma_A)} \|\xi_N(s)\|_{L^2(\Gamma_A)} & \leq \frac{C_2}{\epsilon_2} \|g_D^A(s)\|_{L^2(\Gamma_A)}^2 + \epsilon_2 \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2, \\
\|g_D^R(s)\|_{L^2(\Gamma_R)} \|\xi_N(s)\|_{L^2(\Gamma_R)} & \leq \frac{C_3}{\epsilon_3} \|g_D^R(s)\|_{L^2(\Gamma_R)}^2 + \epsilon_3 \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2, \\
\|\varphi(s)\|_{L^5(\Gamma_R)}^4 \|\xi_N(s)\|_{L^5(\Gamma_R)} & \leq \frac{C_4}{\sqrt[4]{\epsilon_4}} \|\varphi(s)\|_{L^5(\Gamma_R)}^5 + \epsilon_4 \|\xi_N(s)\|_{L^5(\Gamma_R)}^5, \\
\|\xi_N(s) + \zeta_D(s)\|_{L^5(\Gamma_R)}^4 \|\zeta_D(s)\|_{L^5(\Gamma_R)} & \leq \frac{C_5}{\sqrt[4]{\epsilon_5}} \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5 \\
& + \epsilon_5 \|\xi_N(s) + \zeta_D(s)\|_{L^5(\Gamma_R)}^5,
\end{aligned}$$

where  $\epsilon_k$ ,  $k = 1, \dots, 5$ , are arbitrary strictly positive numbers, and  $C_k$ ,  $k = 1, \dots, 5$ , are constants that may depend on the trace operator and Young and Poincaré's inequalities. If we take  $\epsilon_k$ ,  $k = 1, 2, 3$ , such that  $K - \sum_{k=1}^3 \epsilon_k = K/2$  and  $\epsilon_5 = b_2^R/2$ , then (renaming the constants if necessary):



$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\xi_N(s)\|_{L^2(\Omega)}^2 + \frac{K}{2} \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2 + \frac{b_2^R}{2} \|\xi_N(s) + \zeta_D(s)\|_{L^5(\Gamma_R)}^5 \\
& \leq C_1 \|H_D(s)\|_{L^2(\Omega)}^2 + C_2 \|g_D^A(s)\|_{L^2(\Gamma_A)}^2 + C_3 \|g_D^R(s)\|_{L^2(\Gamma_R)}^2 \quad (\text{A.23}) \\
& \quad + \frac{C_4}{\sqrt[4]{\epsilon_4}} \|\varphi(s)\|_{L^5(\Gamma_R)}^5 + \epsilon_4 \|\xi_N(s)\|_{L^5(\Gamma_R)}^5 + C_5 \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5.
\end{aligned}$$

Adding to both sides  $\frac{b_2^S}{2} \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5$ , using the inequality  $(a+b)^5 \leq 16a^5 + 16b^5$  (for  $a, b \geq 0$ ), and taking  $\epsilon_5 = b_2^R/64$ :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\xi_N(s)\|_{L^2(\Omega)}^2 + \frac{K}{2} \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2 + \frac{b_2^R}{64} \|\xi_N(s)\|_{L^5(\Gamma_R)}^5 \\
& \leq C_1 \|H_D(s)\|_{L^2(\Omega)}^2 + C_2 \|g_D^A(s)\|_{L^2(\Gamma_R)}^2 + C_3 \|g_D^R(s)\|_{L^2(\Gamma_R)}^2 \quad (\text{A.24}) \\
& \quad + C_4 \|\varphi(s)\|_{L^5(\Gamma_R)}^5 + C_5 \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5.
\end{aligned}$$

Finally, integrating over the time interval  $[0, t]$  (renaming again the constants):

$$\begin{aligned}
& \|\xi_N(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2 ds + \int_0^t \|\xi_N(s)\|_{L^5(\Gamma_R)}^5 ds \\
& \leq C \left[ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 + \int_0^t \|H_D(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \|g_D^A(s)\|_{L^2(\Gamma_R)}^2 ds \quad (\text{A.25}) \right. \\
& \quad \left. + \int_0^t \|\varphi(s)\|_{L^5(\Gamma_R)}^5 ds + \int_0^t \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5 ds \right],
\end{aligned}$$

where we have used that  $\|\xi_0^N\|_{L^2(\Omega)} \leq \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}$ ,  $\forall N \in \mathbb{N}$ . Finally, applying Gronwall's Lemma, we obtain that there exists a positive constant  $C$  independent of  $N$  such that (A.19) is satisfied.

For obtaining (A.20) it is sufficient to apply Holder inequality and bear in mind the fact that the projection operator  $P_N$  onto  $\tilde{X}_N$  is bounded independently of  $N$ :

$$\begin{aligned}
\left\langle \frac{d\xi_N}{dt}, v \right\rangle &= \left\langle P_N \left( \frac{d\xi_N}{dt} \right), v \right\rangle = \left\langle \frac{d\xi_N}{dt}, P_N(v) \right\rangle \\
&\leq C \left[ \|H_D\|_{L^2(0,T;L^2(\Omega))} + \|g_D^A\|_{L^2(0,T;L^2(\Gamma_A))} \right. \\
&\quad + \|g_D^R\|_{L^2(0,T;L^2(\Gamma_R))} + \|\varphi\|_{L^5(0,T;L^5(\Gamma_R))} \\
&\quad + \|\mathbf{v}\|_{L^{10/3}(0,T;[L^3(\Omega)]^3)} \|\xi_N\|_{L^2(0,T;\tilde{X})} \\
&\quad + \|\xi_N\|_{L^2(0,T;\tilde{X})} + \|\xi_N\|_{L^5(0,T;L^5(\Gamma_R))} \\
&\quad \left. + \|\zeta_D\|_{L^5(0,T;L^5(\Gamma_R))} \right] \|P_N\|_{\mathcal{L}(Z,Z)} \|v\|_{L^5(0,T;Z)}.
\end{aligned}$$

■

### A.3 Part 3: Existence of solution for the Galerkin approximation

Now we will demonstrate that there exists, for each  $N \in \mathbb{N}$ , a unique absolutely continuous solution  $\xi_N$  of equation (A.11).

**Lemma 18** *Within the framework established in Hypothesis 12, there exists a unique absolutely continuous solution defined on the whole time interval  $[0, T]$  of Cauchy problem (A.12).*

**PROOF.** To state this lemma we can apply the Caratheodory theorem for ordinary differential equations (cf., for example, Theorem 5.2 of [13]). Indeed,  $\mathbf{F}(\cdot, t)$  is continuous for any  $t \in [0, T]$ , and  $\mathbf{F}(\mathbf{y}, \cdot) \in L^{5/4}(0, T)$  for any  $\mathbf{y} \in \mathbb{R}^N$ . Then, given an open ball  $B$  in  $\mathbb{R}^N$ , if we prove that there exist two functions  $m_B, l_B \in L^1(0, T)$  such that:

$$\begin{aligned} \|\mathbf{F}(\mathbf{y}, t)\| &\leq m_B(t), \text{ a.e. } t \in ]0, T[, \forall \mathbf{y} \in B, \\ \|\mathbf{F}(\mathbf{y}_1, t) - \mathbf{F}(\mathbf{y}_2, t)\| &\leq l_B(t)\|\mathbf{y}_1 - \mathbf{y}_2\|, \text{ a.e. } t \in ]0, T[, \forall \mathbf{y}_1, \mathbf{y}_2 \in B, \end{aligned} \quad (\text{A.26})$$

we can conclude that problem (A.12) has a unique absolutely continuous solution, which can be extended to the boundary of  $]0, T[ \times B$ . It is worthwhile mentioning here that, if

