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A note on the rate of convergence for Chebyshev-Lobatto and Radau systems

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Abstract: This paper is devoted to Hermite interpolation with Chebyshev-Lobatto and Chebyshev-Radau nodal points. The aim of this piece of work is to establish the rate of convergence for some types of smooth functions. Although the rate of convergence is similar to that of Lagrange interpolation, taking into account the asymptotic constants that we obtain; the use of this method is justified and it is very suitable when we dispose of the appropriate information. 10

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1 Introduction

The nodal systems related to the Jacobi polynomials play an important role in the theory of Hermite interpolation on the bounded interval, (see [1–3]). The hungarian interpolatory school, beginning with Fejér, has used these systems for Lagrange and Hermite interpolation. Szegő, who is one of the more important references in these subjects, 20 proves for Hermite interpolation that the generalized step polynomials converge to continuous functions uniformly on $[-1 + \varepsilon, 1 - \varepsilon]$, for every $\varepsilon > 0$, when the nodes are the zeros of the Jacobi polynomials, with parameters α and β . Moreover, if $\alpha \geq 0$ and the function is merely continuous in $[-1, 1]$, then the step polynomials are in general divergent at $x = 1$, and a similar result holds for $\beta \geq 0$ and $x = -1$, (see [4]). P. Szász improves these results of convergence by adding the endpoints to the nodal system and by using them as Lagrange data points, 25 (see [5, 6]). The most important, among these nodal systems, is that corresponding to the Chebyshev polynomials of the second kind joined with the endpoints ± 1 . This set of points is usually called Chebyshev-Lobatto nodal system. Other useful systems are the so called Chebyshev-Radau nodal systems, which correspond to the zeros of the Chebyshev polynomials of the third and fourth kind joined with the points -1 and 1 , respectively. Since the derivative at the endpoints is not prescribed, this approach improves the results of convergence but it does not 30 solve a proper Hermite interpolation problem. Nevertheless, Hermite and Hermite-Fejér interpolation problems with extended nodal systems are interesting problems that have been subject of study for several researchers, obtaining algorithms for computing the interpolation polynomials and results of convergence. Indeed barycentric formulas

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presented in [7] were improved in [8] and barycentric methods for more general Hermite interpolation problems can be seen in [9]. The convergence of the Hermite-Fejér process has been proved for continuous functions using the Chebyshev-Lobatto nodal systems and the rate of convergence was obtained in terms of the modulus of continuity, (see [10, 11]). The study of the convergence of the Hermite interpolants for continuous functions using Chebyshev-Lobatto and Chebyshev-Radau nodal systems can be seen in [8]. The technique used in [8] is based on the idea to pass the problem to the unit circle by the Szegő transformation $x = \frac{z+\bar{z}}{2}$, to apply the convergence result of Hermite-Fejér interpolation for continuous functions on the circle given in [12], and then to recover the convergence results for the interpolants on the interval $[-1, 1]$.

Hermite interpolation problems have also been studied with more general nodal systems such as normal and strongly normal point systems, that were introduced by G. Grünwald. The zeros of certain Jacobi polynomials satisfy this last condition. Other important sets of zeros of orthogonal polynomial that were used as nodal points are those corresponding to Legendre and ultraspherical polynomials. In the case of unbounded intervals, some results about convergence of interpolation polynomials were obtained by using as nodes the zeros of orthogonal polynomials with respect to the weights of Hermite, Laguerre, Sonin-Markov and Freud-type.

This note attempts to complete the Hermite interpolation theory with Chebyshev-Lobatto and Chebyshev-Radau systems and in order to extend the results in [8] to another more wide class of functions, we use a new technique in this paper. First we obtain a new representation for the Hermite interpolation polynomials related to the Chebyshev polynomials of the first kind. As a consequence we present some results about the rate of convergence for these extended interpolants when apply to some types of smooth functions. Although the rate of convergence is similar to that of Lagrange interpolation, taking into account the asymptotic constants that we obtain; the use of this method is justified when we have more information in the problem to be solved, that is, if we know the values of the derivatives on the nodal points. Really we have proved that with $2n$ interpolation conditions the rate of convergence is $\mathcal{O}(\frac{1}{(2n)^{s-1}})$, while by using Lagrange interpolation with n interpolation conditions the rate of convergence is $\mathcal{O}(\frac{1}{n^{s-1}})$. In both cases, s is a parameter related to the smoothness of the coefficients of functions represented by Chebyshev series. Hence, when we dispose of the appropriate information the use of this method is very suitable. For example, in the numerical solution of differential equations, if the values of the solution and its derivative in these nodal points are known, this type of interpolation could be applied to rebuild the solution.

