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Universal central extensions of braided crossed modules of groups

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ABSTRACT

In this paper, we give a canonical braiding on the universal central extension of a crossed module with a given braiding in the category of crossed modules. We also construct the universal central extension of a braided crossed module in the category of braided crossed modules, showing that when one of these constructions exists, both of them exist and coincide.

0. Introduction

The concept of central extension of groups is highly relevant in mathematics, for instance, in the interpretation of the third cohomology, and it plays a fundamental role in several areas of physics as well, for instance, in the quantization of symmetries. This notion was extended to crossed modules of groups. The study of central extensions in the categories of crossed modules was initiated in [20] for groups, and it remains a current research topic, as indicated by the diverse literature treating this issue.

Crossed modules of groups are algebraic objects equivalent to strict 2-groups, or equivalently categorical groups, internal groups in the category of small categories, and also simplicial objects with associated Moore complex of length 1. Since crossed modules of groups are a generalization of groups, it is natural to search extensions of classical results in the theory of groups in the category of crossed modules of groups.

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Joyal and Street defined in [17] the concept of braiding for monoidal categories as a natural isomorphism $\tau_{X,Y}\colon X\otimes Y\to Y\otimes X$, satisfying the well-known two hexagon diagrams (see [18] or below in Section 1), generalizing the idea of the usual tensor product of vector spaces. The notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups (see [9,17]).

In this paper, we will devise a braided version of the results given by Norrie in [20] for braided crossed modules of groups; more precisely, we will study universal central extensions in the category of braided crossed modules **BCM**.

In [10], Fukushi gave a braided version of the results on universal central extensions of crossed modules of groups provided by Norrie in [20]. For a given braided crossed module, he found a canonical braiding on the universal central extension of the crossed module in the category **CM**, which provides an extension in **BCM**. However, it is different from the archetype of universal central extension in the category of braided crossed modules **BCM** since, in this category, it is necessary to add additional restrictions, including the braiding on the notions of centre and commutator.

For that purpose, in the braided context, we will need the definition of centre and commutator given by Huq in [14]. On the other hand, the forgetful functor $\mathbf{U}\colon \mathbf{BCM}\longrightarrow \mathbf{CM}$, which forgets the braiding, allows us to study the theory of braided crossed modules as crossed modules. In particular, the universal central extension of a perfect braided crossed module can be constructed in the categories **BCM** and **CM**, respectively, making it a natural problem to compare both constructions.

This text is organized as follows. In the first section, we provide some definitions, such as braiding, central extensions in a semi-abelian category, by studying them in detail in the categories **CM** and **BCM** (**B**-central extensions), and the non-abelian tensor product of groups, necessary for developing the work. In Section 2, we construct the universal **B**-central extension for a **B**-perfect braided crossed module and prove that a braided crossed module admits a universal **B**-central extension if and only if it is **B**-perfect. In Section 3, we construct the universal **U**-central extension for a perfect braided crossed module, where $\mathbf{U}\colon \mathbf{BCM}\longrightarrow \mathbf{CM}$ is the forgetful functor. Section 4 studies the relation between the universal **B**-central extension and the universal **U**-central extension of a braided crossed module. Finally, we prove that both universal extensions exist and coincide for a **B**-perfect braided crossed module.

Note that the framework of this paper is different from that given in [8] since the category **CM** is not a (Birkhoff) subcategory of **BCM** (see below Example 1.9).

1. Preliminaries

1.1. Central extensions in a semi-abelian category

An *extension* in a semi-abelian category $\mathcal{C}$ is a regular epimorphism. Following the theory in [15], we consider three particular kinds of extensions, trivial, normal and central, in a semi-abelian category $\mathcal{C}$ with respect to the full subcategory $\mathbf{Ab}\mathcal{C}$ of abelian objects. $\mathbf{Ab}\mathcal{C}$ is a Birkhoff subcategory of $\mathcal{C}$, and we denote the left adjoint (reflector functor) by $I\colon \mathcal{C}\to \mathbf{Ab}\mathcal{C}$ and its unit by $\eta_C\colon C\to I(C)$.

An extension $B\overset{\gamma}{\twoheadrightarrow} A$ is *trivial* if the induced square

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & A \\ {\scriptstyle\eta_B}\downarrow & & \downarrow{\scriptstyle\eta_A} \\ I(B) & \xrightarrow[I(\gamma)]{} & I(A) \end{array}$$

is a pullback in $\mathcal{C}$.

An extension is *normal* if one of the projections of the kernel pair is trivial.

An extension $B\overset{\gamma}{\twoheadrightarrow} A$ is *central* if there exists another extension $C\overset{\omega}{\twoheadrightarrow} A$ such that π_2 (also denoted $\omega^*(\gamma)$) in the pullback

$$\begin{array}{ccc} B\times_A C & \xrightarrow{\pi_1} & B \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\omega]{} & A, \end{array}$$

is trivial.

In the semi-abelian context, normal and central extension concepts are equivalent. Moreover, the notions of (categorically) central extension in the sense of Huq [14], an extension $B \xrightarrow{\gamma} A$ is central if and only if its kernel $\ker(\gamma) \xrightarrow{\gamma^k} B$ Huq-commutes with the identity 1_B, and Galois central extensions with respect to abelianization in the sense of Janelidze and Kelly [15], are equivalent. The equivalence of the two notions is a consequence of [13, Proposition 2.2] combined with [3, Remark 3.1] (see also [11, 1.5.12–1.5.15 Propositions]). Since the centre of B in the sense of Huq, $Z(B)$, is the final object in the category of central subobjects, we have the following characterization: an extension $B \xrightarrow{\gamma} A$ is central if and only if $\ker(\gamma) \subseteq Z(B)$.

The central extensions in $\mathcal{C}$ of an object A constitute another category, where a morphism $\gamma \to \omega$ between two central extensions of A, $B \xrightarrow{\gamma} A$ and $C \xrightarrow{\omega} A$, is a commutative triangle

$$\begin{array}{ccc} B & \xrightarrow{\theta} & C \\ & {\scriptstyle \gamma}\searrow \quad \swarrow{\scriptstyle \omega} & \\ & A & \end{array}$$

in the category $\mathcal{C}$.

A central extension $\mathcal{U} \twoheadrightarrow A$ is said to be *universal* (of A) if it is an initial object in the category of central extensions of A. From the definition, it is clear that a universal central extension is unique up to isomorphisms.

An object P of $\mathcal{C}$ is called *perfect* when $I(P)$ is the zero object of $\mathsf{Ab}\mathcal{C}$. If $B \xrightarrow{\gamma} A$ is an extension and B is $\mathsf{Ab}\mathcal{C}$-perfect, then so is A because the reflector functor I preserves regular epimorphisms, and a regular quotient of zero is zero.

The following statement establishes the relationship between universal central extensions and perfect objects.

Lemma 1.1 *([13, Lemma 2.3]). If $\mathcal{U} \xrightarrow{\upsilon} A$ is a universal $\mathsf{Ab}\mathcal{C}$-central extension then the objects $\mathcal{U}$ and A are $\mathsf{Ab}\mathcal{C}$-perfect.*

1.2. Crossed modules

Crossed modules of groups were introduced by Whitehead in [21] to study the second relative homotopy groups. This algebraic structure has important implications for category theory, homology and homotopy theories, differential geometry and physics.

Definition 1.2. A *crossed module of groups* is a pair $(G \xrightarrow{\partial} H, \cdot)$ where:

- G and H are groups, acting on themselves by conjugation, together with a group action $\cdot$ of H on G, denoted by ${}^h g$, i.e. an automorphism $H \to \mathrm{Aut}(G)$, and
- $\partial\colon G \to H$ is a group homomorphism satisfying:
  - ∂ is an H-equivariant map (we suppose the conjugation action of H on itself), i.e.

$$\partial({}^h g) = h\partial(g)h^{-1}, \qquad g \in G,\ h \in H,$$

– ∂ satisfies the Peiffer identity:

$$^{\partial(g)}g' = {}^{g}g' = gg'g^{-1}, \qquad g, g' \in G.$$

Example 1.3.

1. A canonical example of a crossed module is given by a group G and a normal subgroup N of G. The inclusion $N \overset{i}{\hookrightarrow} G$ with the conjugation action, ${}^{g}n = gng^{-1}$, $g \in G$, $n \in N$, is a crossed module of groups. In particular, the identity map $G \xrightarrow{\mathrm{Id}_G} G$ is a crossed module.
2. Any central extension of groups $G \overset{\partial}{\twoheadrightarrow} H$ (i.e. $\ker(\partial) \subseteq Z(G)$) is a crossed module, with the action ${}^{\partial(g)}g' = [g, g']$. Conversely, a simply connected crossed module (i.e. ∂ is surjective) is a central extension. In particular, $G \overset{\mathrm{conj}}{\twoheadrightarrow} \mathrm{Inn}(G)$, $g \mapsto \mathrm{conj}(g)$, with the action, ${}^{\mathrm{conj}(g)}g' = gg'g^{-1}$, is a crossed module of groups, where $\mathrm{Inn}(G)$ denotes the inner automorphisms of a group G.
3. Another standard example of a crossed module is $M \overset{0}{\to} G$, where M is a G-module.
4. For a group $(G, \cdot)$ consider the semidirect product $G \rtimes G$ with the multiplication $(g_1, g_2)(x_1, x_2) = (g_1 g_2 x_1 g_2^{-1}, g_2 x_2)$. Then $(G \overset{\partial}{\to} G \rtimes G, \cdot)$ is a crossed module, where $\partial(g) = (g^{-1}, g)$, and the action of $G \rtimes G$ on G is given by ${}^{(g_1, g_2)}x = g_2 x g_2^{-1}$.

Definition 1.4. Let $(G \overset{\partial}{\to} H, \cdot)$ and $(G' \overset{\partial'}{\to} H', *)$ be two crossed modules. A *morphism of crossed modules of groups* is a pair of group homomorphisms, $f_1 \colon G \to G'$ and $f_2 \colon H \to H'$, such that:

$$f_1({}^{h}g) = {}^{f_2(h)}f_1(g), \quad \text{for all } \ g \in G, h \in H. \tag{XGH1}$$

$$\partial' \circ f_1 = f_2 \circ \partial. \tag{XGH2}$$

The category of crossed modules of groups is a semi-abelian category in the sense of [16], and it will be denoted by **CM**.