$$\begin{aligned} \text{diam}(B)^2 > C \left[ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 + \|H_D\|_{L^2(0, T; L^2(\Omega))}^2 \right. \\ \left. + \|g_D^A\|_{L^2(0, T; L^2(\Gamma_A))}^2 + \|g_D^R\|_{L^2(0, T; L^2(\Gamma_R))}^2 \right. \\ \left. + \|\zeta_D\|_{L^5(0, T; L^5(\Gamma_R))}^5 + \|\varphi\|_{L^5(0, T; L^5(\Gamma_R))}^5 \right], \end{aligned} \quad (\text{A.27})$$

then the solution  $\mathbf{y}$  cannot reach the boundary of  $B$  because of the *a priori* estimate (A.19) and the fact that  $\|\xi_N(t)\|_{L^2(\Omega)}^2 = \|\mathbf{y}(t)\|^2$ . The first of the Caratheodory conditions (A.26) can be obtained by applying Holder inequality (in fact, we obtain that  $m_B \in L^{5/4}(0, T)$ ). In the other hand, for the second Caratheodory condition, we can use the following inequality (straightforward consequence of the mean value Theorem):

$$b|b|^3 - a|a|^3 = 4|c|^3(b - a), \quad (\text{A.28})$$

for  $c = \lambda b + (1 - \lambda)a$  and  $\lambda \in (0, 1)$ . So, we obtain the following inequality:

$$\begin{aligned}
|F_k(\mathbf{y}_1, t) - F_k(\mathbf{y}_2, t)| &= |a(t; \mathbf{y}_1 \cdot \boldsymbol{\omega}, \omega_k) - a(t; \mathbf{y}_2 \cdot \boldsymbol{\omega}, \omega_k)| \\
&\leq C_1 \left[ \|\mathbf{v}(t)\|_{[L^3(\Omega)]^3} \|\nabla \boldsymbol{\omega}\|_{[L^2(\Omega)]^{N \times 3}} \|\omega_k\|_{L^6(\Omega)} \right. \\
&\quad + \|\nabla \boldsymbol{\omega}\|_{[L^2(\Omega)]^{N \times 3}} \|\nabla \omega_k\|_{L^2(\Omega)} + \|\boldsymbol{\omega}\|_{[L^2(\Gamma_A)]^N} \|\omega_k\|_{L^2(\Gamma_A)} \\
&\quad \left. + \|\boldsymbol{\omega}\|_{[L^2(\Gamma_R)]^N} \|\omega_k\|_{L^2(\Gamma_R)} \right] \|\mathbf{y}_1 - \mathbf{y}_2\| \\
&\quad + C_2 \left[ \left( \max\{\|\mathbf{y}_1\|, \|\mathbf{y}_2\|\} \right)^3 \|\boldsymbol{\omega}\|_{[L^5(\Gamma_R)]^N}^3 \right. \\
&\quad \left. + \|\xi_D(t)\|_{L^5(\Gamma_R)}^3 \right] \|\boldsymbol{\omega}\|_{[L^5(\Gamma_R)]^N} \|\omega_k\|_{L^5(\Gamma_R)} \|\mathbf{y}_1 - \mathbf{y}_2\|.
\end{aligned}$$

Therefore, we can conclude the existence of a function  $l_B \in L^1(0, T)$  (in fact,  $l_B \in L^{5/3}(0, T)$ ) such that the second of the Caratheodory conditions (A.26) is achieved.  $\blacksquare$

#### A.4 Part 4: Convergence of the Galerkin approximation

In previous subsections we have seen that there exists a bounded sequence  $\{\xi_N\}_{N \in \mathbb{N}} \subset \widetilde{W}$  of solutions of problem (A.12). In this subsection we will pass to the limit and obtain a solution of equation (A.4).

**Lemma 19** *There exists a subsequence of  $\{\xi_N\}_{N \in \mathbb{N}}$ , still denoted in the same way, such that:*

- (1)  $\xi_N \rightharpoonup \xi$  in  $L^2(0, T; \widetilde{X})$ ,
- (2)  $\xi_N \rightharpoonup^* \xi$  in  $L^\infty(0, T; L^2(\Omega))$ ,
- (3)  $\xi_N \rightarrow \xi$  in  $L^{10/3-\epsilon}(0, T; L^{10/3-\epsilon}(\Omega))$ ,
- (4)  $\xi_N \rightarrow \xi$  in  $L^2(0, T; L^2(\Gamma_A))$ ,
- (5)  $\xi_N \rightarrow \xi$  in  $L^5(0, T; L^5(\Gamma_R))$ .

**PROOF.** Thanks to the boundedness in  $\widetilde{W}$  of the sequence  $\{\xi_N\}_{N \in \mathbb{N}}$ , we obtain the first two convergences. The third and fourth limits are a direct consequence of Aubin, and Lions Lemma (cf. Lemma 7.7 of [21]) and the compactness of  $H^1(\Omega)$ , respectively, in  $L^{6-\epsilon}(\Omega)$  and  $L^{4-\epsilon}(\partial\Omega)$ ,  $\forall \epsilon > 0$ . Finally, the fifth convergence is a consequence of Lemma 15.  $\blacksquare$

**Lemma 20** *If  $\{\xi_N\}_{N \in \mathbb{N}}$  is a bounded sequence in  $\widetilde{W}$ , then there exists a subsequence of  $\{\xi_N\}_{N \in \mathbb{N}}$ , still denoted in the same way, such that, for all  $\phi \in L^5(0, T; L^5(\Gamma_R))$ :*

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \int_0^T \int_{\Gamma_R} |\xi_N + \zeta_D|^3 (\xi_N + \zeta_D) \phi \, d\gamma \, dt \\
&= \int_0^T \int_{\Gamma_R} |\xi + \zeta_D|^3 (\xi + \zeta_D) \phi \, d\gamma \, dt.
\end{aligned} \tag{A.29}$$

**PROOF.** Using the same technique that we have employed in the proof of Lemma 18, from the strong convergence of  $\{\xi_N\}_{N \in \mathbb{N}}$  to  $\xi$  in  $L^5(0, T; L^5(\Gamma_R))$  and the inequality (A.19) we have:

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^T \int_{\Gamma_R} \left[ |\xi_N + \zeta_D|^3(\xi_N + \zeta_D) - |\xi + \zeta_D|^3(\xi + \zeta_D) \right] \phi \, d\gamma \, dt \\ & \leq \lim_{N \rightarrow \infty} \left( \max\{\|\xi_N\|_{L^5(0, T; L^5(\Gamma_R))}, \|\xi\|_{L^5(0, T; L^5(\Gamma_R))}\}^3 \right. \\ & \left. + \|\zeta_D\|_{L^5(0, T; L^5(\Gamma_R))}^3 \right) \|\phi\|_{L^5(0, T; L^5(\Gamma_R))} \|\xi_N - \xi\|_{L^5(0, T; L^5(\Gamma_R))} = 0. \end{aligned} \quad (\text{A.30})$$

■

### A.5 Part 5: Proof of the main result of the Appendix

Now, we can demonstrate the Theorem 14:

**PROOF.** We will divide the proof into three parts, in the first part we will pass to the limit in the Galerkin approximation in order to obtain a solution for the system (A.4), in the second part, we will derive the estimates (A.6) and (A.7) and, finally, in the third part we will prove the uniqueness of solution.

