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GOODNESS OF CYCLES AND HAMILTONIAN PATHS

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Dedicated with a lot of love to our friend Elena Martín Peinador and our friend and mentor Eusebio Corbacho Rosas on the occassion of their retirements

1. Introduction

The problem of finding an optimal Hamiltonian cycle on a set of complex plane points (TSP) is a well-known open problem. If the number of points is small, we can simply calculate all the possible cycles and stay with a shorter one, but this strategy is not viable when the number of points is large. Historically the problem has been addressed by building partitions of the plane containing a small number of vertices, in these subsets it is easy to determine optimal cycles and finally design appropriate strategies to paste the previous paths thus obtaining an approximation of the solution to the problem posed (see [4])

We have designed (see [1]) an algorithm that, in a reasonable time and in a personal computer, gives us a solution to the problem posed although the number of points is large. In this algorithm we propose a different strategy, we consider a partition of the set of points V in its different levels of convexity $\{V_1, V_2, \cdots, V_p\}$, thinking that the geometry of the distribution of the points would facilitate the connection between the different elements of the partition.

If $co(V)$ is the convex envelope of V and $\delta(co(V))$ is its boundary, we define

$$V_1 = V \cap \delta(co(V)),$$
$$V_2 = V \cap \delta(co(V \setminus V_1)),$$
$$V_3 = V \cap \delta(co(V \setminus (V_1 \cup V_2)),$$
$$\cdots$$
$$V_p = V \cap \delta(co(L \setminus (V_1 \cup \cdots \cup V_{p-1})).$$

The basic ideas of this approach are:

(1) If $V = V_1$, it would be enough to fix an orientation on the border and, according to it, order the points of V. In this way we would construct a Hamiltonian path closed in V of minimum length since it is the only one that has no crosses.
(2) If there is a $z_0 \in V$ such that $V\prime = V \setminus \{z_0\}$ has the property expressed in 1, we can build an optimal Hamiltonian cycle in $V\prime$ and replace the most appropriate edge $z_i z_{i+1}$ for the polygon $z_i z_0 z_{i+1}$ in such a way that the

new cycle, which obviously would remain Hamiltonian, will increase its length as little as possible.

(3) If there were a $\{z_0, \cdots, z_n\} \subset V$ such that $V\prime = V \setminus \{z_0, \cdots, z_n\}$ had the property expressed in 1, we could consider the optimal Hamiltonian cycle in $V\prime$ and study the appropriate order to incorporate the points z_i to this cycle so that the Hamiltonian condition is preserved and its length grows as little as possible.

In any set V you can find your partition in convexity levels $\{V_1, \cdots, V_p\}$ and to the existing Hamiltonian path in V_1 to incorporate the points of V_2 according to the 3 heuristic. To the Hamiltonian cycle thus formed that we designate V_{12}, we incorporate the points of V_3 for the same procedure, and so on, until we reach the cycle $V_{1 \cdots p}$.

We implemented this algorithm in Sage (Software for Algebra and Geometry Experimentation) which is a mathematical software of free access, that has several downloadable versions through its website compatible with the operating systems more common: Windows, Linux, MacOS.

We have applied this algorithm to several concrete problems among which we highlight: Design of optimal routes through the cities of peninsular Spain, or a tour of various countries of the EEC (see [1],[3]), study of the Farnsworth-Munsell colorimetry test (see [2], [5]).

The appropriate framework to address these problems is graphs with weights. A weighted graph is a triplet $G = (V, E, w)$, where V denotes the set of vertices, E the set of edges and $w : V \times V \to \mathbb{R}$ a symmetric kernel verifying: $w(i, j) > 0$ if $(i, j) \in E$ and $w(i, j) = 0$ in another case. The graph is simple if $w(i, i) = 0, \forall i \in V$.

If $|V| = n$ and $|E| = m$ we will say that $\Gamma = [i_1, i_2, \ldots, i_{n+1}]$ is a Hamiltonian cycle in G if $(i_k, i_{k+1}) \in E, \forall k = 1, 2, \cdots, n$, $i_{n+1} = i_1$ and $\forall j \in V$, $\exists k | j = i_k$. A graph is called Hamiltonian if it admits a Hamiltonian cycle. We denote by HCG the set of all Hamiltonian cycles of G.

We call weight of $\Gamma = [i_1, i_2, \ldots, i_{n+1}]$ to number

$$w\Gamma = \sum_{k=1}^{n} w(i_k, i_{k+1}).$$

Let G be a Hamiltonian graph, denoted by

$$MHCG = \min\{w\Gamma \in \mathbb{R} | \Gamma \in HCG\}$$

Γ_0 is an optimal cycle on G if $w\Gamma_0 = MHCG$.