The notions of commutator of a crossed module and perfect crossed module were introduced in [20]. This notion of commutator coincides in the category **CM** with the idea of commutator given by Huq in [14] in a category with products, zero objects, kernels and cokernels. The commutator is characterized as follows.

The *commutator crossed submodule* of a crossed module $\mathcal{G} = (G \overset{\partial}{\to} H, \cdot)$ is $[\mathcal{G}, \mathcal{G}] = ([H, G] \xrightarrow{\partial|_{[H,G]}} [H, H], \cdot_C)$, where $\cdot_C$ is the induced action, and

- $[H, G] = \langle \ \{{}^{h}g \ g^{-1} \mid h \in H, \ g \in G\} \ \rangle$ is the displacement subgroup of G relative to H.
- $[H, H]$ is the commutator subgroup of H.

The commutator crossed submodule is a normal crossed submodule (see [20]).

The quotient of a crossed module $\mathcal{G}$ by its commutator, $\mathcal{G}_{\mathrm{ab}} = \mathcal{G}/[\mathcal{G}, \mathcal{G}]$, is an abelian crossed module; that is, $G/[H, G]$ and $H/[H, H]$ are both abelian groups and $H/[H, H]$ acts trivially on $G/[H, G]$. We will denote the category of abelian crossed modules by **AbCM**, which is a Birkhoff subvariety of **CM**. Now, we have the reflector functor I, called *abelianization*, $(-)_{\mathrm{ab}} \colon$ **CM** $\to$ **AbCM**.

In the framework of the semi-abelian category **CM**, an extension is a surjective morphism, and a crossed module $\mathcal{G} = (G \overset{\partial}{\to} H, \cdot)$ is perfect (**AbCM**-perfect) if it coincides with its commutator crossed submodule ($\mathcal{G} = [\mathcal{G}, \mathcal{G}]$), i.e. $G = [H, G]$ and $H = [H, H]$.

The notion of the centre of an object was defined in [14], in a category with specific properties. This construction only needs that the category has finite products and zero object.

The category **CM** has centres in the sense of Huq [14], they were constructed in [20] and are characterized as follows.

The *centre* of a crossed module $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot)$ is the crossed submodule $Z(\mathcal{G}) = (G^H \xrightarrow{\partial|_{G^H}} Z(H) \cap \mathrm{st}_H(G), \cdot_Z)$, where:

- $G^H = \{g \in G \mid {}^h g = g,\ h \in H\}$,
- $Z(H) = \{h \in H \mid hh' = h'h,\ h' \in H\}$ is the centre of H,
- $\mathrm{st}_H(G) = \{h \in H \mid {}^h g = g,\ g \in G\}$,

and $\cdot_Z$ is the induced action, which means that it is the trivial action by the definition of G^H.

The universal central extension is entirely related to the concept of the non-abelian tensor product. In the case of groups, Brown and Loday in [6] defined the non-abelian tensor product of groups and proved that the universal central extension is the non-abelian tensor product $G \otimes G$ with the epimorphism $G \otimes G \twoheadrightarrow G$ sending $g \otimes g'$ to $[g, g']$.

In the case of crossed modules of groups [20], the notion of the non-abelian tensor product is also needed, which was introduced by Brown and Loday in [6].

Definition 1.5 *([6]).* Let G and H be two groups such that G acts on H with $\cdot$ and H acts on G with $*$, both by automorphisms.

The *non-abelian tensor product of G with H*, denoted by $G \otimes H$, is the group generated by the symbols $g \otimes h$, where $g \in G$, $h \in H$, with the relations

$$gg' \otimes h = (gg'g^{-1} \otimes {}^g h)(g \otimes h), \tag{T1}$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes hh'h^{-1}). \tag{T2}$$

Proposition 1.6 *([5,6]). Let G be a group. Then $(G \otimes G \xrightarrow{\partial} G, \cdot)$ is a crossed module of groups where $G \otimes G$ is the non-abelian tensor product of G with itself using the conjugation action. The action $\cdot\colon G \times (G \otimes G) \to (G \otimes G)$ and the map $\partial\colon G \otimes G \to G$ are defined on generators as ${}^g(g_1 \otimes g_2) = gg_1g^{-1} \otimes gg_2g^{-1}$ and $\partial(g_1 \otimes g_2) = [g_1, g_2]$.*

The following equalities on the non-abelian tensor product are needed in the successive sections.

$${}^{[h,h']}(h'' \otimes h''') = (h \otimes h')(h'' \otimes h''')(h \otimes h')^{-1}. \tag{1.1}$$

$$[h \otimes h', h'' \otimes h'''] = [h, h'] \otimes [h'', h''']. \tag{1.2}$$

Proof. See [5, Proposition 3 (8)] and [5, Proposition 3 (11)], respectively. □

On the other hand, the non-abelian tensor product was used by Norrie in [20] to prove that the universal central extension of a perfect crossed module $(G \xrightarrow{\partial} H, \cdot)$ in **CM** is given by:

$$(G \otimes H \xrightarrow{\partial \otimes \mathrm{Id}_H} H \otimes H, *) \xrightarrow{\upsilon = (\lambda, \xi)} (G \xrightarrow{\partial} H, \cdot), \tag{UCE}$$

where $\lambda(g \otimes h) = g\,{}^h g^{-1}$, $\xi(h \otimes h') = [h, h']$, and the action $*$ of $H \otimes H$ on $G \otimes H$ is given by ${}^{h \otimes h'}(g \otimes h'') = {}^{[h,h']}g \otimes {}^{[h,h']}h''$, for $h, h', h'' \in H, g \in G$.

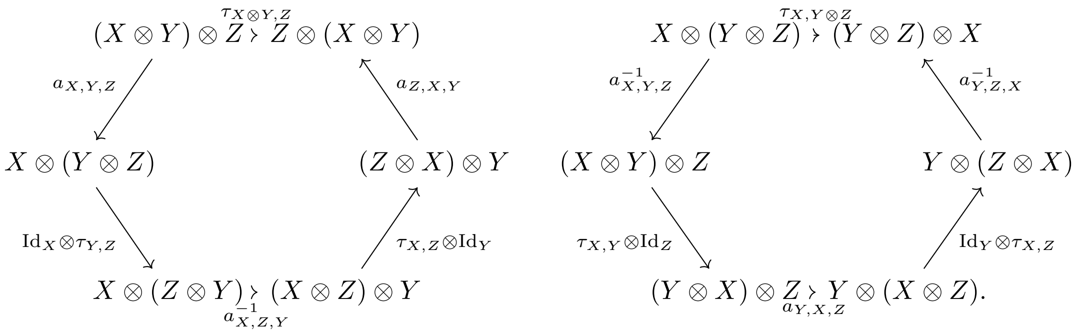
1.3. Braided crossed modules and braided monoidal categories

Now, we focus on the concept of a braided tensor category. Joyal and Street introduced the definition of braiding on a categorical group in [17] and [18].

A *monoidal (or tensor) category* is a 6-tuple $\boldsymbol{C} = (\boldsymbol{C}, \otimes, a, I, l, r)$ where $\boldsymbol{C}$ is a category, $\otimes\colon \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ is a bifunctor, $a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is a natural isomorphism called the associator, which satisfies an associative coherence diagram (pentagon axiom), $I \in \mathrm{Ob}(\boldsymbol{C})$ (unit object), and $l\colon I \otimes X \to X$, $r\colon X \otimes I \to X$ are natural isomorphisms (called the left and right unitors, respectively), which also satisfy for all $X, Y \in \mathrm{Ob}(\boldsymbol{C})$ the unit coherence diagram (triangle equation).

A monoidal category is called *strict* if the isomorphisms a, l and r are the identity morphisms. In this case we have that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, $X \otimes I = X = I \otimes X$.

A *braiding on a strict monoidal category* $\boldsymbol{C}$ (see [17,18]) is a natural isomorphism $\tau_{X,Y}\colon X \otimes Y \to Y \otimes X$ such that for all $X, Y, Z \in \mathrm{Ob}(\boldsymbol{C})$ the following associative coherence diagrams (hexagon axioms) commute:



A *categorical group* $\boldsymbol{C}$ is a monoidal category in which every arrow is invertible and, for each object X, there is an object X^* with an arrow $\varepsilon_X\colon X \otimes X^* \to I$. A categorical group is *strict* when it is strict as a monoidal category, and ε_X can be chosen to be an identity. Strict categorical groups are the internal groups in the category of small categories.

A braiding is called *symmetric* if, in addition, the condition $\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{Id}_{X\otimes Y}$ is satisfied for all objects X and Y of $\boldsymbol{C}$. Furthermore, the braiding is called *Picard* if, in addition to the symmetry condition, it satisfies the condition $\tau_{X,X} = \mathrm{Id}_{X\otimes X}$.

In 1984, Conduché in [9, Equalities (2.12)] introduced the notion of a braided crossed module of groups as a particular case of a 2-crossed module of groups. Later, the notion of a braided crossed module over a groupoid was presented by Brown and Gilbert in [4].

Definition 1.7. A *braiding* (or *Peiffer lifting*) on the crossed module $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot)$ is a map $\{-,-\}\colon H \times H \to G$ satisfying:

$$\partial\{h, h'\} = [h, h'], \tag{BG1}$$

$$\{\partial g, h\} = g\, {}^{h}g^{-1}, \tag{BG2}$$

$$\{h, \partial g\} = {}^{h}g\, g^{-1}, \tag{BG3}$$

$$\{h, h'h''\} = \{h, h'\}\ {}^{h'}\{h, h''\}, \tag{BG4}$$

$$\{hh', h''\} = {}^{h}\{h', h''\}\, \{h, h''\}, \tag{BG5}$$

for $g, g' \in G$, $h, h', h'' \in H$.

If the map $\{-,-\}$ is a braiding on $\mathcal{G}$ we will say that $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a *braided crossed module of groups*.