First, for a fixed index  $k \in \mathbb{N}$ , if we multiply (A.11) by a scalar function  $\psi$  continuously differentiable on  $[0, T]$ , such that  $\psi(T) = 0$ , integrate with respect to  $t$ , and integrate by parts, we have,  $\forall N \geq k$ :

$$\begin{aligned} & - \int_0^T \int_{\Omega} \xi_N(t) \frac{d\psi}{dt}(t) \omega_k \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega} \mathbf{v}(t) \cdot \nabla \xi_N(t) \psi(t) \omega_k \, d\mathbf{x} \, dt \\ & + K \int_0^T \int_{\Omega} \nabla \xi_N(t) \cdot \nabla \omega_k \psi(t) \, d\mathbf{x} \, dt + b_1^A \int_0^T \int_{\Gamma_A} \xi_N(t) \psi(t) \omega_k \, d\gamma \, dt \\ & \quad + b_1^R \int_0^T \int_{\Gamma_R} \xi_N(t) \psi(t) \omega_k \, d\gamma \, dt \\ & \quad + b_2^R \int_0^T \int_{\Gamma_R} |\xi_N(t) + \zeta_D(t)|^3 (\xi_N(t) + \zeta_D(t)) \psi(t) \omega_k \, d\gamma \, dt \\ & = \int_{\Omega} \xi_0^N \omega_k \psi(0) \, d\mathbf{x} + \int_0^T \int_{\Omega} H_D(t) \psi(t) \omega_k \, d\mathbf{x} \, dt + b_1^A \int_0^T \int_{\Gamma_A} g_D^A(t) \psi(t) \omega_k \, d\gamma \, dt \\ & \quad + b_1^R \int_0^T \int_{\Gamma_R} g_D^R(t) \psi(t) \omega_k \, d\gamma \, dt + b_2^R \int_0^T \int_{\Gamma_R} \varphi^4(t) \psi(t) \omega_k \, d\gamma \, dt. \end{aligned}$$

The passage to the limit for  $N \rightarrow \infty$  in the integrals of the left-hand side is due to the Lemmas 19 and 20. We observe also that  $\xi_0^N \rightarrow \theta^0 - \zeta_D(0)$  in  $L^2(\Omega)$ . Hence, we find in the limit:

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \xi(t) \frac{d\psi}{dt}(t) \eta \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega} \mathbf{v}(t) \cdot \nabla \xi(t) \psi(t) \eta \, d\mathbf{x} \, dt \\
& + K \int_0^T \int_{\Omega} \nabla \xi(t) \cdot \nabla \omega_k \psi(t) \, d\mathbf{x} \, dt + b_1^A \int_0^T \int_{\Gamma_A} \xi(t) \psi(t) \eta \, d\gamma \, dt \\
& \quad + b_1^R \int_0^T \int_{\Gamma_R} \xi(t) \psi(t) \eta \, d\gamma \, dt \\
& \quad + b_2^R \int_0^T \int_{\Gamma_R} |\xi(t) + \zeta_D(t)|^3 (\xi(t) + \zeta_D(t)) \psi(t) \eta \, d\gamma \, dt \quad (\text{A.31}) \\
& = \int_{\Omega} (\theta^0 - \zeta_D(0)) \eta \psi(0) \, d\mathbf{x} + \int_0^T \int_{\Omega} H_D(t) \psi(t) \eta \, d\mathbf{x} \, dt \\
& + b_1^A \int_0^T \int_{\Gamma_A} g_D^A(t) \psi(t) \eta \, d\gamma \, dt + b_1^R \int_0^T \int_{\Gamma_R} g_D^R(t) \psi(t) \eta \, d\gamma \, dt \\
& \quad + b_2^R \int_0^T \int_{\Gamma_R} \varphi^A(t) \psi(t) \eta \, d\gamma \, dt,
\end{aligned}$$

for each  $\eta \in \widetilde{X}$  which is a finite lineal combination of elements  $\omega_k$ . Since each term of above expression depends linearly and continuously on  $\eta$ , for the norm of  $\widetilde{X}$ , previous equality remains still valid, by continuity, for each  $\eta \in \widetilde{X}$ . Now, writing in particular (A.31) for  $\psi = \phi \in \mathcal{D}(0, T)$ , we obtain the variational formulation (A.4). Finally, we can prove that  $\xi(0) = \theta^0 - \zeta_D(0)$  multiplying (A.4) by the same  $\psi$  as before, integrating by parts with respect to  $t$ , and comparing with (A.31).

Then, multiplying inequality (A.25) by  $\phi$ , with  $\phi \in \mathcal{D}(0, T)$ ,  $\phi \geq 0$ , and integrating in  $[0, T]$  we have:

$$\begin{aligned}
& \int_0^T \left\{ \|\xi_N(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \xi_N(s)\|_{[L^2(\Omega)]^3}^2 \, ds + \right. \\
& \left. \int_0^t \|\xi_N(s)\|_{L^5(\Gamma_R)}^5 \, ds \right\} \phi(t) \, dt \leq C \int_0^T \left\{ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 \right. \\
& \quad + \int_0^t \|H_D(s)\|_{L^2(\Omega)}^2 \, ds + \int_0^t \|g_D^A(s)\|_{L^2(\Gamma_R)}^2 \, ds \\
& \quad \left. + \int_0^t \|\varphi(s)\|_{L^5(\Gamma_R)}^5 \, ds + \int_0^t \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5 \, ds \right\} \phi(t) \, dt, \quad (\text{A.32})
\end{aligned}$$

from which, taking into account that the norm of a reflexive Banach space is weakly lower semicontinuous, we can pass to the inferior limit thanks to convergences of Lemma 19:

$$\begin{aligned}
& \int_0^T \left\{ \|\xi(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \xi(s)\|_{[L^2(\Omega)]^3}^2 \, ds + \right. \\
& \left. \int_0^t \|\xi(s)\|_{L^5(\Gamma_R)}^5 \, ds \right\} \phi(t) \, dt \leq C \int_0^T \left\{ \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 \right. \\
& + \int_0^t \|H_D(s)\|_{L^2(\Omega)}^2 \, ds + \int_0^t \|g_D^A(s)\|_{L^2(\Gamma_R)}^2 \, ds + \int_0^t \|\varphi(s)\|_{L^5(\Gamma_R)}^5 \, ds \\
& \quad \left. + \int_0^t \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5 \, ds \right\} \phi(t) \, dt, \quad \forall \phi \in \mathcal{D}(0, T), \phi \geq 0. \quad (\text{A.33})
\end{aligned}$$

Thus, we obtain the following energy inequality for a.e.  $t \in (0, T)$ :

$$\begin{aligned} & \|\xi(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \xi(s)\|_{[L^2(\Omega)]^3}^2 ds + \int_0^t \|\xi(s)\|_{L^5(\Gamma_R)}^5 ds \\ & \leq C \left[ \int_0^T \|\theta^0 - \zeta_D(0)\|_{L^2(\Omega)}^2 ds + \int_0^t \|H_D(s)\|_{L^2(\Omega)}^2 ds \right. \\ & \left. + \int_0^t \|g_D^A(s)\|_{L^2(\Gamma_R)}^2 ds + \int_0^t \|\varphi(s)\|_{L^5(\Gamma_R)}^5 ds + \int_0^t \|\zeta_D(s)\|_{L^5(\Gamma_R)}^5 ds \right]. \end{aligned} \quad (\text{A.34})$$

Finally, (A.6) can be derived from above expression thanks to the Gronwall's Lemma, and estimate (A.7) is a direct consequence of Holder inequality.