2 Chebyshev-Lobatto Hermite interpolation. Rate of convergence for smooth functions

Let us consider the nodal system $\{x_j\}_{j=0}^n = \{\cos \frac{\pi j}{n}\}_{j=0}^n$, that is, $x_0 = 1, x_1, \dots, x_{n-1}$ the zeros of the Chebyshev polynomial of the second kind $U_{n-1}(x)$ and $x_n = -1$. This nodal system is named Chebyshev-Lobatto system and the nodal polynomial is

$$N_{n+1}(x) = U_{n-1}(x)(1-x^2). \quad (1)$$

Now our aim is to obtain results about the rate of convergence when we interpolate some types of smooth functions. So if f is a differentiable function defined on $[-1, 1]$ we denote by $\mathcal{L}_{2n+1}(f, x)$ a polynomial in the space \mathbb{P}_{2n+1} characterized by satisfying

$$\mathcal{L}_{2n+1}(f, x_j) = f(x_j), \quad \mathcal{L}'_{2n+1}(f, x_j) = f'(x_j) \quad \text{for } j = 0, \dots, n. \quad (2)$$

To reach our goal we use some well known results about the Chebyshev polynomials of the first kind $\{T_n\}$ and the second kind $\{U_n\}$, that can be seen in [13, 14].

First we examine the auxiliary polynomials, closely related with the nodal system, and defined by $E_0(x) = \frac{(N_{n+1}(x))^2}{4n^2(x-1)}$ and $E_n(x) = \frac{(N_{n+1}(x))^2}{4n^2(x+1)}$.

Lemma 2.1. *The polynomials $E_0, E_n \in \mathbb{P}_{2n+1}$ satisfy the following properties:*

- (i) $E_0(x_j) = E_n(x_j) = 0$ for $j = 0, \dots, n$.
- (ii) $E'_0(1) = 1, E'_0(x_j) = 0$ for $j = 1, \dots, n$.

(iii) $E'_n(-1) = 1$, $E'_n(x_j) = 0$ for $j = 0, \dots, n-1$.

(iv) If $x \in [-1, 1]$ then $|E_0(x)|$, $|E_n(x)| \leq \frac{1}{2n^2}$.

Proof. (i) - (iii) can be seen in [8].

(iv) If we take into account the definition for E_0 we have

$$E_0(x) = \frac{(U_{n-1}(x)(1-x^2))^2}{4n^2(x-1)} = \frac{\left(\frac{\sin(n \arccos x)}{\sin(\arccos x)}(1-x^2)\right)^2}{4n^2(x-1)} = \frac{(x+1)(\sin(n \arccos x))^2}{4n^2},$$

from which it follows (iv). One can obtain the same bound for E_n proceeding in a similar way. \square 5

The following result establishes a new representation for the interpolation polynomial corresponding to the Chebyshev polynomial of the first kind T_h , when $h \geq 2n$.

Proposition 2.2. *Let h be a natural number, $h = 2n(\ell + 1) + k$, with ℓ and k nonnegative integers, n a positive integer and $0 \leq k \leq 2n - 1$. Then*

$$\mathcal{L}_{2n+1}(T_h, x) = a_k T_k(x) + b_{2n-k} T_{2n-k}(x) + c_0 E_0(x) + d_n E_n(x), \quad (3)$$

where:

(i) If $k \neq 0$ then $a_k = 2 + \ell$, $b_{2n-k} = -1 - \ell$, $c_0 = 4(2n^2 + 3n^2\ell + n^2\ell^2)$ and $d_n = (-1)^{k-1}4(2n^2 + 3n^2\ell + n^2\ell^2)$. 10

(ii) If $k = 0$ then $a_0 = -2\ell - \ell^2$, $b_{2n} = (\ell + 1)^2$, $c_0 = 0$ and $d_n = 0$.

Actually the coefficients depend on h but we omit it in the notation for the sake of simplicity.

Proof. Taking into account that both expressions in the representation (3) belong to \mathbb{P}_{2n+1} , we only have to prove 15 that the expression for $\mathcal{L}_{2n+1}(T_h, x)$ given in (3) fulfills the corresponding interpolation conditions.

(i) Let $k \neq 0$. Now the interpolation conditions for $\mathcal{L}_{2n+1}(T_h, x)$ are:

$$T_h(x_j) = \cos(h \arccos x_j) = \cos\left(\frac{\pi j k}{n}\right), \text{ for } 0 \leq j \leq n,$$

$$T'_h(x_j) = \frac{h \sin(h \arccos x_j)}{\sin \arccos x_j} = \frac{h \sin\left(\frac{\pi j k}{n}\right)}{\sqrt{1-x_j^2}}, \text{ for } 1 \leq j \leq n-1,$$

$$T'_h(1) = h^2 \text{ and } T'_h(-1) = (-1)^{h-1}h^2 = (-1)^{k-1}h^2.$$

On the other hand the next relations hold. 20

For $j \in \{0, \dots, n\}$

$$a_k T_k(x_j) + b_{2n-k} T_{2n-k}(x_j) + c_0 E_0(x_j) + d_n E_n(x_j) = a_k \cos\left(k \frac{\pi j}{n}\right) + b_{2n-k} \cos\left((2n-k) \frac{\pi j}{n}\right) = (a_k + b_{2n-k}) \cos\left(\frac{\pi j k}{n}\right) = \cos\left(\frac{\pi j k}{n}\right).$$