Joyal and Street proved in [18] that the notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups. For the reader's convenience, we recall here the link idea of a braiding on a monoidal category and a braiding on a crossed module (see [18]).

A braided strict categorical group $\boldsymbol{C}$ gives a braided crossed module as follows: H is the set of objects of $\boldsymbol{C}$, G is the set of morphisms $X \xrightarrow{f} I$ into the unit object, ∂ is the source function $f \mapsto X$, and the braiding is given by

$$\{X, Y\} \coloneqq \tau_{X,Y} \otimes \mathrm{Id}_{X^* \otimes Y^*} \colon X \otimes Y \otimes X^* \otimes Y^* \to I.$$

Conversely, given a braided crossed module $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$, the corresponding braided strict categorical group $\boldsymbol{C}$ is the category whose objects are the elements $h \in H$, a morphism $h \xrightarrow{g} h'$ is an element $g \in G$ with $h = \partial(g)h'$, the composition is the multiplication in G, the tensor product is given by

$$h \otimes h' \coloneqq hh' \quad \text{and} \quad (h_1 \xrightarrow{g_1} h_1') \otimes (h_2 \xrightarrow{g_2} h_2') \coloneqq h_1 h_2 \xrightarrow{g_1\, {}^{h_1'} g_2} h_1' h_2',$$

and the braiding

$$\tau_{h,h'} \coloneqq \{h, h'\} \colon hh' = h \otimes h' \to h' \otimes h = h'h.$$

A braided crossed module $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is *symmetric* if for every $h, h' \in H$ we have $\{h, h'\}\{h', h\} = 1$. If, in addition, we have $\{h, h\} = 1$ for every $h \in H$, we say that the braided crossed module is *Picard*.

If the braided crossed module $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is Picard, then the braiding is identically equal to 1, and therefore G and H are abelian, and the action of H on G is trivial, i.e. $(G \xrightarrow{\partial} H, \cdot)$ is an abelian crossed module (see [2]).

In [9, Equalities (2.12)], the action is redundant, but it can be recovered as Conduché already explains previously as ${}^{h}g = g\{\partial(g^{-1}), h\}$. We will take this action into account, duplicate one of the equalities and use the last two equalities (2.11) of [9] instead of the last two of (2.12). This is consistent because they are equivalent (see [9]).

As a consequence of (BG2) we have

$$\{\partial g, \partial g'\} = [g, g'],$$

and as a consequence of (BG2) and (BG3), we deduce:

$$\{1, h\} = \{h, 1\} = 1. \tag{1.3}$$

It has been proven in [9, Equation (2.14)] (see also [12]) that:

$${}^{h_1}\{h_2, h_3\} = \{h_1 h_2 h_1^{-1}, h_1 h_3 h_1^{-1}\}. \tag{1.4}$$

Example 1.8.

1. Let N be a normal subgroup of a group G such that the quotient group G/N is abelian, i.e. $[G, G] \subseteq N$. Then $(N \overset{i}{\hookrightarrow} G, \mathrm{conj})$ is a braided crossed module, where the braiding is given by $\{g, g'\} = [g, g']$. In particular, $(G \xrightarrow{\mathrm{Id}_G} G, \mathrm{conj}, [-,-])$ is a braided crossed module.
2. There is a canonical braiding on the crossed module $(G \otimes G \xrightarrow{\partial} G, \cdot)$ given by $\{g, g'\} = g \otimes g'$ (see [10] and also [6, Proposition 2.3] and [19, Proposition 1.2.8]).
3. Let $(G \overset{\partial}{\twoheadrightarrow} H, \cdot)$ be a simply connected crossed module. There is a canonical braiding on $(G \overset{\partial}{\twoheadrightarrow} H, \cdot)$, given by $\{\partial(g), \partial(g')\} = [g, g']$.
In particular, $G \overset{\mathrm{conj}}{\twoheadrightarrow} \mathrm{Inn}(G)$, with the braiding $\{\mathrm{conj}(g), \mathrm{conj}(g')\} = [g, g']$, is a braided crossed module.

4. If $\partial\colon G \to H$ is a homomorphism of abelian groups, i.e. an abelian crossed module, then $(G \xrightarrow{\partial} H, \odot)$ is a braided crossed module, where the braiding is given by $\{h, h'\} = 1$ and $\odot$ is the trivial action.
5. Let $G = \langle g \mid g^4 = 1\rangle \cong C_4$ and $H = \langle h \mid h^4 = 1\rangle \cong C_4$ be cyclic groups of order 4. The crossed module $(G \xrightarrow{\partial} H, \cdot)$, where $\partial(g) = h^2$ and the action is trivial, admits the following braiding:

$$\begin{aligned}&\{h,h\} = \{h,h^3\} = \{h^3,h\} = \{h^3,h^3\} = g^2,\\ &\{h,h^2\} = \{h^2,h\} = \{h^2,h^2\} = \{h^2,h^3\} = \{h^3,h^2\} = 1,\\ &\{h',1\} = \{1,h'\} = 1, \text{ for all } h' \in H.\end{aligned}$$

6. If $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ and $(G' \xrightarrow{\partial'} H', \cdot', (\!(-,-)\!))$ are two braided crossed modules, then the product $(G \times G' \xrightarrow{\partial\times\partial'} H \times H', *, \langle\!\langle -,-\rangle\!\rangle)$ is a braided crossed module, where the braiding is given by $\langle\!\langle (h_1,h_1'),(h_2,h_2')\rangle\!\rangle = \big(\{h_1,h_2\}, (\!(h_1',h_2')\!)\big)$.

A consequence of Example 1.8 (5) is that the category **BCM** is not a subcategory of **CM** since such a crossed module admits two braidings: the given and the trivial braiding. Moreover, we will see that the abelian objects in the category **BCM** are the abelian crossed modules with the trivial braiding (see Proposition 1.14).

Next, we will see examples of crossed modules that do not accept any braiding. Consequently, the category **CM** is not a (Birkhoff) subcategory of **BCM**.

Example 1.9.

1. Let $G = \langle g \mid g^4 = 1\rangle \cong C_4$ and $H = \langle h \mid h^4 = 1\rangle \cong C_4$ be cyclic groups of order 4. The crossed module $(G \xrightarrow{\partial} H, \cdot)$, where $\partial(g) = h^2$ and the action is given by ${}^h g = g^{-1}$, does not admit any braiding, since neither $\{h,h'\} = 1$ (the crossed module is not abelian) nor $\{h,h'\} = g^2$ can be braidings. If $\{h,h'\} = 1$ then the crossed module would be abelian, and if $\{h,h'\} = g^2$ then

$$\begin{aligned}&\{h,h^2\} = \{h,h\}\,{}^h\{h,h\} = g^2 g^{-2} = 1,\\ &\{h,h^2\} = \{h,\partial(g)\} = {}^h g g^{-1} = g^{-2} = g^2,\end{aligned}$$

which is an absurd.
2. Let G be a group with a normal subgroup N such that $[G,G] \nsubseteq N$, for example, $G = S_3 \times C_2$ and $N = \{1\} \times C_2$. Then, the crossed module $(N \xhookrightarrow{i} G, \mathrm{conj})$ does not admit any braiding since $\{g,g'\} = [g,g'] \notin N$.
3. As a consequence of the previous item, the crossed module of Example 1.3 (4) does not admit any braiding since $[G \rtimes G, G \rtimes G] \nsubseteq \partial(G)$.

Definition 1.10. A *morphism of braided crossed modules*

$$f = (f_1, f_2)\colon (G \xrightarrow{\partial} H, \cdot, \{-,-\}) \longrightarrow (G' \xrightarrow{\partial'} H', \cdot', (\!(-,-)\!))$$

is a homomorphism of crossed modules that preserves the braiding, i.e.

$$f_1(\{h,h'\}) = (\!(f_2(h), f_2(h'))\!), \qquad \text{for } h,h' \in H. \tag{BXGH3}$$

We will denote the category of braided crossed modules by **BCM**. Similar to **CM**, we will see that the category **BCM** is semi-abelian.

The category **BCM** has binary coproducts, since in [7] it is proved that there are binary coproducts of 2-crossed modules, $(G \xrightarrow{\partial} H \xrightarrow{\partial'} N)$, and a braided crossed module is a particular case of 2-crossed module, with $N = 1$. Moreover, it is pointed, Barr exact and Bourn protomodular, since the category **CM** is too. Therefore, **BCM** is semi-abelian. We also have a faithful forgetful functor $\mathbf{U}\colon \mathbf{BCM} \to \mathbf{CM}$.

In the case of the braiding category **BCM**, the idea of braiding changes slightly the concepts of centre and commutator from the category of crossed modules **CM**, which makes appear the following subobjects using the definition given by Huq [14] in the general case. We will call the commutator and centre of Huq in the braiding category **BCM** as the **B**-*commutator* and **B**-*centre*, respectively.

Proposition 1.11. *The* **B**-*commutator (commutator in the sense of Huq [14] in the category* **BCM**) *of a braided crossed module* $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is given by*

$$[\mathcal{G},\mathcal{G}]_{\mathbf{B}} = \big(\mathbf{B}_H(G) \xrightarrow{\partial|_{\mathbf{B}_H(G)}} [H,H], \cdot_C, \{-,-\}_C\big),$$

where $\cdot_C$ *and* $\{-,-\}_C$ *are the induced operations, and* $\mathbf{B}_H(G) = \big\langle \{\ \{h,h'\} \mid h,h' \in H\} \big\rangle$.

Remark 1.12. The **B**-commutator is a normal braided crossed submodule (cf. [20]). $\mathbf{B}_H(G)$ is a normal subgroup of G, and moreover we have the following inclusions of subgroups:

$$[G,G] \subset [H,G] \subset \mathbf{B}_H(G).$$

The proof of Proposition 1.11 is a consequence of Lemma 1.13 and Proposition 1.14 presented below.