Now, we will prove the uniqueness of solution. Let us assume the existence of two solutions  $\underline{\xi}_1$  and  $\xi_2$  for problem (A.4), and define  $\xi_{12} = \xi_1 - \xi_2$ . We have that  $\xi_{12} \in \widetilde{W}$ ,  $\xi_{12}(0) = 0$ , a.e.  $\mathbf{x} \in \Omega$ , and that satisfies the following variational formulation:

$$\begin{aligned} & \int_{\Omega} \frac{\partial \xi_{12}}{\partial t} \eta \, d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot \nabla \xi_{12} \eta \, d\mathbf{x} \\ & + K \int_{\Omega} \nabla \xi_{12} \cdot \nabla \eta \, d\mathbf{x} + b_1^A \int_{\Gamma_A} \xi_{12} \eta \, d\gamma + b_1^R \int_{\Gamma_R} \xi_{12} \eta \, d\gamma \\ & + b_2^R \int_{\Gamma_R} \left[ |\xi_1 + \zeta_D|^3 (\xi_1 + \zeta_D) - |\xi_2 + \zeta_D|^3 (\xi_2 + \zeta_D) \right] \eta \, d\gamma \\ & = 0, \quad \text{a.e. } t \in ]0, T[, \quad \forall \eta \in \widetilde{X}. \end{aligned} \quad (\text{A.35})$$

From a direct computation, we observe that  $\frac{d\xi_{12}}{dt} = \alpha + \beta + \gamma$ , where  $\alpha \in L^2(0, T; \widetilde{X}')$ ,  $\beta \in L^{5/4}(0, T; L^{5/4}(\Gamma))$  and  $\gamma \in L^{4/3}(0, T; L^{5/6}(\Omega))$ , defined by:

$$\langle \alpha, z \rangle = -K \int_0^T \int_{\Omega} \nabla \xi_{12} \cdot \nabla z \, d\mathbf{x} \, dt - b_1^A \int_0^T \int_{\Gamma_A} \xi_{12} z \, d\gamma \, dt - b_1^R \int_0^T \int_{\Gamma_R} \xi_{12} z \, d\gamma \, dt,$$

$\forall z \in \widetilde{X}$ ,  $\beta = -b_2^R \left[ |\xi_1|_{\Gamma_R} + \zeta_D|_{\Gamma_R} \right]^3 (\xi_1|_{\Gamma_R} + \zeta_D|_{\Gamma_R}) - |\xi_2|_{\Gamma_R} + \zeta_D|_{\Gamma_R} \right]^3 (\xi_2|_{\Gamma_R} + \zeta_D|_{\Gamma_R})$ , and  $\gamma = -\mathbf{v} \cdot \nabla \xi_{12}$ .

Now, for each  $\delta > 0$ , we define the following function:

$$\kappa_{\delta}(r) = \begin{cases} 1 & \text{if } r \geq \delta, \\ \frac{r}{\delta} & \text{if } -\delta \leq r \leq \delta, \\ -1 & \text{if } r \leq -\delta, \end{cases} \quad (\text{A.36})$$

and its primitive,

$$K_{\delta}(r) = \int_0^r \kappa_{\delta}(s) \, ds = \begin{cases} r - \frac{\delta}{2} & \text{if } r \geq \delta, \\ \frac{r^2}{2\delta} & \text{if } -\delta \leq r \leq \delta, \\ -r - \frac{\delta}{2} & \text{if } r \leq -\delta. \end{cases} \quad (\text{A.37})$$

We can extend the results proved in Corollary 9 of [2] to our case (we just need to add the term in  $L^{5/4}(0, T; L^{5/4}(\Gamma_R))$  in the proof of Lemma 6 of [2] and then we can use this result to prove Corollary 9) and then, we have:

$$\begin{aligned} & \int_{\Omega} K_{\delta}(\xi_{12}(t)) dx - \int_{\Omega} K_{\delta}(\xi_{12}(0)) d\mathbf{x} = \int_0^t \langle \alpha(s), \kappa_{\delta}(\xi_{12}(s)) \rangle_{\tilde{X}', \tilde{X}} ds \\ + \int_0^t \int_{\Gamma_R} \beta(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma ds + \int_0^t \int_{\Omega} \gamma(s) \kappa_{\delta}(\xi_{12}(s)) d\mathbf{x} ds, \quad a.e. t \in ]0, T[. \end{aligned} \quad (\text{A.38})$$

By a direct evaluation of previous expression, we have:

$$\begin{aligned} & \int_{\Omega} K_{\delta}(\xi_{12}(t)) d\mathbf{x} + K \int_0^t \int_{\Omega} \nabla \xi_{12}(s) \cdot \nabla \kappa_{\delta}(\xi_{12}(s)) d\mathbf{x} ds \\ & + \int_0^t \int_{\Omega} \mathbf{v}(s) \cdot \nabla \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\mathbf{x} dt + b_1^A \int_0^t \int_{\Gamma_A} \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma ds \\ & + b_1^R \int_0^t \int_{\Gamma_R} \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma ds + b_2^R \int_0^t \int_{\Gamma_R} [|\xi_1(s) + \zeta_D(s)|^3 (\xi_1(s) + \zeta_D(s)) \\ & - |\xi_2(s) + \zeta_D(s)|^3 (\xi_2(s) + \zeta_D(s))] \kappa_{\delta}(\xi_{12}(s)) d\gamma ds = 0, \end{aligned}$$

a.e.  $t \in ]0, T[$ . Moreover, it is obvious that:

$$\begin{aligned} & \int_{\Omega} \mathbf{v}(s) \cdot \nabla \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\mathbf{x} = 0, \quad a.e. s \in ]0, T[, \\ & \int_{\Omega} \nabla \xi_{12}(s) \cdot \nabla \kappa_{\delta}(\xi_{12}(s)) d\mathbf{x} = \int_{\Omega} \kappa'_{\delta}(\xi_{12}(s)) \|\nabla \xi_{12}(s)\|^2 d\mathbf{x} \geq 0, \quad a.e. s \in ]0, T[, \\ & \int_{\Gamma_A} \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma \geq 0, \quad a.e. s \in ]0, T[, \\ & \int_{\Gamma_R} \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma \geq 0, \quad a.e. s \in ]0, T[, \\ & \int_{\Gamma_R} [|\xi_1(s) + \zeta_D(s)|^3 (\xi_1(s) + \zeta_D(s)) - |\xi_2(s) + \zeta_D(s)|^3 (\xi_2(s) + \zeta_D(s))] \kappa_{\delta}(\xi_{12}(s)) d\gamma \\ & = 4 \int_{\Omega} |c(s)|^3 \xi_{12}(s) \kappa_{\delta}(\xi_{12}(s)) d\gamma \geq 0, \quad a.e. s \in ]0, T[, \end{aligned}$$

for  $c(s) = \lambda(s)\xi_1(s) + (1 - \lambda(s))\xi_2(s)$  and  $\lambda(s) \in (0, 1)$ , a.e.  $s \in ]0, T[$ . Thus, we can deduce that:

$$\int_{\Omega} K_{\delta}(\xi_{12}(t)) d\mathbf{x} = 0, \quad a.e. t \in ]0, T[. \quad (\text{A.39})$$

Finally, from the definition (A.37) of  $K_{\delta}$  it is straightforward that  $0 \leq |r| - K_{\delta}(r) \leq \frac{\delta}{2}$ ,  $\forall r \in \mathbb{R}$ , and then:

$$\|\xi_{12}(t)\|_{L^1(\Omega)} \leq \frac{\delta}{2} |\Omega| + \int_{\Omega} K_{\delta}(\xi_{12}(t)) d\mathbf{x}, \quad \forall \delta > 0, \quad (\text{A.40})$$

with  $|\Omega|$  denoting the volume of  $\Omega$ , which implies that  $\xi_{12}(\mathbf{x}, t) = 0$ , a.e.  $(\mathbf{x}, t) \in \Omega \times ]0, T[$ , and, consequently,  $\xi_1 = \xi_2$ .  $\blacksquare$