For $j \in \{1, \dots, n-1\}$

$$a_k T'_k(x_j) + b_{2n-k} T'_{2n-k}(x_j) + c_0 E'_0(x_j) + d_n E'_n(x_j) = a_k k \frac{\sin\left(k \frac{\pi j}{n}\right)}{\sqrt{1-x_j^2}} + b_{2n-k} (2n-k) \frac{\sin\left((2n-k) \frac{\pi j}{n}\right)}{\sqrt{1-x_j^2}} = \frac{\sin\left(\frac{\pi j k}{n}\right)}{\sqrt{1-x_j^2}} (ka_k - b_{2n-k}(2n-k)) = \frac{h \sin\left(\frac{\pi j k}{n}\right)}{\sqrt{1-x_j^2}}.$$

For $j = 0$ and $j = n$ it holds

$$a_k T'_k(1) + b_{2n-k} T'_{2n-k}(1) + c_0 E'_0(1) + d_n E'_n(1) = a_k k^2 + b_{2n-k} (2n-k)^2 + c_0 = h^2,$$

$$a_k T'_k(-1) + b_{2n-k} T'_{2n-k}(-1) + c_0 E'_0(-1) + d_n E'_n(-1) = a_k k^2 + b_{2n-k} (2n-k)^2 + d_n = (-1)^{k-1} h^2.$$

Thus for $k \neq 0$ equality (3) is proved.

(ii) When $k = 0$ we have a similar situation. On the one hand we have the interpolation conditions:

$$T_h(x_j) = \cos(h \arccos x_j) = \cos((2 + 2\ell)\pi j) = 1, \text{ for } 0 \leq j \leq n,$$

$$T'_h(x_j) = h \frac{\sin(h \arccos x_j)}{\sin(\arccos x_j)} = h \frac{\sin\left((2n + 2\ell n) \frac{\pi j}{n}\right)}{\sin(\arccos x_j)} = 0, \text{ for } 1 \leq j \leq n - 1,$$

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$$T'_h(1) = h^2 \text{ and } T'_h(-1) = (-1)^{h-1} h^2 = -h^2.$$

On the other hand the following relations hold.

For $j \in \{0, \dots, n\}$

$$a_0 T_0(x_j) + b_{2n} T_{2n}(x_j) = a_0 + b_{2n} \cos\left(2n \frac{\pi j}{n}\right) = a_0 + b_{2n} = 1.$$

For $j \in \{1, \dots, n - 1\}$

$$a_0 T'_0(x_j) + b_{2n} T'_{2n}(x_j) = b_{2n} 2n \frac{\sin(2n \arccos x_j)}{\sin \arccos x_j} = 0.$$

For $j = 0$ and for $j = n$ it holds

$$a_0 T'_0(1) + b_{2n-k} T'_{2n-k}(1) = b_{2n} (2n)^2 = h^2,$$

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$$a_0 T'_0(-1) + b_{2n} T'_{2n}(-1) = -b_{2n} (2n)^2 = -h^2.$$

So for $k = 0$ the statement is also proved. □

Remark 2.3. Notice that the preceding representation is valid for $h \geq 2n$ and for $h = 2n + 1$ it gives an alternative representation of the polynomial $T_{2n+1}(x)$.

Corollary 2.4. Let h be a natural number, $h = 2n(\ell + 1) + k$, with ℓ and k nonnegative integers, n a positive integer and $0 \leq k \leq 2n - 1$. Then

- (i) If $k \neq 0$ it holds that $|\mathcal{L}_{2n+1}(T_h, x)| \leq 4\ell^2 + 14\ell + 11, \forall x \in [-1, 1]$.
- (ii) If $k = 0$ it holds that $|\mathcal{L}_{2n+1}(T_h, x)| \leq 2\ell^2 + 4\ell + 1, \forall x \in [-1, 1]$.

Proof. (i) and (ii) are straightforward consequences of the previous representations. □

Now we are in conditions to study the rate of convergence of the interpolation polynomials for some kind of smooth functions, (see [15]).

Proposition 2.5. Let f be a function defined on $[-1, 1]$ by $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$ with $|a_k| \leq K \frac{1}{k^s}$ for $k \neq 0$, $s \geq 4$ and K a positive constant. Then $\mathcal{L}_{2n+1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ with rate of convergence $\mathcal{O}\left(\frac{1}{n^{s-1}}\right)$.

Proof. We decompose f as follows $f = f_{1,2n+1} + f_{2,2n+1}$ with $f_{1,2n+1} = \sum_{k=0}^{2n+1} a_k T_k$ and $f_{2,2n+1} = \sum_{k=2n+2}^{\infty} a_k T_k$. Since $\mathcal{L}_{2n+1}(f_{1,2n+1}, \cdot) = f_{1,2n+1}$, we study the behavior of $\mathcal{L}_{2n+1}(f_{2,2n+1}, \cdot) - f_{2,2n+1}$.

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If we denote by $H_{n,s}$ the generalized harmonic number defined by $H_{n,s} = \sum_{k=1}^n \frac{1}{k^s}$ and by $H(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ then

$$|f_{2,2n+1}(x)| \leq \sum_{k=2n+2}^{\infty} |a_k| \leq \sum_{k=2n+2}^{\infty} K \frac{1}{k^s} = K(H(s) - H_{2n+1,s}) \leq \frac{K}{(s-1)(2n+1)^{s-1}}, \tag{4}$$

where the last inequality comes from the classical method of the integral for approximating the sum of the series.