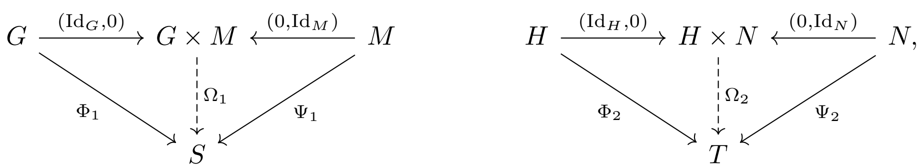
Lemma 1.13. *Suppose* $\mathcal{X} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$, $\mathcal{Y} = (M \xrightarrow{\delta} N, *, \{\!\langle -,- \rangle\!\})$ *and* $\mathcal{Z} = (S \xrightarrow{\rho} T, \star, (\!(-,-)\!))$ *are braided crossed modules of groups.*

$\mathcal{X} \xrightarrow{\Phi} \mathcal{Z} \xleftarrow{\Psi} \mathcal{Y}$ *commute if, and only if:*

(a) $\Phi_1(g)\Psi_1(m) = \Psi_1(m)\Phi_1(g)$.
(b) $\Phi_2(h)\Psi_2(n) = \Psi_2(n)\Phi_2(h)$.
(c) ${}^{\Phi_2(h)}\Phi_1(g)\ {}^{\Psi_2(n)}\Psi_1(m) = {}^{\Phi_2(h)\Psi_2(n)}(\Phi_1(g)\Psi_1(m))$.
(d) $(\!(\Phi_2(h_1),\Phi_2(h_2))\!)\ (\!(\Psi_2(n_1),\Psi_2(n_2))\!) = (\!(\Phi_2(h_1)\Psi_2(n_1),\Phi_2(h_2)\Psi_2(n_2))\!)$,

with $g \in G$, $m \in M$, $n, n_1, n_2 \in N$, *and* $h, h_1, h_2 \in H$.

Proof. Φ and Ψ commute means that we have the following diagrams:



where the morphism $\Omega = (\Omega_1, \Omega_2)$ is a morphism in **BCM**.

The two diagrams are in the category of groups, which means that Φ_1 commutes with Ψ_1 and Φ_2 commutes with Ψ_2 in groups.

(a) and (b) are the characterization that the morphisms commute in groups, (c) is equivalent to (XGH1), and (d) is equivalent to (BXGH3). Moreover, (XGH2) is always true for our hypotheses. □

Proposition 1.14. *If* $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a braided crossed module, then its abelianization is*

$$\overline{\mathcal{G}} = (\frac{G}{\mathbf{B}_H(G)} \xrightarrow{\overline{\partial}} \frac{H}{[H,H]}, \overline{\cdot}, \{-,-\}).$$

Note that $\{\overline{h_1}, \overline{h_2}\} = \overline{\{h_1, h_2\}} = \overline{1}$, *i.e. the braiding is trivial.*

Proof. It is easy to show that

$$[\mathcal{G}, \mathcal{G}]_{\mathbf{B}} = \big(\mathbf{B}_H(G) \xrightarrow{\partial|_{\mathbf{B}_H(G)}} [H,H], \cdot_C, \{-,-\}_C\big)$$

is a braided crossed submodule (and hence a normal braided crossed submodule). We take the cokernel of the inclusion, which gives us the regular epimorphism $\mathcal{G} \xrightarrow{\pi} \overline{\mathcal{G}} = \frac{\mathcal{G}}{[\mathcal{G},\mathcal{G}]_{\mathbf{B}}}$. We will prove that it is the abelianization.

It is easy to see that π commutes with itself using Lemma 1.13, so we need to prove that it satisfies the universal property.

Let us take another regular epimorphism that commutes with itself. Since it is a regular epimorphism, it is a quotient, so it is

$$\mathcal{G} \xrightarrow{\epsilon} (\frac{G}{I_1} \xrightarrow{\overline{\partial}} \frac{H}{I_2}, \overline{\cdot}, \{-,-\}),$$

where I_1 and I_2 are normal subgroups of G and H, respectively.

Since $\frac{\mathcal{G}}{[\mathcal{G},\mathcal{G}]_{\mathbf{B}}}$ is a cokernel, the universal property follows from $\epsilon \circ i_{[\mathcal{G},\mathcal{G}]_{\mathbf{B}}} = 0$. We will denote $i = i_{[\mathcal{G},\mathcal{G}]_{\mathbf{B}}}$.

Next, we will see that the image of all the generators is 1.

- For $\{h_1, h_2\} \in \mathbf{B}_H(G)$,

$$\epsilon_1 \circ i_1(\{h_1, h_2\}) = \epsilon_1(\{h_1, h_2\}) = \{\epsilon_2(h_1), \epsilon_2(h_2)\}.$$

  Since ϵ commutes with itself,

$$\begin{aligned}\{\epsilon_2(h_1), \epsilon_2(h_1)\} &= \{\epsilon_2(h_1)\epsilon_2(1), \epsilon_2(1)\epsilon_2(h_1)\} \\ &= \{\epsilon_2(h_1), \epsilon_2(1)\}\{\epsilon_2(1), \epsilon_2(h_1)\} \\ &= \epsilon_1(\{h_1, 1\})\epsilon_1(\{1, h_2\}) = \epsilon_1(1)\epsilon_1(1) = \epsilon_1(1) = 1.\end{aligned}$$

- For $[h_1, h_2] \in [H, H]$,

$$\epsilon_2([h_1, h_2]) = [\epsilon_2(h_1), \epsilon_2(h_2)].$$

  Since ϵ commutes with itself, we have that $\epsilon_2(h_1)\epsilon_2(h_2) = \epsilon_2(h_2)\epsilon_2(h_1)$, hence $[\epsilon_2(h_1), \epsilon_2(h_2)] = 1$. □

The quotient of a braided crossed module $\mathcal{G}$ by its $\mathbf{B}$-commutator, $\mathcal{G}_{\mathrm{ab}} = \mathcal{G}/[\mathcal{G},\mathcal{G}]_{\mathbf{B}}$, is an abelian crossed module with trivial braiding; that is, $G/\mathbf{B}_H(G)$ and $H/[H,H]$ are both abelian groups, $H/[H,H]$ acts trivially on $G/\mathbf{B}_H(G)$, and the braiding is trivial, $\{\overline{h}, \overline{h'}\} = 1$. Now, we have the category of abelian crossed modules with trivial braiding, **AbBCM**, a Birkhoff subvariety of **BCM**, and the reflector functor, $I = (-)_{\mathrm{ab}} \colon \mathbf{BCM} \to \mathbf{AbBCM}$.

Proposition 1.15. *The* $\mathbf{B}$*-centre (centre in the sense of Huq [14] in the category* $\mathbf{BCM}$*) of a braided crossed module* $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is the braided crossed submodule* $Z_{\mathbf{B}}(\mathcal{G}) = (G^H \xrightarrow{\partial|_{G^H}} \mathrm{Z}_{\mathbf{B}}(H), \cdot_Z, \{-,-\}_Z)$*, where*

$$\mathrm{Z}_{\mathbf{B}}(H) = \{h \in H \mid \{h, h'\} = 1 = \{h', h\},\ h' \in H\},$$

$\cdot_Z$ *is the induced action and* $\{-,-\}_Z$ *is the induced braiding, i.e. the trivial action and the trivial braiding by the definition of* G^H *and* $\mathrm{Z}_{\mathbf{B}}(H)$*.*

Remark 1.16. It is easy to show that the following inclusions of subgroups are true:

$$G^H \subset Z(G), \qquad \mathrm{Z}_{\mathbf{B}}(H) \subset \mathrm{st}_H(G) \cap Z(H).$$

Besides, if we use the properties (BG2) and (BG3), then we have that $G^H = \{g \in G \mid \partial(g) \in Z_{\mathbf{B}}(H)\}$.

The proof of Proposition 1.15 is a consequence of Lemma 1.13, and Lemmas 1.17 and 1.18 presented below.

Lemma 1.17. *Let* $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *be a braided crossed module. If* $z, z' \in \mathrm{Z}_{\mathbf{B}}(H)$ *then*

$$\{h, h'\} = \{zh, z'h'\}.$$

Proof.

$$\begin{aligned}\{z\,h, z'\,h'\} &= {}^{z}\{h, z'\,h'\}\,\{z, z'\,h'\} = \{h, z'\,h'\}\,\{z, z'\,h'\} \\ &= \{h, z'\}\ {}^{z'}\{h, h'\}\ \ \{z, z'\}\ {}^{z'}\{z, h'\} \\ &= \{h, z'\}\ \{h, h'\}\ \{z, z'\}\ \{z, h'\} = \{h, h'\},\end{aligned}$$

where we have used that $z, z' \in \mathrm{Z}_{\mathbf{B}}(H) \subset \mathrm{st}_H(G)$, (BG4) and (BG5). □

Lemma 1.18. *Let* $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *be a braided crossed module of groups and* $\mathcal{Z} = (G_1 \xrightarrow{\partial} H_1, \cdot, \{-,-\})$ *a braided crossed submodule of* $\mathcal{G}$*.*

$\mathcal{Z}$ *is central if, and only if,* $G_1 \subset G^H$ *and* $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H)$*.*

Proof. $\mathcal{Z}$ is central if, and only if, $i_{\mathcal{Z}}$ and $\mathrm{Id}_{\mathcal{G}}$ commute.

Lemma 1.13 provides that $\mathcal{Z}$ is central if, and only if,

(a) $g_1 g = g g_1$, for $g \in G$, $g_1 \in G_1$.
(b) $h_1 h = h h_1$, for $h \in$, $h_1 \in H_1$.
(c) ${}^{h_1}g_1\,{}^{h}g = {}^{h_1 h}(g_1 g)$, for $g \in G$, $g_1 \in G_1$, $h \in$, $h_1 \in H_1$.
(d) $\{h_1, h_1'\}\{h, h'\} = \{h_1 h, h_1' h'\}$, for $h, h' \in H$, $h_1, h_1' \in H_1$.