## References

- [1] L.J. Alvarez-Vázquez, F.J. Fernández, and R. Muñoz-Sola. Mathematical analysis of a three-dimensional eutrophication model. *J. Math. Anal. Appl.*, 349:135–155, 2009.
- [2] L.J. Alvarez-Vázquez, F.J. Fernández, and R. Muñoz-Sola. Analysis of a multistate control problem related to food technology. *J. Diff. Equations*, 245:130–153, 2008.
- [3] A. Bahlaoui, A. Raji, and M. Hasnaoui. Multiple Steady State Solutions Resulting From Coupling Between Mixed Convection And Radiation In An Inclined Channel. *Heat Mass Transfer*, 41:899–908, 2005.
- [4] M. Benes and P. Kucera. On the Navier-Stokes flows for heat-conducting fluids with mixed boundary conditions. *J. Math. Anal. Appl.*, 389:769–780, 2012.
- [5] T.L. Bergman, A.S. Lavine, F.P. Incropera, and D.P. DeWitt. *Fundamentals of Heat and Mass Transfer*. John Wiley & Sons, New York, 2011.
- [6] A. Bermúdez de Castro. *Continuum Thermomechanics*. Birkhauser, Basel, 2005.
- [7] J.B. Conway. *A course in functional analysis*. Springer-Verlag, New York, 1990.
- [8] M.C. Delfour, G. Payre, and J.P. Zolesio. Approximation of nonlinear problems associated with radiating bodies in space. *SIAM J. Numer. Anal.*, 24:1077–1094, 1987.
- [9] F.J. Fernández, L.J. Alvarez-Vázquez, and A. Martínez. On the existence and uniqueness of solution of an hydrodynamic problem related to water artificial circulation in a lake. *Submitted*, 2018.
- [10] F.J. Fernández, L.J. Alvarez-Vázquez, A. Martínez, and M.E. Vázquez-Méndez. A 3D optimal control problem related to the urban heat islands. *J. Math. Anal. Appl.*, 446:1571–1605, 2017.
- [11] A. Fursikov, M. Gunzburger, and L. Hou. Trace theorems for three-dimensional, time-dependent solenoidal vector fields and their applications. *Trans. Amer. Math. Soc.*, 354:1079–1116, 2002.
- [12] V. Girault and P.A. Raviart. *Finite element methods for Navier-Stokes equations*. Springer-Verlag, Berlin, 1986.
- [13] J.K. Hale. *Ordinary differential equations*. Robert E. Krieger Publishing Co., Huntington, 1980.
- [14] F. Hecht. New development in Freefem++. *J. Numer. Math.*, 20:251–265, 2012.
- [15] D.D. Joseph. On the stability of the Boussinesq equations. *Arch. Rational Mech. Anal.*, 20:59–71, 1965.



- [16] A.E. Kovtanyuk, A.Y. Chebotarev, N.D. Botkin, and K.H. Hoffmann. Theoretical analysis of an optimal control problem of conductive-convective-radiative heat transfer. *J. Math. Anal. Appl.*, 412:520–528, 2014.
- [17] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. American Mathematical Society, Providence, 1968.
- [18] M. Lambert and R. Kessler. Radiation Heat Transfer in Mixed Convection Flow. In: A. Dillmann et al. (eds.) *New Results in Numerical and Experimental Fluid Mechanics VII*, Springer, Berlin, 2010.
- [19] A. Martínez, F.J. Fernández, and L.J. Alvarez-Vázquez. Water artificial circulation for eutrophication control. *Math. Control Rel. Fields*, 8:277–313, 2018.
- [20] M. Metzger. Existence for a time-dependent heat equation with non-local radiation terms. *Math. Meth. Appl. Sci.*, 22:1101–1119, 1999.
- [21] T. Roubíček. *Nonlinear partial differential equations with applications*. Birkhäuser, Basel, 2013.