Proceeding in a similar way we obtain

$$\begin{aligned} |\mathcal{L}_{2n+1}(f_{2,2n+1}, \cdot)| &\leq \sum_{k=2n+2}^{\infty} |a_h| |\mathcal{L}_{2n+1}(T_h, \cdot)| \leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{2n-1} |a_{2n(\ell+1)+k}| |\mathcal{L}_{2n+1}(T_{2n(\ell+1)+k}, \cdot)| \\ &\leq \underbrace{\sum_{\ell=0}^{\infty} \sum_{k=1}^{2n-1} |a_{2n(\ell+1)+k}| |\mathcal{L}_{2n+1}(T_{2n(\ell+1)+k}, \cdot)|}_{*} + \underbrace{\sum_{\ell=0}^{\infty} |a_{2n(\ell+1)}| |\mathcal{L}_{2n+1}(T_{2n(\ell+1)}, \cdot)|}_{**} \end{aligned} \tag{5}$$

and taking into account the previous corollary we get

$$\begin{aligned} * &\leq \sum_{\ell=0}^{\infty} \sum_{k=1}^{2n-1} \frac{K}{(2n(\ell+1)+k)^s} (4\ell^2 + 14\ell + 11) \leq \sum_{\ell=0}^{\infty} \frac{2nK}{(2n(\ell+1))^s} (4\ell^2 + 14\ell + 11) \leq \\ &\frac{K}{(2n)^{s-1}} \sum_{\ell=0}^{\infty} \frac{4\ell^2 + 14\ell + 11}{(\ell+1)^s} \leq \frac{11K}{(2n)^{s-1}} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)^{s-2}} \end{aligned} \tag{6}$$

and

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$$** \leq \sum_{\ell=0}^{\infty} \frac{K}{(2n(\ell+1))^s} (2\ell^2 + 4\ell + 1) \leq \frac{K}{(2n)^s} \sum_{\ell=0}^{\infty} \frac{2\ell^2 + 4\ell + 1}{(\ell+1)^s} \leq \frac{2K}{(2n)^s} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)^{s-2}}. \tag{7}$$

Hence the interpolation error can be bounded as follows:

$$|f(x) - \mathcal{L}_{2n+1}(f, x)| = |f_{2,2n+1}(x) - \mathcal{L}_{2n+1}(f_{2,2n+1}, x)| \leq |f_{2,2n+1}(x)| + |\mathcal{L}_{2n+1}(f_{2,2n+1}, x)|$$

and using (4), (5), (6) and (7), the result is obtained. □

Remark 2.6. It is well known that the Chebyshev-Fourier coefficients of functions in L^2 converge to zero, and for smooth functions they behave like in Proposition 2.5. Indeed some kind of smooth functions satisfy the preceding requirements. For example, it is easy to conclude that a function with the third derivative of bounded variation on $[-1, 1]$ fulfills the hypothesis of Proposition 2.5. It can also be proved that functions s times continuously differentiable on $[-1, 1]$, with $s \geq 4$, fulfill the hypothesis of the preceding Proposition. Another interesting question is that we can weaken the hypothesis on the parameter s asking only for $s > 3$. Moreover, for infinitely differentiable functions their Chebyshev-Fourier coefficients converge to zero geometrically, that is, exponentially with k .

Remark 2.7. The strategy used in the preceding proposition is different from the one used in [8]; hence by passing the results to the unit circle one can obtain similar results.

Next we study the case of analytic functions on $[-1, 1]$.

Proposition 2.8. If f is an analytic function on $[-1, 1]$, then $\mathcal{L}_{2n+1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ with a geometric rate of convergence.

Proof. Let f be an analytic function on $[-1, 1]$. Then f can be represented as $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$ and it holds that $|a_k| \leq Kr^k$ for some $K > 0$ and $0 < r < 1$, as can be seen in [4]. 20

We decompose f as follows $f = f_{1,2n+1} + f_{2,2n+1}$, with $f_{1,2n+1} = \sum_{k=0}^{2n+1} a_k T_k$ and $f_{2,2n+1} = \sum_{k=2n+2}^{\infty} a_k T_k$. Since $\mathcal{L}_{2n}(f_{1,2n+1}, \cdot) = f_{1,2n+1}$ we study the behavior of $|f_{2,2n+1} - \mathcal{L}_{2n+1}(f_{2,2n+1}, \cdot)|$. Proceeding like in the previous proposition we get

$$|f_{2,2n+1}| \leq \sum_{k=2n+2}^{\infty} |a_k| \leq \frac{K}{1-r} r^{2n+2}, \quad (8)$$

and taking into account Corollary 2.4 it is clear that for $h \geq 2n$ it holds $|\mathcal{L}_{2n+1}(T_h, \cdot)| \leq 4h^2 + 14h + 11$. Therefore

$$|\mathcal{L}_{2n+1}(f_{2,2n+1}, \cdot)| \leq \sum_{h=2n+2}^{\infty} |a_h| |\mathcal{L}_{2n+1}(T_h, \cdot)| \leq \sum_{h=2n+2}^{\infty} K r^h (4h^2 + 14h + 11) = p_2(n) r^{2n+2}, \quad (9)$$

5 where $p_2(n)$ is a well determined polynomial of degree 2. Then by using (8) and (9) the result is proved. \square

Remark 2.9. We want to point out that the results in Propositions 2.5 and 2.8 justify the practical use of these interpolants for smooth functions when we have information about the values of the function and its first derivative on the nodal points.