First, we will prove that (d) is equivalent to $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H)$. On the one hand, if $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H)$, and by using Lemma 1.17, we have:

$$\{h_1, h_1'\}\{h, h'\} = 1\{h, h'\} = \{h, h'\} = \{h_1 h, h_1' h'\}.$$

On the other hand, $\mathcal{Z}$ is central and we have equality (d). So, we have that $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H)$ since

$$\{h_1, h\} = 1\{h_1, h\} = \{h_1^{-1}, 1\}\{h_1, h\} = \{h_1^{-1} h_1, h\} = \{1, h\} \overset{(1.3)}{=} 1.$$

$$\{h,h_1\} = 1\{h,h_1\} = \{1,h_1^{-1}\}\{h,h_1\} = \{h,h_1^{-1}h_1\} = \{h,1\} \overset{(1.3)}{=} 1,\ h_1 \in H_1, h \in H.$$

Now, assuming that $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H)$, we will prove that (c) is equivalent to $G_1 \subset G^H$. If $G_1 \subset G^H$, then we have:

$$^{h_1h}(g_1g) = {}^{h_1h}g_1\,{}^{h_1h}g = {}^{h_1}\big({}^{h}g_1\big)\,{}^{h_1}\big({}^{h}g\big) = {}^{h_1}g_1\,{}^{h}g, \quad g \in G, g_1 \in G_1, h \in H, h_1 \in H_1.$$

The last equality follows from $g_1 \in G^H$ and $h_1 \in \mathrm{Z}_{\mathbf{B}}(H) \subset \mathrm{st}_H(G)$

On the other hand, if $\mathcal{Z}$ is central, by using (c) we can prove that $G_1 \subset G^H$:

$$g_1^{-1}\,{}^{h}g_1 \overset{(c)}{=} {}^{h}(g_1^{-1}g_1) = {}^{h}1 = 1 \Rightarrow {}^{h}g_1 = g_1, \quad g_1 \in G_1, h \in H.$$

(a) and (b) are equivalent to $G_1 \subset \mathrm{Z}(G)$ and $H_1 \subset \mathrm{Z}(H)$. Since $G_1 \subset G^H \subset \mathrm{Z}(G)$ and $H_1 \subset \mathrm{Z}_{\mathbf{B}}(H) \subset \mathrm{Z}(H)$, the result is obtained. □

In the framework of the semi-abelian category **BCM**, an extension $\mathcal{X} \overset{\gamma}{\twoheadrightarrow} \mathcal{Y}$ is a surjective morphism, and it is central in the category **BCM** if it satisfies $\ker\gamma \subseteq Z_{\mathbf{B}}(\mathcal{X})$. In this case it is called **B**-*central.* A braided crossed module $\mathcal{G} = (G \overset{\partial}{\to} H, \cdot, \{-,-\})$ is **B**-perfect (**AbBCM**-perfect) if it coincides with its **B**-commutator braided crossed submodule ($\mathcal{G} = [\mathcal{G},\mathcal{G}]_{\mathbf{B}}$), i.e. $G = \mathbf{B}_H(G)$ and $H = [H,H]$.

The forgetful functor $\mathbf{U}\colon \mathbf{BCM} \to \mathbf{CM}$ provides the following external notion of a central extension of braided crossed modules.

Definition 1.19. We will say that an extension $\mathcal{X} \xrightarrow{(\gamma_1,\gamma_2)} \mathcal{Y}$ of braided crossed modules in **BCM** is a **U**-*central extension* if $\mathbf{U}(\mathcal{X}) \xrightarrow{\mathbf{U}(\gamma_1,\gamma_2)} \mathbf{U}(\mathcal{Y})$ is central in **CM**, i.e. $\ker\big(\mathbf{U}(\gamma_1,\gamma_2)\big)$ is a crossed submodule of the centre of $\mathbf{U}(\mathcal{X})$.

It is immediate that every **B**-central extension in the category **BCM** is a **U**-central extension. The following example shows that the converse does not hold, i.e. not every **U**-central extension is a **B**-central extension. Furthermore, it emphasizes that the concepts of **B**-centre and **B**-commutator of a braided crossed module $\mathcal{X}$ are different from the notions of centre and commutator of $\mathbf{U}(\mathcal{X})$.

Example 1.20. Let $G \neq 0$ be an abelian group. Note that $G \otimes G \cong G \otimes_{\mathbb{Z}} G$ as abelian groups. Here, we will denote the trivial group by 0.

Using Example 1.8 (2), we have that $(G \otimes G \overset{\partial}{\to} G, \cdot)$, with $\partial = 0$, ${}^{g}(g_1 \otimes g_2) = g_1 \otimes g_2$ and $\{g,g'\} = g \otimes g'$ is a braided crossed module.

(i) Let $\mathcal{G} = (G \otimes G \overset{0}{\to} G, \odot, - \otimes -)$ be the braided crossed module, where the tensor product is the usual as abelian groups and $\odot$ is the trivial action.

The centre $Z\big(\mathbf{U}(\mathcal{G})\big) = (G \otimes G \overset{0}{\to} G)$ and the **B**-centre $Z_{\mathbf{B}}(\mathcal{G}) = (G \otimes G \overset{0}{\to} 0, - \otimes -)$ are different in **CM**, i.e. $Z\big(\mathbf{U}(\mathcal{G})\big) \neq \mathbf{U}(Z_{\mathbf{B}}(\mathcal{G}))$, since

$$(G \otimes G)^G = G \otimes G, \quad \mathrm{st}_G(G \otimes G) = G, \quad Z(G) = G, \quad \text{and } Z_{\mathbf{B}}(G) = 0.$$

On the other hand, the commutator $[\mathbf{U}(\mathcal{G}),\mathbf{U}(\mathcal{G})] = (0 \overset{0}{\to} 0)$ and the **B**-commutator $[\mathcal{G},\mathcal{G}]_{\mathbf{B}} = (G \otimes G \overset{0}{\to} 0, - \otimes -)$ are also different in **CM**, i.e. $[\mathbf{U}(\mathcal{G}),\mathbf{U}(\mathcal{G})] \neq \mathbf{U}([\mathcal{G},\mathcal{G}]_{\mathbf{B}})$, since

$$[G, G \otimes G] = 0, \quad \mathbf{B}_G(G \otimes G) = G \otimes G, \quad \text{and } [G,G] = 0.$$

(ii) Now, we will show a **U**-central extension that is not a **B**-central extension.

In particular, let be the braided crossed modules $\mathcal{X} = (\mathbb{Z}^3 \otimes \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3, \odot, - \otimes -)$ and $\mathcal{Y} = (\mathbb{Z}^2 \otimes \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^2, \odot, - \otimes -)$, for $G = \mathbb{Z}^3$ and $G = \mathbb{Z}^2$, respectively.

By taking the projection $\mathbb{Z}^3 \xrightarrow{\pi} \mathbb{Z}^2$, $(x, y, z) \mapsto (x, y)$, we have that $\pi \otimes \pi \colon \mathbb{Z}^3 \otimes \mathbb{Z}^3 \to \mathbb{Z}^2 \otimes \mathbb{Z}^2$ is surjective and $\mathcal{X} \xrightarrow{(\pi\otimes\pi,\pi)} \mathcal{Y}$ is an extension of braided crossed modules.

It is immediate that $\ker(\pi \otimes \pi) \subset (\mathbb{Z}^3 \otimes \mathbb{Z}^3)^{\mathbb{Z}^3} = \mathbb{Z}^3 \otimes \mathbb{Z}^3$ and $\ker(\pi) \subset Z(\mathbb{Z}^3) \cap \mathrm{st}_{\mathbb{Z}^3}(\mathbb{Z}^3 \otimes \mathbb{Z}^3) = \mathbb{Z}^3$, i.e. $\ker(\pi \otimes \pi, \pi) \subset Z\big(\mathbf{U}(\mathcal{X})\big)$, and so the extension is a **U**-central extension.

However $0 \neq \ker(\pi) = \{(x, y, z) \in \mathbb{Z}^3 \mid x = y = 0\} \nsubseteq Z_{\mathbf{B}}(\mathbb{Z}^3) = 0$, and therefore the extension $\mathcal{X} \xrightarrow{(\pi\otimes\pi,\pi)} \mathcal{Y}$ is not a **B**-central extension.

2. The universal B-central extension for B-perfect braided crossed modules

In this section, we will find the expression of a universal **B**-central extension when it exists, and we will try to characterize this fact.

Lemma 2.1. *If $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-, -\})$ is a braided crossed module, then $H \otimes H \xrightarrow{\beta_1} G$ and $H \otimes H \xrightarrow{\beta_2} H$ defined by $\beta_1(h \otimes h') = \{h, h'\}$, and $\beta_2(h \otimes h') = [h, h']$, are homomorphisms of groups.*

*Besides, β_1 and β_2 are simultaneously surjective if and only if the braided crossed module $\mathcal{G}$ is **B**-perfect.*

Proof. By Proposition 1.6, β_2 is a homomorphism of groups. Since β_1 is determined by generators, we only need to prove it is well defined.

We will prove that β_1 preserves the relations (T1)–(T2).

Since the two actions in $H \otimes H$ are the conjugation of H, we have in (T1)

$$\begin{aligned}
\beta_1(h_1h_2 \otimes h_3) &= \{h_1h_2, h_3\} = {}^{h_1}\{h_2, h_3\}\{h_1, h_3\} \\
&= \{h_1h_2h_1^{-1}, h_1h_3h_1^{-1}\}\{h_1, h_3\} \\
&= \beta_1(h_1h_2h_1^{-1} \otimes h_1h_3h_1^{-1})\beta_1(h_1 \otimes h_3) \\
&= \beta_1(h_1h_2h_1^{-1} \otimes {}^{h_1}h_3)\beta_1(h_1 \otimes h_3) \\
&= \beta_1\Big((h_1h_2h_1^{-1} \otimes {}^{h_1}h_3)(h_1 \otimes h_3)\Big),
\end{aligned}$$

where we have used (1.4) and (BG5).

For (T2) we have

$$\begin{aligned}
\beta_1(h_1 \otimes h_2h_3) &= \{h_1, h_2h_3\} = \{h_1, h_2\}({}^{h_2}\{h_1, h_3\}) \\
&= \{h_1, h_2\}\{h_2h_1h_2^{-1}, h_2h_3h_2^{-1}\} \\
&= \beta_1(h_1 \otimes h_2)\beta_1(h_2h_1h_2^{-1} \otimes h_2h_3h_2^{-1}) \\
&= \beta_1(h_1 \otimes h_2)\beta_1({}^{h_2}h_1 \otimes h_2h_3h_2^{-1}) \\
&= \beta_1\Big((h_1 \otimes h_2)({}^{h_2}h_1 \otimes h_2h_3h_2^{-1})\Big),
\end{aligned}$$

where we have used (1.4) and (BG4).