The calculation of an optimal cycle (TSP) is a classic NP-hard problem, but there are different algorithms that give us good approximations as we mentioned before. The natural problem that we propose is to determine the goodness of an approximation Γ_0, calculated with one of these methods. The classic technique used is to compare its weight with the weight of a minimum spaning tree ($wMST$), which we can easily calculate using the Dijkstra algorithm. The ultimate reason for this procedure is that by deleting any edge of Γ_0 we get a tree and so we have

$$wMST < w\Gamma_0$$

The $wMST$ is therefore a lower bound of $MHCG$, of easy calculation, but not too natural to the problem that we pose, since in a cycle all the vertices are of order 2 and a minimum expansion tree, although it can have vertices of order 2, it also contains vertices of order 1 and can contain vertices of order 3, 4 or even higher order. We set out to obtain a good lower bound of $MHCG$ more natural than $wMST$ that allows us to calibrate the goodness of a cycle Γ_0 on a graph G.

2. Goodness of a Hamiltonian Cycle

Let $G = (V, E, w)$ be a simple Hamiltonian graph, $|V| = n$ and $\Gamma = [i_1, i_2, \ldots, i_{n+1}]$ a cycle in G of weight $w\Gamma = \sum_{k=1}^{n} w(i_k, i_{k+1})$. We consider the lists

$$L_i = sorted[w(i,j) \in \mathbb{R} | w(i,j) > 0,\ j = 1, 2, \ldots, n],\ i = 1, 2, \ldots, n$$

ordered from lowest to highest, it is obvious that L_i is not empty, even more $|L_i| \geq 2$ as a list, although not necessarily as a set, because G is Hamiltonian. We denote by $l_i^1 = L_i[1]$ and $l_i^2 = L_i[2]$ the first and second elements of L_i respectively. We define the lower bound of the weights cycles of G

$$LBHCG = \sum_{i=1}^{n} \frac{l_i^1 + l_i^2}{2}$$

We have the following result.

Theorem 1. *Let $G = (V, E, w)$ be a simple Hamiltonian graph and $\Gamma \in HCG$. It is verified*

$$LBHCG \leq w\Gamma$$

Proof
Giving $\Gamma \in HCG$, $\Gamma = [i_1, i_2, \cdots, i_{n+1}]$ we have

$$2w\Gamma = \sum_{k=1}^{n} w(i_{k-1}, i_k) + w(i_k, i_{k+1}) \geq \sum_{k=1}^{n} (l_i^1 + l_i^2)$$

Taking $i_0 = i_n$, and so we obtain

$$w\Gamma \geq \sum_{i=1}^{n} \frac{l_i^1 + l_i^2}{2} = LBHCG$$

Corollary 2. *Let $G = (V, E, w)$ be a simple Hamiltonian graph and $\Gamma_0 \in HCG$ verifying*

$$w\Gamma_0 = LBHCG$$

So Γ_0 is optimal.

Proof
As $LBHCG \leq MHCG \leq w\Gamma_0$ from the hypothesis follows

$$LBHCG = MHCG = w\Gamma_0$$

and so Γ_0 is optimal.

Example 3. Let $G = (V, E, w)$ be the complete simple metric graph over the set of vertices

$$V = \{(k, l) \in \mathbb{R}^2 \mid k, l \in \mathbb{N},\ 1 \leq k, l \leq 20\}$$

The cycle Γ_0 of the Figure 1 is optimal.

It is obvious that $l_i^1 = l_i^2 = 1,\ \forall i = 1, 2, \ldots, 400$ and so

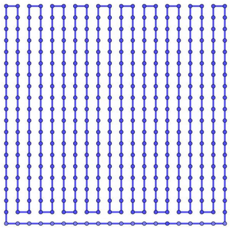

FIGURE 1

$$LBHCG = 400 = MHCG = w\Gamma_0$$

and therefore Γ_0 is optimal.

Example 4. Let $G = (V, E, w)$ be the simple complete metric graph with vertices on the set of 20 evenly distributed points on the unit circumference, which we order in an anti-clockwise way, the first one being $(1, 0)$. The cycle $\Gamma_0 = [1, 2, \ldots, 20, 1]$ of the Figure 2 is optimal on G.

Denoting by l the side length of the polygon of 20 sides inscribed in the unit circumference, it is obvious that $l_i^1 = l_i^2 = l,\ \forall i = 1, 2, \ldots, 20$ and so

$$LBHCG = 20l = MHCG = w\Gamma_0$$

From the last identity it follows that Γ_0 is optimal.

Remark 5. $LBHCG$ is a lower bound of $MHCG$ more natural than $wMST$ because it does not measure more than the weight of a cycle of the lowest possible weight, without prejudice to whether it can be or not be constructed on G. $LBHCG$ is therefore more natural than $wMST$ to calibrate the goodness of a cycle Γ on G.

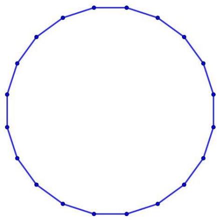

FIGURE 2

To calculate the goodness of a cycle Γ on G we can think of calculating the difference $w\Gamma - MHCG$ but as the calculation of $MHCG$ is also very complex we can approximate this difference by $w\Gamma - LBHCG$ or better yet by the relative difference

$$\frac{w\Gamma - LBHCG}{LBHCG}$$

and define the goodness of Γ as

$$\mathcal{G}\Gamma = 1 - \frac{w\Gamma - LBHCG}{LBHCG}$$

Thus a cycle Γ will be good if $\mathcal{G}\Gamma \sim 1$. If $\mathcal{G}\Gamma = 1$ the cycle is optimal, but a cycle can be optimal and $\mathcal{G}\Gamma < 1$.