3 Chebyshev-Radau Hermite interpolation.

10 Rate of convergence for smooth functions

3.1 The case of Chebyshev polynomials of the fourth kind

Let us consider the nodal system $\{x_j\}_{j=0}^{n-1} = \{\cos \frac{2\pi j}{2n-1}\}_{j=0}^{n-1}$, that is, $x_0 = 1$ and x_1, \dots, x_{n-1} the zeros of the Chebyshev polynomial of the fourth kind $W_{n-1}(x)$. Then, the nodal polynomial is

$$M_n(x) = W_{n-1}(x)(1-x).$$

Our aim is to obtain results about the rate of convergence when we interpolate some types of smooth functions. So we consider f a differentiable function defined on $[-1, 1]$ and we denote by $\mathcal{H}_{2n-1}(f, x)$ the interpolation polynomial in the space \mathbb{P}_{2n-1} characterized by satisfying the interpolation conditions

$$\mathcal{H}_{2n-1}(f, x_j) = f(x_j), \quad \mathcal{H}'_{2n-1}(f, x_j) = f'(x_j) \quad \text{for } j = 0, \dots, n-1. \quad (10)$$

15 Next we examine some auxiliary polynomials, closely related with the nodal system.

Lemma 3.1. The polynomial $D_0(x) = \frac{(M_n(x))^2}{(2n-1)^2(x-1)} \in \mathbb{P}_{2n-1}$ satisfies the following properties:

- (i) $D_0(x_j) = 0$ for $j = 0, \dots, n-1$.
- (ii) $D'_0(1) = 1$, $D'_0(x_j) = 0$ for $j = 1, \dots, n-1$.
- (iii) If $x \in [-1, 1]$ then it holds $|D_0(x)| \leq \frac{4}{(2n-1)^2}$.

20 *Proof.* (i) It is immediate.

(ii) It can be seen in [8].

(iii) It is a straightforward consequence of the definition of D_0 and W_{n-1} , (see [13, 14]):

$$D_0(x) = \frac{(M_n(x))^2}{(2n-1)^2(x-1)} = \frac{(W_{n-1}(x))^2(x-1)}{(2n-1)^2} = \frac{2 \sin^2(n \frac{1}{2} \arccos x)(x-1)}{(2n-1)^2}. \quad \square$$

Next we obtain a new representation of the interpolation polynomials related to the Chebyshev polynomials of the first kind.

Proposition 3.2. *Let h be a natural number $h = (2n - 1)(\ell + 1) + k$ with n a positive integer, ℓ and k nonnegative integers and $0 \leq k < 2n - 1$. If $T_h(x)$ is the Chebyshev polynomial of degree h then $\mathcal{H}_{2n-1}(T_h, x)$ can be represented as:*

$$\mathcal{H}_{2n-1}(T_h, x) = a_k T_k(x) + b_{2n-1-k} T_{2n-1-k}(x) + c_0 D_0(x), \quad (11)$$

where:

- (i) If $k \neq 0$ then $a_k = 2 + \ell$, $b_{2n-1-k} = -1 - \ell$ and $c_0 = (2 + 3\ell + \ell^2)(2n - 1)^2$.
- (ii) If $k = 0$ then $a_0 = -2\ell - \ell^2$, $b_{2n-1} = (\ell + 1)^2$ and $c_0 = 0$.

Actually the coefficients depend on h but we omit it in the notation for the sake of simplicity.

Proof. Taking into account that both representations in (11) belong to \mathbb{P}_{2n-1} , we only have to prove that $\mathcal{H}_{2n-1}(T_h, x)$ given in (11) fulfills the corresponding interpolations conditions.

- (i) If $k \neq 0$, on the one hand we have that the interpolation conditions are

$$T_h(x_j) = \cos(h \arccos x_j) = \cos\left(\left((2n - 1)(\ell + 1) + k\right) \frac{2\pi j}{2n - 1}\right) = \cos\left(\frac{2\pi j k}{2n - 1}\right), \quad \text{for } 0 \leq j \leq n,$$

$$T'_h(x_j) = h \frac{\sin(h \arccos x_j)}{\sin \arccos x_j} = h \frac{\sin\left(\left((2n - 1)(\ell + 1) + k\right) \frac{2\pi j}{2n - 1}\right)}{\sin \arccos x_j} = h \frac{\sin\left(\frac{2\pi j k}{2n - 1}\right)}{\sqrt{1 - x_j^2}}, \quad \text{for } 0 \leq j \leq n - 1,$$

$$T'_h(x_0) = T'_h(1) = h^2.$$

On the other hand we have:

For $j \in \{0, \dots, n\}$

$$\begin{aligned} & a_k T_k(x_j) + b_{2n-1-k} T_{2n-1-k}(x_j) + c_0 D_0(x_j) = \\ & a_k \cos\left(k \frac{2\pi j}{2n - 1}\right) + b_{2n-1-k} \cos\left((2n - 1 - k) \frac{2\pi j}{2n - 1}\right) = \\ & (a_k + b_{2n-1-k}) \cos\left(\frac{2\pi j k}{2n - 1}\right) = \cos\left(\frac{2\pi j k}{2n - 1}\right). \end{aligned}$$

For $j \in \{1, \dots, n - 1\}$

$$\begin{aligned} & a_k T'_k(x_j) + b_{2n-1-k} T'_{2n-1-k}(x_j) + c_0 D'_0(x_j) = \\ & a_k k \frac{\sin(k \arccos x_j)}{\sin \arccos x_j} + b_{2n-1-k} (2n - 1 - k) \frac{\sin((2n - 1 - k) \arccos x_j)}{\sin \arccos x_j} = \\ & a_k k \frac{\sin\left(k \frac{2\pi j}{2n - 1}\right)}{\sqrt{1 - x_j^2}} + b_{2n-1-k} (2n - 1 - k) \frac{\sin\left((2n - 1 - k) \frac{2\pi j}{2n - 1}\right)}{\sqrt{1 - x_j^2}} = \\ & \frac{\sin\left(\frac{2\pi j k}{2n - 1}\right)}{\sqrt{1 - x_j^2}} (a_k k - b_{2n-1-k} (2n - 1 - k)) = h \frac{\sin\left(\frac{2\pi j k}{2n - 1}\right)}{\sqrt{1 - x_j^2}}. \end{aligned}$$

For $j = 0$ it holds

$$a_k T'_k(x_0) + b_{2n-1-k} T'_{2n-1-k}(x_0) + c_0 D'_0(x_0) = a_k k^2 + b_{2n-1-k} (2n - 1 - k)^2 + c_0 = h^2.$$

Hence, for $k \neq 0$, we have proved expression (11).

- (ii) When $k = 0$ we have a similar situation. On the one hand we get:

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$$T_h(x_j) = \cos(h \arccos x_j) = \cos\left((2n-1)(\ell+1)\frac{2\pi j}{2n-1}\right) = 1, \text{ for } 0 \leq j \leq n,$$

$$T'_h(x_j) = h \frac{\sin(h \arccos x_j)}{\sin \arccos x_j} = h \frac{\sin\left((2n-1)(\ell+1)\frac{2\pi j}{2n-1}\right)}{\sin \arccos x_j} = 0, \text{ for } 0 \leq j \leq n-1,$$

$$T'_h(x_0) = T'_h(1) = h^2.$$

On the other hand the following relations hold.

For $j \in \{0, \dots, n\}$

$$\begin{aligned} a_0 T_0(x_j) + b_{2n-1} T_{2n-1}(x_j) &= a_0 + b_{2n-1} \cos((2n-1) \arccos x_j) = \\ &= a_0 + b_{2n-1} \cos\left((2n-1)\frac{2\pi j}{2n-1}\right) = a_0 + b_{2n-1} = 1. \end{aligned}$$

5 For $j \in \{1, \dots, n-1\}$

$$a_0 T'_0(x_j) + b_{2n-1} T'_{2n-1}(x_j) = b_{2n-1}(2n-1) \frac{\sin((2n-1) \arccos x_j)}{\sin \arccos x_j} = 0.$$

For $j = 0$

$$a_0 T'_0(1) + b_{2n-1} T'_{2n-1}(1) = b_{2n-1}(2n-1)^2 = h^2.$$

So, for $k = 0$, the statement has also been proved. □

Corollary 3.3. Let h be $h = (2n-1)(\ell+1) + k$ with n a positive integer, ℓ and k nonnegative integers and $0 \leq k < 2n-1$. Then we have:

- 10 (i) If $k \neq 0$ then $|\mathcal{H}_{2n-1}(T_h, x)| \leq 4\ell^2 + 14\ell + 11 \forall x \in [-1, 1]$.
 (ii) If $k = 0$ then $|\mathcal{H}_{2n-1}(T_h, x)| \leq 2\ell^2 + 4\ell + 1 \forall x \in [-1, 1]$.

Proof. Both statements are straightforward consequences of the previous representations. □

Now we are in conditions for proving our main results concerning with some kind of smooth functions.

Proposition 3.4. Let f be a function defined for $x \in [-1, 1]$ by $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$ with $|a_k| \leq K \frac{1}{k^s}$ for $k \neq 0$
 15 and $s \geq 4$. Then $\mathcal{H}_{2n-1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ and the rate of convergence is $\mathcal{O}(\frac{1}{n^{s-1}})$.

Proof. By using the same technique as in Proposition 2.5, we can write $f = f_{1,2n-1} + f_{2,2n-1}$ with $f_{1,2n-1} = \sum_{k=0}^{2n-1} a_k T_k$ and $f_{2,2n-1} = \sum_{k=2n}^{\infty} a_k T_k$.