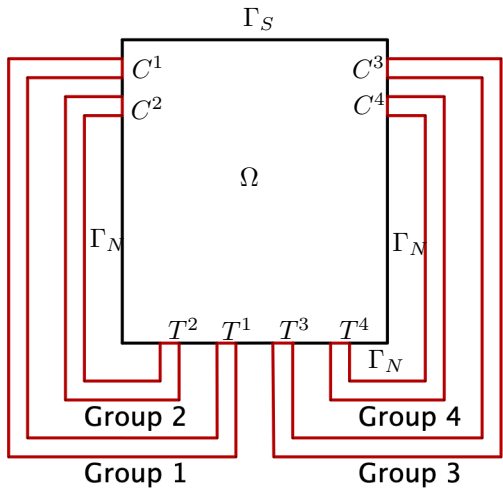


Figure 2  
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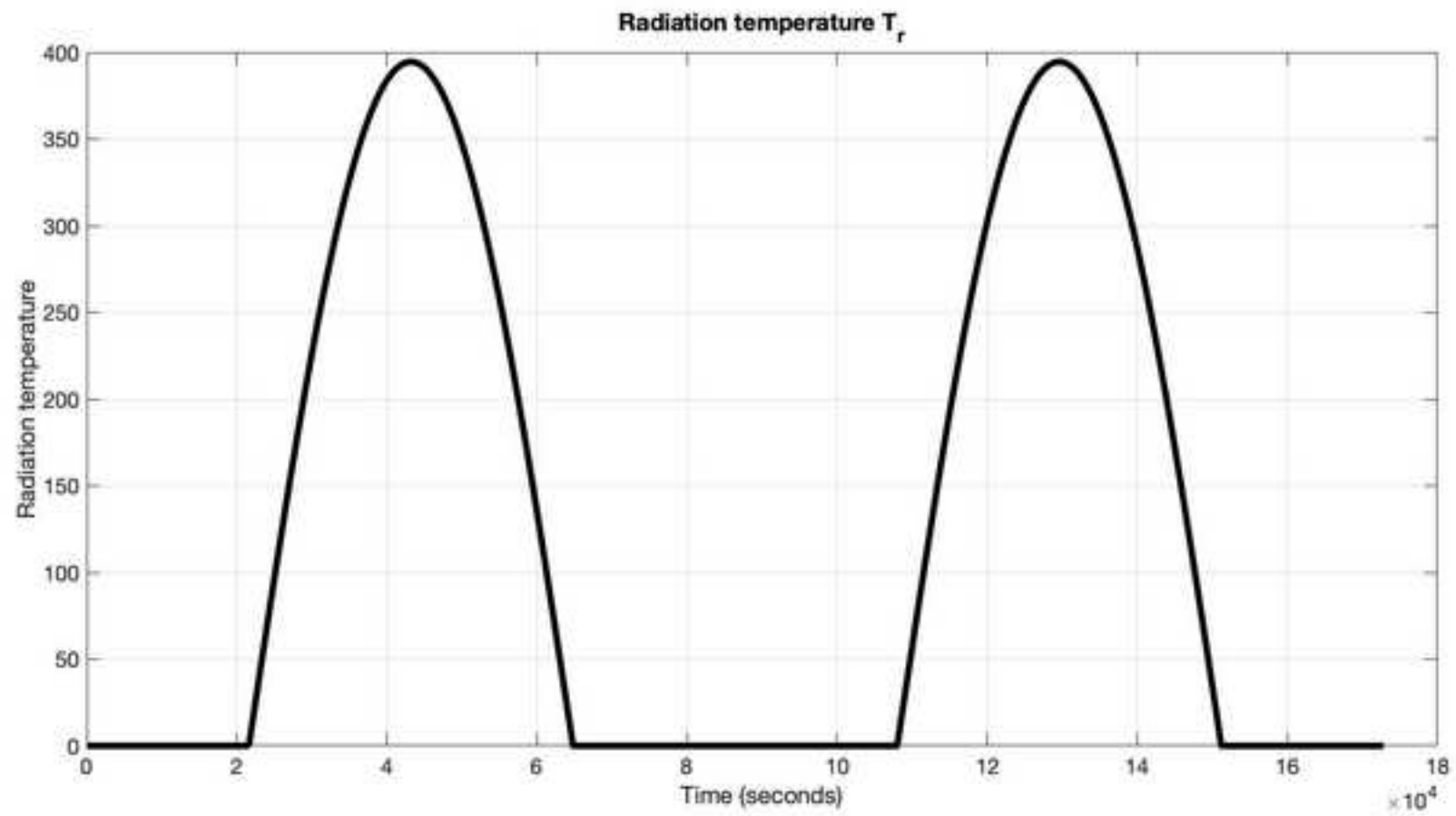
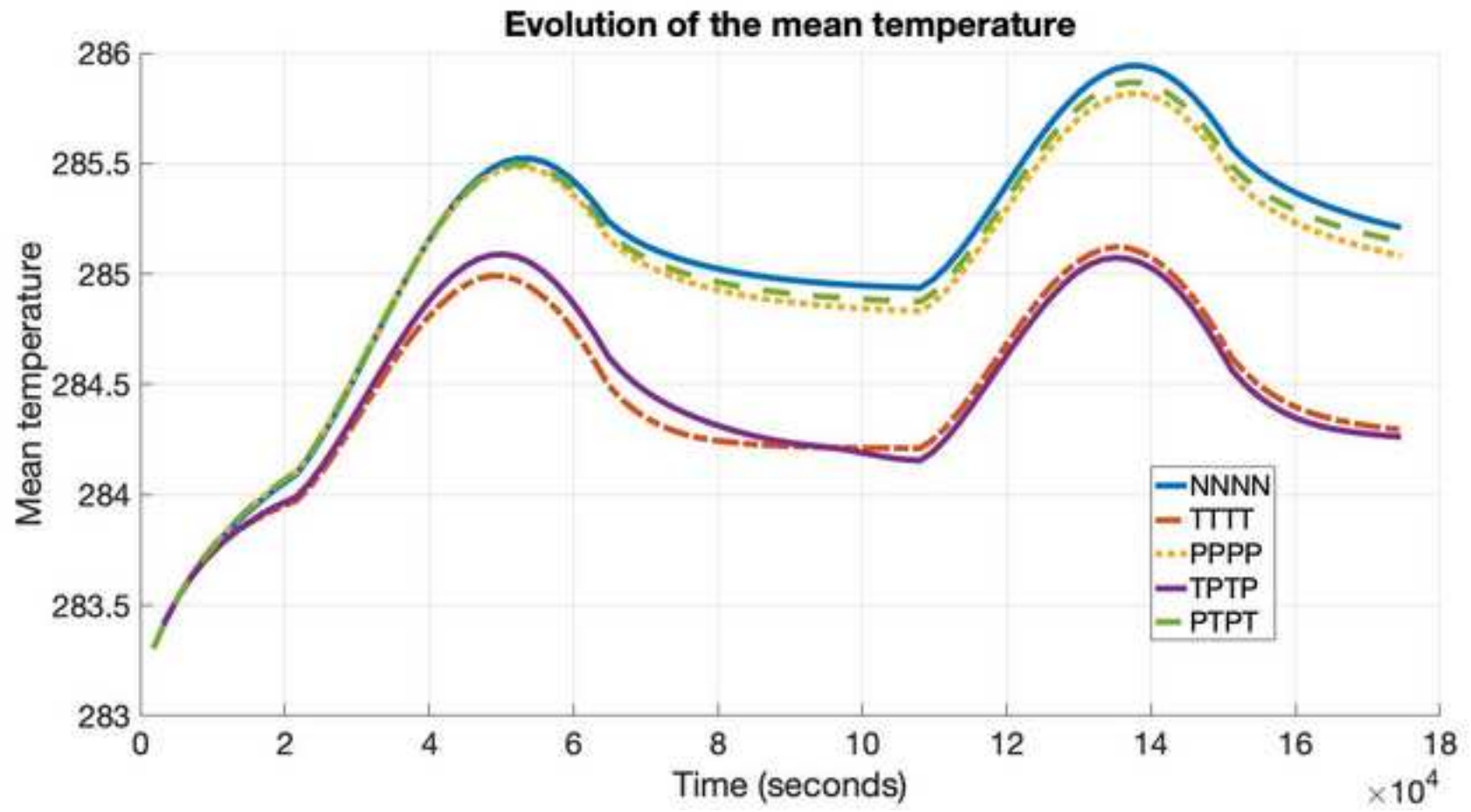
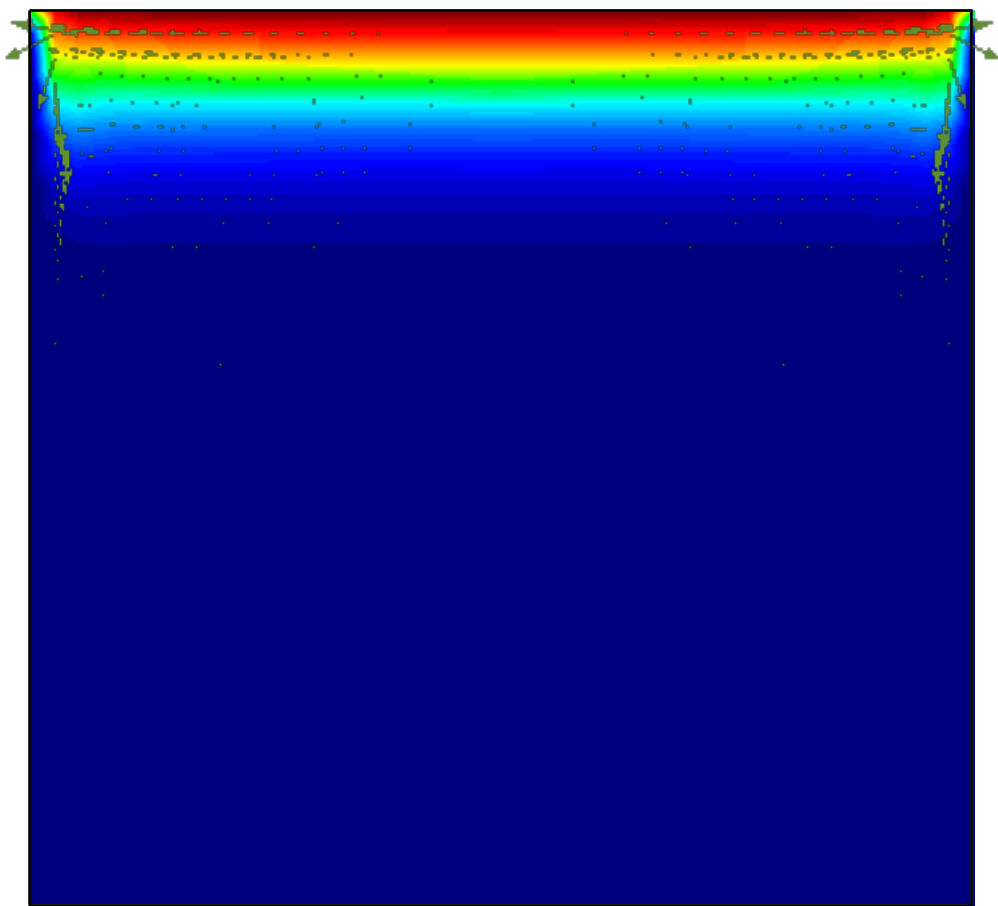