The concept of goodness that we have just presented is related to the one introduced in [5], although they do not coincide.

Example 6. Using our algorithm we calculate a Hamiltonian cycle for the 58 main cities of the Iberian Peninsula as shown in Figure 3. The goodness of the obtained cycle is 0.8525.

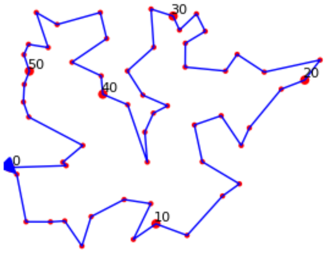

FIGURE 3

We can use the same idea to study Hamiltonian paths.

3. Goodness of a Hamiltonian Path

Let $G = (V, E, w)$ be a connected graph with $|V| = n$ and $|E| = m$ let us say that $P = [i_0, i_1, \ldots, i_{n-2}, j_0]$ is a Hamiltonian path with beginning at i_0 and ending in j_0 if $\forall j \in V \setminus \{i_0, j_0\},\ \exists k = 1, \ldots, n-2 \,|\, j = i_k$. We call weight of P at

$$wP = \sum_{k=0}^{n-3} w(i_k, i_{k+1}) + w(i_{n-2}, j_0).$$

If $G = (V, E, w)$ is a graph that admits a Hamiltonian path with extremes in i_1 and i_n, we denote by

$$HPG_{i_1 i_n} = \{P = [i_1, i_2, \cdots, i_n] | P \text{ is a path on G of extremes } i_1\, i_n\} \text{ and by}$$

$$MHPG_{i_1 i_n} = min\{wP \in \mathbb{R} | P \in HPG_{i_1 i_n}\}$$

A path of extremes i_1, i_n, $P = [i_1, i_2, \cdots, i_n]$, it is said optimal if $wP = MHPG_{i_1 i_n}$.

We define the lower bound of the weights of the Hamiltonian paths with start and end at the vertices i_0 and j_0 respectively as

$$LBHPG_{i_0 j_0} = l^1_{i_0} + l^1_{j_0} + \sum_{k=1, k \neq i_0, k \neq j_0}^{n} \frac{l^1_k + l^2_k}{2}$$

where l^j_k are the constants constructed for the calculation of the $LBHCG$ level.

The following result, for Hamiltonian paths, is analogous to Theorem 1.

Theorem 7. *Let $G = (V, E, w)$ be a simple connected graph and $P = [i_1, i_2, \cdots, i_n]$ a Hamiltonian path on G. It is verified*

$$LBHPG_{i_1 i_n} \leq wP$$

Proof
Proceeding as in Theorem 1, we have

$$2wP = \sum_{k=2}^{n-2} (w(i_{k-1}, i_k) + w(i_k, i_{k+1})) + w(i_1, i_2) + w(i_{n-1}, i_n) \geq$$

$$\sum_{i_k, i_k \neq i_1, i_k \neq i_n} (l^1_{i_k} + l^2_{i_k}) + 2l^1_{i_1} + 2l^1_{i_n} = 2LBHPG_{i_1 i_n}.$$

Corollary 8. *Let $G = (V, E, w)$ be a simple connected graph and $P_0 = [i_1, i_2, \cdots, i_n]$ a Hamiltonian path on G. If*

$$wP_0 = LBHPG_{i_1 i_n}$$

Then P_0 is optimal.

Proof
As $LBHPG_{i_1 i_n} \leq MHPG_{i_1 i_n} \leq wP$ for any path P, from the hypothesis follows $wP_0 = LBHPG_{i_1 i_n} = MHPG_{i_1 i_n}$ and thus P_0 is optimal.

Example 9. From the previous corollary it follows that the Hamiltonian path of Figure 4 is optimal over the simple connected metric graph determined by the vertices of the figure. because it is optimal in each vertex.

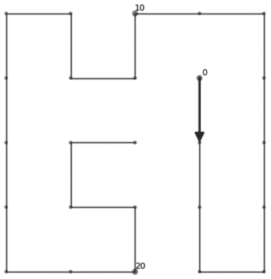

FIGURE 4

Determining an optimal path in a connected graph $G = (V, E, w)$ is a problem of difficult resolution, but in analogy to the study of the optimality of Hamiltonian cycles we can define the goodness of a Hamiltonian path with the extremes i_1, i_n as

$$\mathcal{G}P = 1 - \frac{wP - LBHPG_{i_1 i_n}}{LBHPG_{i_1 i_n}}$$

Thus a Hamiltonian path P will be good if $\mathcal{G}P \sim 1$. If $\mathcal{G}P = 1$ the path is optimal, but a path can be optimal and $\mathcal{G}P < 1$.

Example 10. In Figure 5 we represent the layout of the French Way of Santiago de Compostela from León. It is obvious that this path is optimal at each vertex and therefore optimal and has goodness 1.

FIGURE 5

We can extend this idea to non-Hamiltonian graphs but this will be treated in a future work.

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