Since $\mathcal{H}_{2n-1}(f_{1,2n-1}, \cdot) = f_{1,2n-1}$, we study the behavior of $\mathcal{H}_{2n-1}(f_{2,2n-1}, \cdot) - f_{2,2n-1}$. If we denote by $H_{n,s}$ the generalized harmonic number defined by $H_{n,s} = \sum_{k=1}^n \frac{1}{k^s}$ and by $H(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ then

$$|f_{2,2n-1}(x)| \leq \sum_{k=2n}^{\infty} |a_k| \leq \sum_{k=2n}^{\infty} K \frac{1}{k^s} = K (H(s) - H_{2n-1,s}) \leq \frac{K}{(s-1)(2n-1)^{s-1}}. \tag{12}$$

20 Proceeding in a similar way

$$|\mathcal{H}_{2n-1}(f_{2,2n-1}, \cdot)| \leq$$

$$\begin{aligned} \sum_{h=2n}^{\infty} |a_h| |\mathcal{H}_{2n-1}(T_h, \cdot)| &\leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{2n-2} |a_{2n-1+\ell(2n-1)+k}| |\mathcal{H}_{2n-1}(T_{2n-1+\ell(2n-1)+k}, \cdot)| \leq \\ &\underbrace{\sum_{\ell=0}^{\infty} \sum_{k=1}^{2n-2} |a_{2n-1+\ell(2n-1)+k}| |\mathcal{H}_{2n-1}(T_{2n-1+\ell(2n-1)+k}, \cdot)|}_{*} + \\ &\underbrace{\sum_{\ell=0}^{\infty} |a_{2n-1+\ell(2n-1)}| |\mathcal{H}_{2n-1}(T_{2n-1+\ell(2n-1)}, \cdot)|}_{**} \end{aligned} \quad (13)$$

and taking into account the previous corollary we obtain

$$\begin{aligned} * &\leq \sum_{\ell=0}^{\infty} \sum_{k=1}^{2n-2} \frac{K}{((2n-1)(\ell+1)+k)^s} (4\ell^2 + 14\ell + 11) \leq \\ &\sum_{\ell=0}^{\infty} \frac{(2n-1)K}{((2n-1)(\ell+1))^s} (4\ell^2 + 14\ell + 11) \leq \frac{K}{(2n-1)^{s-1}} \sum_{\ell=0}^{\infty} \frac{4\ell^2 + 14\ell + 11}{(\ell+1)^s} \leq \\ &\frac{11K}{(2n-1)^{s-1}} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)^{s-2}} \end{aligned} \quad (14)$$

and

$$\begin{aligned} ** &\leq \sum_{\ell=0}^{\infty} \frac{K}{((2n-1)(\ell+1))^s} (2\ell^2 + 4\ell + 1) \leq \frac{K}{(2n-1)^s} \sum_{\ell=0}^{\infty} \frac{2\ell^2 + 4\ell + 1}{(\ell+1)^s} \leq \\ &\frac{2K}{(2n-1)^s} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)^{s-2}}. \end{aligned} \quad (15)$$

Therefore, taking into account

$$|f(x) - \mathcal{H}_{2n-1}(f, x)| = |f_{2,2n-1}(x) - \mathcal{H}_{2n-1}(f_{2,2n-1}, x)| \leq |f_{2,2n-1}(x)| + |\mathcal{H}_{2n-1}(f_{2,2n-1}, x)|,$$

if we use (12), (13), (14) and (15), the result is obtained. □

Next we study the case of analytic functions on $[-1, 1]$.

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Proposition 3.5. *Let f be an analytic function on $[-1, 1]$. Then $\mathcal{H}_{2n-1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ with a geometric rate of convergence.*

Proof. Proceeding like in Proposition 2.8, f can be represented as $f = \sum_{k=0}^{\infty} a_k T_k$, with $|a_k| \leq Kr^k$ for some $K > 0$ and $0 < r < 1$. We decompose f as follows $f = f_{1,2n-1} + f_{2,2n-1}$ with $f_{1,2n-1} = \sum_{k=0}^{2n-1} a_k T_k$ and $f_{2,2n-1} = \sum_{k=2n}^{\infty} a_k T_k$. Since $\mathcal{H}_{2n-1}(f_{1,2n-1}, \cdot) = f_{1,2n-1}$ we study the behavior of $|f_{2,2n-1} - \mathcal{H}_{2n-1}(f_{2,2n-1}, \cdot)|$. Thus, we have

$$|f_{2,2n-1}| = \left| \sum_{k=2n}^{\infty} a_k T_k \right| \leq \sum_{k=2n}^{\infty} |a_k| \leq \frac{K}{1-r} r^{2n}, \quad (16)$$

and

$$|\mathcal{H}_{2n-1}(f_{2,2n-1}, \cdot)| = \left| \sum_{k=2n}^{\infty} a_k \mathcal{H}_{2n-1}(T_k, \cdot) \right| \leq \sum_{k=2n}^{\infty} |a_k| |\mathcal{H}_{2n-1}(T_k, \cdot)| \leq$$

$$\sum_{k=2n}^{\infty} Kr^k(4k^2 + 14k + 11) = p_2(n)r^{2n}, \quad (17)$$

where $p_2(n)$ denotes a polynomial of degree 2.