Figure 3  
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# Temperature

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286

285.7

285.4

285.1

284.8

284.5

284.2

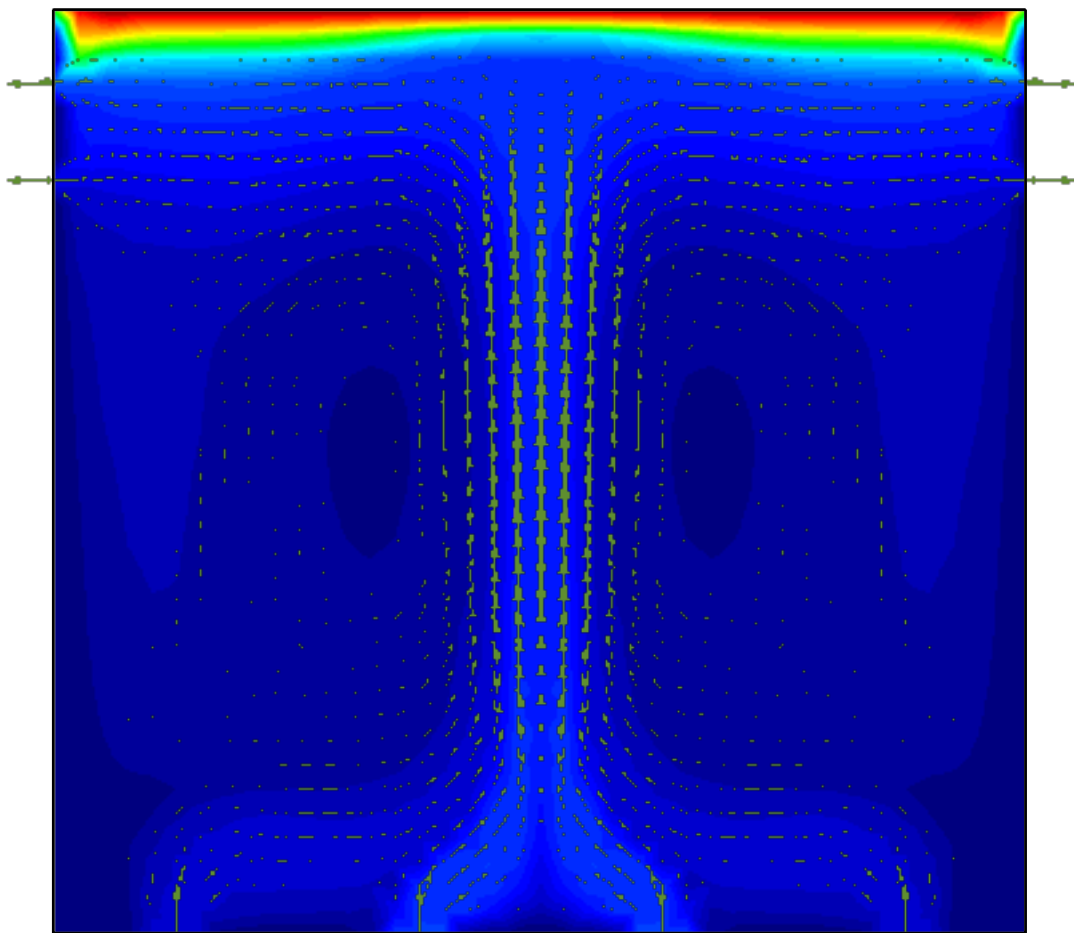
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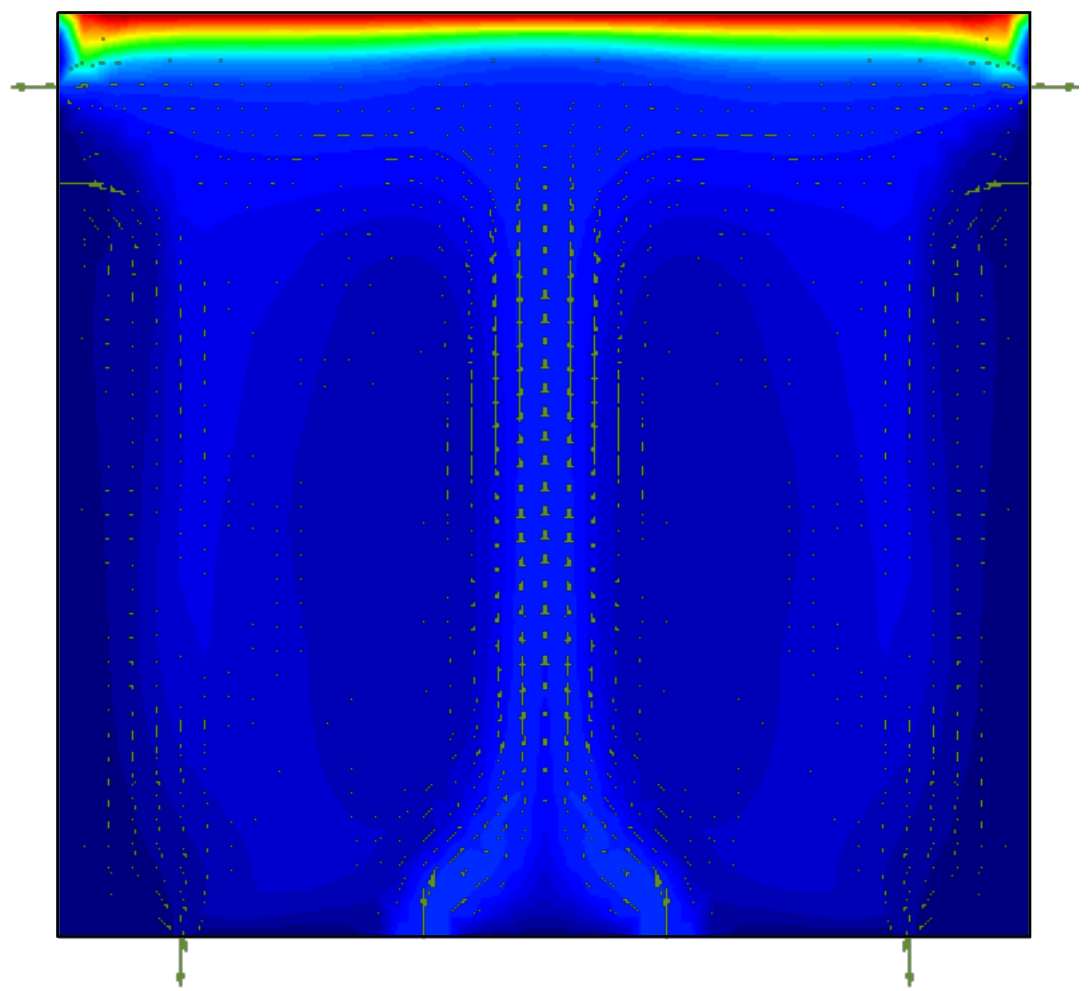
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283.3

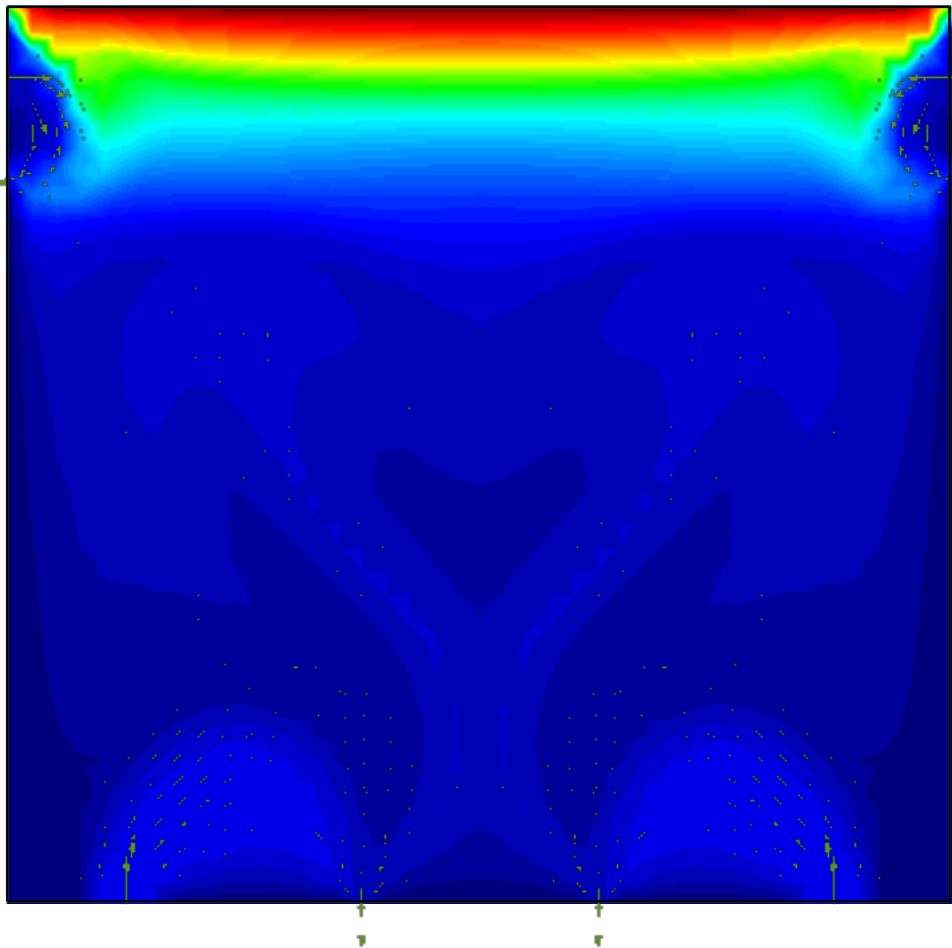
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Dear editor of Journal of Computational and Applied Mathematics:

I wish to submit this revised version of the manuscript “Mathematical analysis and numerical resolution of a heat transfer problem arising in water recirculation”, where we have addressed all the interesting suggestions of the two referees, as listed below in the letters to both reviewers.