So, β_1 is well defined, and therefore it is a homomorphism of groups.

For the second part, we have that $\operatorname{Im} \beta_1 = \mathbf{B}_H(G)$ and $\operatorname{Im} \beta_2 = [H, H]$. Therefore, β_1 and β_2 are simultaneously surjective if and only if the braided crossed module is **B**-perfect. □

Lemma 2.2. *Let $(G \xrightarrow{\partial} H, \cdot, \{-, -\})$ be a braided crossed module, and the braided crossed module $(H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-, -])$ (see Example 1.8 (1)).*

Then $(\beta_1, \beta_2)\colon (H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-]) \longrightarrow (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a morphism in* **BCM**, *where* β_1 *and* β_2 *are defined in Lemma 2.1.*

Besides, $\ker(\beta_1) \subset (H \otimes H)^{(H\otimes H)}$ *and* $\ker(\beta_2) \subset \mathrm{Z}_{\mathbf{B}}(H \otimes H)$, *i.e.* $\ker(\beta_1, \beta_2) \subseteq Z_{\mathbf{B}}(H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-])$.

Proof. In order to avoid confusion, we will denote the braiding of $H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H$ as $[\![-,-]\!]$.

First, we will show (XGH1). Let $h \otimes h', h'' \otimes h''' \in H \otimes H$.

$$\begin{aligned}\beta_1\Big({}^{(h\otimes h')}(h''\otimes h''')\Big) &= \beta_1\Big((h\otimes h')(h''\otimes h''')(h\otimes h')^{-1}\Big)\\ &= \beta_1\Big({}^{[h,h']}(h''\otimes h''')\Big)\\ &= \beta_1\Big({}^{[h,h']}h''\otimes {}^{[h,h']}h'''\Big)\\ &= \Big\{{}^{[h,h']}h'', {}^{[h,h']}h'''\Big\}\\ &= {}^{[h,h']}\{h'', h'''\}\\ &= {}^{\beta_2(h\otimes h')}\beta_1(h''\otimes h'''),\end{aligned}$$

where we have used (1.1) and (1.4).

Now, we will show (XGH2).

$$\partial \circ \beta_1(h\otimes h') = \partial\{h,h'\} = [h,h'] = \beta_2(\mathrm{Id}_{H\otimes H}(h\otimes h')),$$

where we have used (BG1).

Now, we will prove (BXGH3).

$$\begin{aligned}\beta_1([\![h\otimes h', h''\otimes h''']\!]) &= \beta_1([h\otimes h', h''\otimes h''']) = \beta_1([h,h']\otimes[h'',h'''])\\ &= \{[h,h'],[h'',h''']\} = \{\beta_2(h\otimes h'), \beta_2(h''\otimes h''')\},\end{aligned}$$

where we have used (1.2).

So, (β_1, β_2) is a morphism in **BCM**. We will now prove that the inclusions hold.

If $h \otimes h' \in \ker(\beta_1)$ then $\{h, h'\} = 1$. By using (BG1) we have that $1 = \partial\{h,h'\} = [h,h'] = \beta_2(h\otimes h')$, and so $h \otimes h' \in \ker(\beta_2)$. Since $(H \otimes H \xrightarrow{\beta_2} H, \cdot)$ is a crossed module, see Proposition 1.6, we have $h\otimes h' \in \ker\beta_2 \subseteq Z(H\otimes H)$.

So, we have for generators $x = h \otimes h'$

$$\begin{aligned}{}^{h''\otimes h'''}(h\otimes h') &= (h''\otimes h''')(h\otimes h')(h''\otimes h''')^{-1}\\ &= (h\otimes h')(h''\otimes h''')(h''\otimes h''')^{-1}\\ &= h\otimes h'.\end{aligned}$$

Therefore, we have that $h\otimes h' \in (H\otimes H)^{(H\otimes H)} = \{x \in H\otimes H \mid {}^{h''\otimes h'''}x = x,\ h''\otimes h''' \in H\otimes H\}$ (it is enough to work on generators), and $\ker(\beta_1) \subset (H\otimes H)^{(H\otimes H)}$.

For the second inclusion, we take $h\otimes h' \in \ker(\beta_2)$, i.e. $[h,h'] = 1$.

Since it is enough to work on generators, we have that

$$\mathrm{Z}_{\mathbf{B}}(H\otimes H) = \{x \in H\otimes H \mid [\![x, h''\otimes h''']\!] = 1 = [\![h''\otimes h''', x]\!],\ h''\otimes h''' \in H\otimes H\}.$$

Taking into account that for generators $x = h \otimes h'$

$$[\![h \otimes h', h'' \otimes h''']\!] = [h \otimes h', h'' \otimes h'''] = [h, h'] \otimes [h'', h'''] = 1 \otimes [h'', h'''] = 1,$$
$$[\![h'' \otimes h''', h \otimes h']\!] = [h'' \otimes h''', h \otimes h'] = [h'', h'''] \otimes [h, h'] = [h'', h'''] \otimes 1 = 1,$$

we deduce $h \otimes h' \in Z_{\mathbf{B}}(H \otimes H)$, which proves that $\ker(\beta_2) \subset Z_{\mathbf{B}}(H \otimes H)$. □

Corollary 2.3. *The morphism given in Lemma 2.2 is a* **B***-central extension if and only if* $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a* **B***-perfect braided crossed module.*

Proof. It follows from Lemmas 2.1 and 2.2. □

Proposition 2.4. *If* $(X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!)) \xrightarrow{\gamma=(\gamma_1,\gamma_2)} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a* **B***-central extension, then we have a morphism in* **BCM**, $\phi = (\phi_1, \phi_2)\colon (H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-]) \longrightarrow (X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!))$, *defined by:*

- $\phi_1\colon H \otimes H \to X_1$, $h \otimes h' \mapsto (\!(s(h), s(h'))\!)$, *where s is a section of* γ_2;
- $\phi_2\colon H \otimes H \to X_2$, $h \otimes h' \mapsto [s(h), s(h')]$, *where s is a section of* γ_2.

Besides, $\gamma \circ \phi = \beta = (\beta_1, \beta_2)$, *i.e.* ϕ *is a morphism between the extensions* $\beta = (\beta_1, \beta_2)$ *and* $\gamma = (\gamma_1, \gamma_2)$ *if* $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is* **B***-perfect (see Lemmas 2.1 and 2.2).*

Proof. We need to prove that ϕ_1 and ϕ_2 are well defined.

We will start with ϕ_1. We will prove that the map ϕ_1 is independent of the choice of the section, i.e. $(\!(s(h), s(h'))\!) = (\!(\varrho(h), \varrho(h'))\!)$, where s and ϱ are sections of γ_2.

Since $\gamma_2(s(h)) = \gamma_2(\varrho(h)) = h$ and $\gamma_2(s(h')) = \gamma_2(\varrho(h')) = h'$, we have $\gamma_2\big(s(h)\varrho(h)^{-1}\big) = \gamma_2\big(s(h')\varrho(h')^{-1}\big) = 1$.

Since $\gamma = (\gamma_1, \gamma_2)$ is a **B**-central extension, we have that $s(h)\varrho(h)^{-1}, s(h')\varrho(h')^{-1} \in \ker(\gamma_2) \subset \mathrm{Z}_{\mathbf{B}}(X_2)$. Now, by using Lemma 1.17,

$$(\!(s(h), s(h'))\!) = (\!(s(h)\varrho(h)^{-1}\ \varrho(h), s(h')\varrho(h')^{-1}\ \varrho(h'))\!) = (\!(\varrho(h), \varrho(h'))\!).$$

Since $\mathrm{Z}_{\mathbf{B}}(X_2) \subset \mathrm{Z}(X_2)$ we can change the proof for ϕ_1 taking the equalities for $[-,-]$ instead of $(\!(-,-)\!)$ which proves that ϕ_2 is independent of the choice of the section.

We can use an analogue argument as in Lemma 2.1 to prove that ϕ_1 and ϕ_2 are well defined, i.e. they preserve the relations. So, they are homomorphisms of groups since they are determined by generators.

To prove that $\phi = (\phi_1, \phi_2)$ is a morphism of braided crossed modules, we also use similar reasoning as the one done in Lemma 2.2, since we can make the changes in the choice inside the braidings and brackets.

To finish, if $h \otimes h' \in H \otimes H$, then

$$\gamma_1 \circ \phi_1(h \otimes h') = \gamma_1((\!(s(h), s(h'))\!)) = \{\gamma_2(s(h)), \gamma_2(s(h'))\} = \{h, h'\} = \beta_1(h \otimes h'),$$
$$\gamma_2 \circ \phi_2(h \otimes h') = \gamma_2([s(h), s(h')]) = [\gamma_2(s(h)), \gamma_2(s(h'))] = [h, h'] = \beta_2(h \otimes h').$$

Therefore, $\gamma \circ \phi = \beta$. □

Lemma 2.5. *If H is a perfect group, i.e.* $H = [H, H]$, *then* $(H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-])$ *is a* **B***-perfect braided crossed module.*

In particular, if $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a* **B***-perfect braided crossed module, then* $(H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-])$ *is a* **B***-perfect braided crossed module.*

Proof. Since the braiding in $(H\otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H\otimes H, \mathrm{conj}, [-,-])$ is the bracket, we have that $[H\otimes H, H\otimes H] = \mathbf{B}_{H\otimes H}(H \otimes H)$, and so it is enough to prove that $[H \otimes H, H \otimes H] = H \otimes H$.

Moreover, it is enough to prove that the generators $[h_1, h_2] \otimes [h_3, h_4]$ are inside $[H \otimes H, H \otimes H]$ since $H = [H, H]$. Using (1.2) we have that $[h_1, h_2] \otimes [h_3, h_4] = [h_1 \otimes h_2, h_3 \otimes h_4] \in [H \otimes H, H \otimes H]$.