Hence using (16) and (17) the result is proved. \square

3.2 The case of Chebyshev polynomials of the third kind

Let us consider the nodal system $\{x_j\}_{j=1}^n = \{\cos \frac{(2j-1)\pi}{2n-1}\}_{j=1}^n$, that is, x_1, \dots, x_{n-1} are the zeros of the Chebyshev polynomial of the third kind $V_{n-1}(x)$ and $x_n = -1$. Then the nodal polynomial is $V_{n-1}(x)(1+x)$.

If f is a differentiable function defined on $[-1, 1]$, we denote by $\mathcal{K}_{2n-1}(f, x)$ the interpolation polynomial in the space \mathbb{P}_{2n-1} characterized by satisfying the interpolation conditions

$$\mathcal{K}_{2n-1}(f, x_j) = f(x_j), \quad \mathcal{K}'_{2n-1}(f, x_j) = f'(x_j) \quad \text{for } j = 1, \dots, n.$$

It is clear that all the asserts related with the extended nodal system corresponding to W_{n-1} can be reproduced with the extended nodal system corresponding to V_{n-1} . The Hermite interpolation polynomial corresponding to a smooth function $f(x)$ with the extended nodal system of V_{n-1} is the Hermite interpolation polynomial corresponding to a smooth function $g(x) = f(-x)$ on the extended nodal system corresponding to W_{n-1} and viceversa.

Indeed it is easy to obtain that if h is a natural number $h = (2n-1)(\ell+1) + k$ with n a positive integer, ℓ and k nonnegative integers and $0 \leq k < 2n-1$, then $\mathcal{K}_{2n-1}(T_h, x)$ can be represented as:

$$\mathcal{K}_{2n-1}(T_h, x) = \mathcal{H}_{2n-1}(T_h(-x), x) = a_k(-1)^k T_k(x) + b_{2n-1-k}(-1)^{2n-1-k} T_{2n-1-k}(x) + c_0 D_0(-x),$$

where the sequences of coefficients are given in Proposition 3.2.

Moreover, proceeding like in the previous subsection one can obtain similar results as those in Propositions 3.4 and 3.5 in a straight way as follows.

Proposition 3.6. *Let f be a function defined for $x \in [-1, 1]$ by $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$ with $|a_k| \leq K \frac{1}{k^s}$ for $k \neq 0$ and $s \geq 4$. Then $\mathcal{K}_{2n-1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ and the rate of convergence is $\mathcal{O}(\frac{1}{n^{s-1}})$.*

Proposition 3.7. *Let f be an analytic function on $[-1, 1]$. Then $\mathcal{K}_{2n-1}(f, \cdot)$ uniformly converges to f on $[-1, 1]$ with a geometric rate of convergence.*

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References

- [1] Davis, P.J., Interpolation and Approximation, Dover Publications, New York, 1975
- [2] Henrici, P., Essentials of Numerical Analysis with Pocket Calculator Demonstrations, Wiley, New York, 1982
- [3] Szabados, J., Vértesi, P., Interpolation of Functions, World Scientific, Singapore, 1990
- [4] Szegő, G., Orthogonal Polynomials, Amer. Math. Soc. Coll. Publ., Vol. 23, 4th ed., Amer. Math. Soc., Providence, 1975
- [5] Szász, P., On quasi-Hermite-Fejér interpolation, Acta Math. Acad. Sci. Hungar., 1959, 10, 413-439
- [6] Szász, P., The extended Hermite-Fejér interpolation formula with application to the theory of generalized almost-step parabolas, Publ. Math. Debrecen, 1964, 11, 85-100
- [7] Schneider, C., Werner, W., Hermite Interpolation: The Barycentric Approach, Computing, 1991, 46, 35-51
- [8] Berriochoa, E., Cachafeiro, A., Díaz, J., Illán, J., Algorithms and convergence for Hermite interpolation based on extended Chebyshev nodal systems, Appl. Math. Comput., 2014, 234, 223-236
- [9] Corless, R.M., Fillion, N., A Graduate Introduction to Numerical methods, Springer, New York, 2013
- [10] Bojanic, R., Prasad, J., Saxena, R.B., An upper bound for the rate of convergence of the Hermite-Fejér process on the extended Chebyshev nodes of the second kind, J. Approx. Theory, 1979, 26, 195-203

- [11] Saxena, R.B., The Hermite-Fejér process on the Tchebycheff matrix of second kind, Stud. Sci. Math. Hungar., 1974, 9, 223-232
- [12] Darius, L., González-Vera, P., A Note on Hermite-Fejér Interpolation for the Unit Circle, Appl. Math. Letters, 2001, 14, 997-1003
- [13] Mason, J.C., Handscomb, D.C., Chebyshev Polynomials, Chapman & Hall/CRC, Boca Raton, 2003
- [14] Rivlin, T., The Chebyshev Polynomials, Pure and Applied Mathematics, John Wiley & Sons, New York, 1974
- [15] Trefethen, L.N., Approximation Theory and Approximation practice, SIAM, Philadelphia, 2013