We feel confident that with these changes the manuscript could be reconsidered for publication.

Thank you very much.

Sincerely yours,

Lino J. Alvarez-Vázquez

## LETTER TO REVIEWER #1 (LIST OF CHANGES):

We thank you the careful reading of the manuscript and the interesting suggestions on it, which have helped us to improve it. We also thank your comments on the interest of the work.

Following your kind remarks, we have addressed all of your recommendations on modifications and/or corrections in the following way:

- We agree with the referee that the choice of the Smagorinsky approach needs to be justified (in opposition to other classical alternatives as, for instance, the  $k$ - $\epsilon$  model). This has been made in the rewritten second paragraph of the Introduction, where we have included several sentences on the topic.
- With respect to the nonlinear boundary condition imposed on  $\theta$  on the boundary  $\Gamma_S$  in contact with air, we have added an explanation at the beginning of page 5, where we justify the use of this condition, and give a related reference [5] for the combination of convective and radiative heat transfer.
- The first time that it is used (in page 3), the Euclidean measure  $\mu(S)$  is rigorously defined as the area measure of any part of the surface  $\partial\Omega$  (in our case, collectors  $C^k$  and injectors  $T^k$ ). The volume measure of  $\Omega$  (which only appears at the final proof of the Appendix) is now denoted by  $|\Omega|$ , in order to avoid mistakes.
- Following your kind suggestion, we have added an explicit expression (10) for the term  $\Xi$  before the last paragraph of Section 2.
- We agree with the reviewer in the fact that some details on the proof of Theorem 6 should be given. So, we have included a few sentences explaining the main details of the demonstration.
- We also think that the study of some additional regularity results could be interesting (for instance, to know what happens if data or second members are smoother). However, this analysis would exceed the main aim of this paper, enlarging it unnecessarily. In all the results presented here we give the regularity obtained for the solutions of the problem under the minimal necessary regularity conditions (in order to achieve solutions). Obviously, when assuming more regularity on data, more regularity on solutions should be expected.
- Finally, thanks for the indications about the margins of some formulas. In this revised version of the paper we have tried to arrange all of them.

We feel confident that with this new redaction, all the weaknesses and deficiencies of the manuscript have been overcome.

For an easier review, these changes have been highlighted in blue (and in brown for those common to both reviewers).

## LETTER TO REVIEWER #2 (LIST OF CHANGES):

We are very grateful to the interesting and helpful suggestions on the paper, which have greatly helped us to improve the manuscript, making it more clear, complete and readable.

In the following we explain how we address remarks/questions proposed in the review:

- We agree with the reviewer that a more exhaustive state-of-art in the study of the heat/hydrodynamics problems would be necessary, in order to give more relevance to the coupled system considered here. With this aim in mind, we have rewritten the second paragraph of Section 1: Introduction (page 2), including several recent references related to these problems: [3,4,16,18]. Also -following your suggestions- we have added the motivations and difficulties related to the choice of the Smagorinsky turbulence model.
- The nonlinear boundary condition of Stefan-Boltzmann type arising from solar radiation in heat equation (1) has been explained in more detail, and the new related reference [5] has been added in page 5.
- Following your interesting remark, the coupling between the heat equation (1) and the Navier-Stokes system (6) through the thermal expansion coefficient  $\alpha_0$  is now explained at the end of page 5, where a commentary on its importance has been added.
- We also agree with the fact that a more detailed explanation on the previous results of the authors [9] about turbulence model is needed. So, at the end of Section 2, we specify the main conclusions of this previous work, outstanding the additional difficulties for the case considered here. We hope that, with this new redaction, this point is now clearer to readers.
- Definition 5 (now in page 11) has been reformulated, explaining now the details of the proposed variational formulations (30) and (31): substitution of decompositions for  $\theta$  and  $v$  in heat equation (1) and Navier-Stokes system (6), integration by parts after multiplying by suitable test functions, etc. We have also remarked some important points (useful for reasoning in following sections).
- In the first paragraph of Section 4 (now in page 12), we have added a new sentence about the new difficulties appearing here with respect to the proof of existence for previous cases (based on standard fixed-point theory).
- We have clarified, for a better understanding of the end of the proof of Lemma 8 (now in page 16), several details for the derivation of the convergence limits “in the spirit of the techniques used in Appendix A”.

- In order to clarify the second paragraph of Section 5 (related to the meaning of a change in the sign of  $g^k(t)$ ), an explanation about why we restrict our study to the positive case has been added. In fact, we explain that -for the sake of simplicity- we only deal with this case, but that it would be possible to study the generic case (including the negative and the null case) under the only assumption on the fact that the flow rates do not change their sign along each time subinterval. In addition, following the suggestion of the referee, we have referred to following section for the computational details.
- We agree with the referee in the fact that the presentation of the numerical resolution is a little confusing (specially subsection 5.1 devoted to the space-time discretizations). In particular:
  - the notation (52) for the discretized control ( $g^{\{k,n\}}$  representing the value of  $g^k(t_n)$ ) is detailed.
  - the discrete characteristic  $X^0$  is explicitly defined after (53).
  - discrete characteristics  $X^n$  for  $n=1, \dots, N$  are defined when appearing, after (52).
  - a clearer definition of discretized space  $Z^n_h$  is given.
  - Remark 10 is clarified.
- Figure 3 (old Figure 2) represents a standard profile for radiation temperature in the sense that, for the time period corresponding to two days, we can find here the null periods corresponding to two nights, and the two non-null periods of daytime (with maximum peaks at noon). A full explanation is given through new formula (63), added in page 22.
- Following your kind suggestion, we have included in Subsection 5.2 a final paragraph with several remarks on the influence of the different parameters in the numerical results (for instance, the importance of the turbulence term, the role of the radiation term  $T_r$ , the values of parameter  $\alpha_0$ , the possibility of refining the mesh in the neighbourhoods of injectors and collectors, etc.). On the other hand, a comment on the suitable values of parameter  $\varepsilon$  has been already included in previous section (page 19).
- We agree with the reviewer in the fact that Appendix A is a little confusing, and it misses some general information, in the first paragraph, on the general purposes of the whole section. So, following your remarks, we have completely reordered the content of the Appendix in order to make it clearer to readers.
- As the referee comments, in generic system (A.1), the water velocity  $v$  is now a given data, and the other parameters are equal to those in system (1). In this way, the old parameter  $D$  would represent the thermal diffusivity and second member  $f$  the external forcing. However, in order to avoid misunderstandings from the readers, we have now changed the old notations to the same as in the coupled system (1), defining them again for the sake of completeness. We feel confident that that with this new redaction this issue is now clearer to reader, emphasizing the necessity of studying in deep the generic system (A.1), in order to obtain existence/uniqueness/regularity results for the coupled problem.

We hope that with this new redaction, the article becomes more useful and clear to readership.

For an easier review, these changes have been highlighted in red (and in brown for those common to both reviewers).

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