For the second part, if $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is $\mathbf{B}$-perfect, then $H = [H, H]$. □

Proposition 2.6. *Let* $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\}) \xrightarrow{\sigma=(\sigma_1,\sigma_2)} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *be a morphism of braided crossed modules such that* $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\})$ *is* $\mathbf{B}$*-perfect.*

If $(X_1 \xrightarrow{\delta} X_2, *, \langle\!\langle -,-\rangle\!\rangle) \xrightarrow{\gamma=(\gamma_1,\gamma_2)} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a* $\mathbf{B}$*-central extension and there exists a homomorphism of braided crossed modules* $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\}) \xrightarrow{\tau=(\tau_1,\tau_2)} (X_1 \xrightarrow{\delta} X_2, *, \langle\!\langle -,-\rangle\!\rangle)$ *such that* $\sigma = \gamma \circ \tau$*, then* τ *is the unique homomorphism that satisfies the equality.*

Proof. Suppose that there are morphisms of braided crossed modules $\varphi = (\varphi_1, \varphi_2), \tau = (\tau_1, \tau_2) \colon (Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\}) \longrightarrow (X_1 \xrightarrow{\delta} X_2, *, \langle\!\langle -,-\rangle\!\rangle)$ such that $\sigma = \gamma \circ \tau = \gamma \circ \varphi$, i.e. $\sigma_1 = \gamma_1 \circ \tau_1 = \gamma_1 \circ \varphi_1$ and $\sigma_2 = \gamma_2 \circ \tau_2 = \gamma_2 \circ \varphi_2$.

If $y \in Y_2$ then $\gamma_2 \circ \tau_2(y) = \gamma_2 \circ \varphi_2(y)$, i.e. $\tau_2(y)\varphi_2(y)^{-1} \in \ker(\gamma_2)$. Then there is $k_y \in \ker(\gamma_2)$ such that $\tau_2(y) = k_y\, \varphi_2(y)$. Since γ is a $\mathbf{B}$-central extension, then we have that $\ker(\gamma_2) \subset \mathrm{Z}_{\mathbf{B}}(X_2) \subset \mathrm{st}_{X_2}(X_1) \cap \mathrm{Z}(X_2)$, and so for $y, z \in Y_2$, we have:

$$\tau_2([y, z]) = [\tau_2(y), \tau_2(z)] = [k_y\, \varphi_2(y), k_z\, \varphi_2(z)] = [\varphi_2(y), \varphi_2(z)] = \varphi_2([y, z]).$$

So, $\varphi_2 = \tau_2$ since $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\})$ is $\mathbf{B}$-perfect.

Besides, by using Lemma 1.17, we have

$$\tau_1(\{\langle y, z\rangle\}) = \langle\!\langle \tau_2(y), \tau_2(z)\rangle\!\rangle = \langle\!\langle k_y\, \varphi_2(y), k_z\, \varphi_2(z)\rangle\!\rangle = \langle\!\langle \varphi_2(y), \varphi_2(z)\rangle\!\rangle = \varphi_1(\{\langle y, z\rangle\}).$$

Therefore, $\varphi_1 = \tau_1$ because $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\})$ is $\mathbf{B}$-perfect, i.e. $Y_1 = \mathbf{B}_{Y_2}(Y_1)$ is generated by the images of the braiding. □

Corollary 2.7. *If* $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *is a* $\mathbf{B}$*-perfect braided crossed module, then*

$$\mathcal{B} = (H \otimes H \xrightarrow{\mathrm{Id}_{H\otimes H}} H \otimes H, \mathrm{conj}, [-,-]) \xrightarrow{\beta=(\beta_1,\beta_2)} \mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\}) \qquad \text{(UBCE)}$$

is the universal $\mathbf{B}$*-central extension of* $\mathcal{G}$*, where* β_1, β_2 *were defined in Lemma 2.1.*

Proof. Since $\mathcal{G}$ is $\mathbf{B}$-perfect, Corollary 2.3 states that the morphism $\mathcal{B} \xrightarrow{\beta} \mathcal{G}$ is a $\mathbf{B}$-central extension. Moreover, Lemma 2.5 implies that $\mathcal{B} \xrightarrow{\beta} \mathcal{G}$ is a $\mathbf{B}$-perfect braided crossed module.

We need to prove that it is universal.

If we have another $\mathbf{B}$-central extension $\mathcal{X} \xrightarrow{\gamma} \mathcal{G}$, then by Proposition 2.4 there is τ such that $\beta = \gamma \circ \tau$, which is unique by Proposition 2.6. □

Let us see the converse of Corollary 2.7.

Corollary 2.8. *A braided crossed module admits a universal* $\mathbf{B}$*-central extension if and only if it is* $\mathbf{B}$*-perfect.*

Proof. It is a consequence of Lemma 1.1 and Corollary 2.7. □

Remark 2.9. A possible alternative to obtain this result, based on the referee's communication, is to consider that **BCM** could be seen as a variety of multi-sorted algebras because a braided crossed module consists of two groups, a homomorphism between them, an action from one to the other, plus a braiding, which amounts to a pair of sets and several operations on, or between, them satisfying a set of equations. By [1, 14.14 Remark], **BCM** would have enough projectives, and therefore, the statement of the previous corollary would follow from [8, Theorem 3.5].

3. Braiding on a universal central extension of crossed modules

Universal central extensions of braided crossed modules of groups are not studied in [10]. However, the author constructed a canonical braiding on the universal central extension of a crossed module of groups [20], when the given crossed module is braided as well, and showed that it is universal in a sense that we will explain in this section.

In this part of the paper, we will consider braided crossed modules extensions, but unlike the previous section, we will construct a braiding on the universal central extension of a braided crossed module though as a crossed module by means of the forgetful functor $\mathbf{U}\colon \mathbf{BCM} \longrightarrow \mathbf{CM}$ and with the centre in **CM**, which we have called **U**-central extension. In this sense, we will obtain similar results given by Fukushi in [10] for crossed modules of groups in the category **BCM**.

If $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a braided crossed module, then there exists a canonical braiding for the crossed module $(G \otimes H \xrightarrow{\partial\otimes \mathrm{Id}_H} H \otimes H, *)$ (see [10, Proposition 3]).

Proposition 3.1 *([10]). If $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a braided crossed module then $\{\!\{-,-\}\!\}\colon (H\otimes H)\times(H\otimes H) \to G\otimes H$, defined on generators by $\{\!\{h\otimes h', h''\otimes h'''\}\!\} = \{h, h'\}\otimes[h'', h''']$, is a braiding for the crossed module $(G \otimes H \xrightarrow{\partial\otimes \mathrm{Id}_H} H \otimes H, *)$.*

Proposition 3.2. *If $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a braided crossed module such that $\mathbf{U}(\mathcal{G})$ is perfect, then $(G \otimes H \xrightarrow{\partial\otimes \mathrm{Id}_H} H \otimes H, *, \{\!\{-,-\}\!\}) \overset{\upsilon=(\lambda,\xi)}{\twoheadrightarrow} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a **U**-central extension, where $\lambda(g\otimes h) = g\,{}^{h}g^{-1}$ (see (UCE)) and $\xi(h\otimes h') = [h, h']$, and $\{\!\{-,-\}\!\}$ is defined in Proposition 3.1.*

Proof. Since $\mathbf{U}(\mathcal{G}) = (G \to H, \cdot)$ is a perfect crossed module we have the universal central extension $(G \otimes H \xrightarrow{\partial\otimes \mathrm{Id}_H} H \otimes H, *) \overset{\upsilon=(\lambda,\xi)}{\twoheadrightarrow} (G \xrightarrow{\partial} H, \cdot)$ in **CM** (UCE).

Now, we will prove that υ respects the braiding.

$$
\begin{aligned}
\lambda\big(\{\!\{h_1\otimes h_1', h_2\otimes h_2'\}\!\}\big) &= \lambda\big(\{h_1, h_1'\}\otimes[h_2, h_2']\big) = \{h_1, h_1'\}\,{}^{[h_2,h_2']}\{h_1, h_1'\}^{-1}\\
&= \big\{\,\partial(\{h_1, h_1'\}), [h_2, h_2']\big\} = \big\{[h_1, h_1'], [h_2, h_2']\big\} = \{\xi(h_1\otimes h_1'), \xi(h_2\otimes h_2')\},
\end{aligned}
$$

where we have used (BG2) and (BG1). So, υ is a **U**-central extension. □

Now, we will provide a universal object for the case of central extensions (in the category **CM**) of braided crossed modules of groups. A **U**-central extension $\mathcal{U} \overset{\upsilon}{\twoheadrightarrow} \mathcal{G}$ in **BCM** is *universal* if it is the initial object in the category of **U**-central extensions of $\mathcal{G}$.

Lemma 3.3. *Let $(G \xrightarrow{\partial} H, \cdot)$ be a crossed module. If $g_1 g^{-1} \in G^H$ and $h_1 h^{-1} \in \mathrm{st}_H(G)$, then*

$$
g\,{}^{h}g^{-1} = g_1\,{}^{h_1}g_1^{-1}, \quad g, g_1 \in G,\ h, h_1 \in H.
$$

Proof. If $g_1g^{-1} \in G^H$ and $h_1h^{-1} \in \mathrm{st}_H(G)$, then $g_1 = gx$, with $x \in G^H$, and $h_1 = hy$, with $y \in \mathrm{st}_H(G)$. So,

$$\begin{aligned} g_1\,{}^{h_1}g_1^{-1} &= gx\,{}^{hy}(x^{-1}g^{-1}) = gx\,{}^{h}(x^{-1}g^{-1}) \\ &= gx\,{}^{h}x^{-1}\,{}^{h}g^{-1} = gxx^{-1}\,{}^{h}g^{-1} = g\,{}^{h}g^{-1}. \quad \square \end{aligned}$$

Proposition 3.4. *If $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a braided crossed module such that $\mathbf{U}(\mathcal{G})$ is perfect, then*

$$\mathcal{U} = (G \otimes H \xrightarrow{\partial \otimes \mathrm{Id}_H} H \otimes H, *, [\![-,-]\!]) \overset{\upsilon=(\lambda,\xi)}{\twoheadrightarrow} \mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\}), \tag{UUCE}$$

is the universal $\mathbf{U}$-central extension of $\mathcal{G}$.

Proof. Let $(X_1 \xrightarrow{\delta} X_2, \circledast, (\!(-,-)\!)) \overset{\gamma=(\gamma_1,\gamma_2)}{\twoheadrightarrow} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a $\mathbf{U}$-central extension of braided crossed modules.

Since $\mathbf{U}(\mathcal{G})$ is perfect, we have that $\mathbf{U}(\mathcal{U}) \overset{\upsilon=(\lambda,\xi)}{\twoheadrightarrow} \mathbf{U}(\mathcal{G})$ is the universal central extension in **CM** (UCE). Therefore, there exists a unique morphism in **CM**

$$(G \otimes H \xrightarrow{\partial \otimes \mathrm{Id}_H} H \otimes H, *) \xrightarrow{\phi=(\phi_1,\phi_2)} (X_1 \xrightarrow{\delta} X_2, \circledast),$$

defined as $\phi_1(g \otimes h) = s_1(g)\ {}^{s_2(h)}s_1(g)^{-1}$ and $\phi_2(h \otimes h') = [s_2(h), s_2(h')]$, where s_1 and s_2 are sections of γ_1 and γ_2, respectively. The maps ϕ_1 and ϕ_2 are independent of the choice of the sections, since for two sections $(s_1, s_2), (\varrho_1, \varrho_2)$ of (γ_1, γ_2) we have $\gamma_1(s_1(g)) = \gamma_1(\varrho_1(g)) = g$ and $\gamma_2(s_2(h)) = \gamma_2(\varrho_2(h)) = h$, hence $s_1(g)\,\varrho_1(g)^{-1} \in \ker\gamma_1 \subset X_1^{X_2}$, and $s_2(h)\,\varrho_2(h)^{-1} \in \ker\gamma_2 \subset Z(X_2) \cap \mathrm{st}_{X_2}(X_1)$. Now, using Lemma 3.3, we have that ϕ_1 is independent of the choice of the sections, and the well definition of ϕ_2 is a consequence of that $s_2(h)\,\varrho_2(h)^{-1} \in Z(X_2)$.

ϕ is the unique morphism in **CM** satisfying $\upsilon = \gamma \circ \phi$ (see [20]).

We check that ϕ is a morphism in **BCM** showing that preserves the braidings $[\![-,-]\!]$ and $(\!(-,-)\!)$.

Let $h_1, h_1', h_2, h_2' \in H$. Then

$$\begin{aligned} \phi_1\big([\![h_1 \otimes h_1', h_2 \otimes h_2']\!]\big) &= \phi_1(\{h_1, h_1'\} \otimes [h_2, h_2']) \\ &= s_1(\{h_1, h_1'\})\ {}^{s_2([h_2,h_2'])}s_1(\{h_1, h_1'\})^{-1} \\ &= \Big(\!\!\Big(\delta\big(s_1(\{h_1, h_1'\})\big), s_2([h_2, h_2'])\Big)\!\!\Big). \end{aligned}$$

On the other hand

$$\begin{aligned} (\!(\phi_2(h_1 \otimes h_1'), \phi_2(h_2 \otimes h_2'))\!) &= \Big(\!\!\Big([s_2(h_1), s_2(h_1')], [s_2(h_2), s_2(h_2')]\Big)\!\!\Big) \\ &= \Big(\!\!\Big(\delta\big((\!(s_2(h_1), s_2(h_1'))\!)\big), [s_2(h_2), s_2(h_2')]\Big)\!\!\Big) \\ &= (\!(s_2(h_1), s_2(h_1'))\!)\ {}^{[s_2(h_2),s_2(h_2')]}(\!(s_2(h_1), s_2(h_1'))\!)^{-1}. \end{aligned}$$

Now, we have

$$\begin{aligned} \gamma_1\big(s_1(\{h_1, h_1'\})\big) &= \{h_1, h_1'\} \\ \gamma_1\Big((\!(s_2(h_1), s_2(h_1'))\!)\Big) &= \big\{\gamma_2 s_2(h_1), \gamma_2 s_2(h_1')\big\} = \{h_1, h_1'\}, \end{aligned}$$

and so $s_1(\{h_1, h_1'\})\ (\!(s_2(h_1), s_2(h_1'))\!)^{-1} \in \ker\gamma_1 \subset X_1^{X_2}$.

We also have

$$\gamma_2\big(s_2([h_2,h_2'])\big) = [h_2,h_2'] = \gamma_2\big([s_2(h_2),s_2(h_2')]\big),$$

and so

$$s_2([h_2,h_2'])\ [s_2(h_2),s_2(h_2')]^{-1} \in \ker\gamma_2 \subset Z(X_2)\cap \mathrm{st}_{X_2}(X_1).$$

Now, by using Lemma 3.3, we have that ϕ preserves the braidings.

The uniqueness of ϕ in **BCM** is a consequence of the faithfulness of the forgetful functor $\mathbf{U}\colon \mathbf{BCM} \to \mathbf{CM}$. □

Let $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a braided crossed module. In the following, we will prove that the universal $\mathbf{U}$-central extension of $\mathcal{G}$ exists if and only if $\mathbf{U}(\mathcal{G})$ is perfect in **CM**.

Proposition 3.5. *Let $\mathcal{Y} \overset{\sigma}{\twoheadrightarrow} \mathcal{G}$ be an extension of braided crossed modules such that $\mathbf{U}(\mathcal{Y})$ is perfect in* **CM**. *Then $\mathbf{U}(\mathcal{G})$ is also perfect in* **CM**.

Proof. Since $\mathbf{U}(\mathcal{Y}) \overset{\mathbf{U}(\sigma)}{\twoheadrightarrow} \mathbf{U}(\mathcal{G})$ is an extension in **CM**, by [20, Proposition 2] $\mathbf{U}(\mathcal{G})$ is perfect. □

Lemma 3.6. *Let $\mathcal{Y} = (Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\}) \overset{\sigma}{\twoheadrightarrow} \mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a $\mathbf{U}$-central extension of braided crossed modules such that $\mathbf{U}(\mathcal{Y})$ is not perfect. Then there exists another $\mathbf{U}$-central extension $\mathcal{X} \overset{\gamma}{\twoheadrightarrow} \mathcal{G}$ in* **BCM** *and two different morphisms $\varphi, \psi\colon \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\sigma = \gamma\circ\varphi = \gamma\circ\psi$.*

Proof. If $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\})$ is a braided crossed module, then we know that the crossed module $[\mathbf{U}(\mathcal{Y}), \mathbf{U}(\mathcal{Y})] = ([Y_2,Y_1] \xrightarrow{\rho|_{[Y_2,Y_1]}} [Y_2,Y_2], \star_C)$ is a normal crossed submodule of $(Y_1 \xrightarrow{\rho} Y_2, \star)$. But it is itself a braided crossed submodule of $(Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\})$ since, if we have $[y,y'],[z,z'] \in [Y_2,Y_2]$, then:

$$\{\langle\ [y,y'],[z,z']\ \rangle\} = \{\langle\ [y,y'], \rho(\{\langle z,z'\rangle\})\ \rangle\} = {}^{[y,y']}\{\langle z,z'\rangle\}\,\{\langle z,z'\rangle\}^{-1} \in [Y_2,Y_1],$$

and therefore we obtain a braiding in $[\mathbf{U}(\mathcal{Y}), \mathbf{U}(\mathcal{Y})]$ which coincides with $\{\langle -,-\rangle\}$, denoted by $\{\langle -,-\rangle\}_C$.

Let us denote

$$i\colon ([Y_2,Y_1] \xrightarrow{\rho|_{[Y_2,Y_1]}} [Y_2,Y_2], \star_C, \{\langle -,-\rangle\}_C) \longrightarrow (Y_1 \xrightarrow{\rho} Y_2, \star, \{\langle -,-\rangle\}).$$

Since $[\mathbf{U}(\mathcal{Y}), \mathbf{U}(\mathcal{Y})]$ is a normal crossed submodule, we can consider $\mathcal{G}\times \mathrm{coker}(i) \overset{\pi^1}{\twoheadrightarrow} \mathcal{G}$ the extension given by the first projection. This extension is a $\mathbf{U}$-central extension and since $\mathbf{U}(\mathcal{Y})$ is not perfect, there are two morphisms in **CM**, $\varphi, \psi\colon \mathcal{Y} \longrightarrow \mathcal{G}\times\mathrm{coker}(i)$ such that $\sigma = \gamma\circ\varphi = \gamma\circ\psi$ (see [20, Lemma 2.62]). The product in **BCM** is the same as in **CM** with induced braiding, so the morphisms are in **BCM**. □

Corollary 3.7. *If the universal $\mathbf{U}$-central extension $\mathcal{U}$ of a braided crossed module $\mathcal{G}$ exists, then $\mathbf{U}(\mathcal{U})$ is perfect in* **CM**.

Proof. If the universal extension is not perfect, then using Lemma 3.6, we have another $\mathbf{U}$-central extension and two different morphisms from the universal $\mathbf{U}$-central extension, which contradicts the universality. □

Corollary 3.8. *A braided crossed module admits a universal $\mathbf{U}$-central extension if and only if it is perfect as a crossed module.*

Proof. If the braided crossed module is perfect in **CM**, then using Proposition 3.4, we have its universal **U**-central extension.

If the braided crossed module has a universal **U**-central extension, then using Corollary 3.7, we have that the universal **U**-central extension is perfect as a crossed module. Since it is an extension, we can use Proposition 3.5 and conclude that our braided crossed module is perfect as a crossed module. □

4. Relationship between the universal B-central extension and the universal U-central extension of braided crossed modules

This section will show the relation between the notions of universal **B**-central extension and universal **U**-central extension in the case of braided crossed modules.

Lemma 4.1. *Let $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a braided crossed module. Then, $\mathcal{G}$ is* **B***-perfect if and only if* $\mathbf{U}(\mathcal{G})$ *is perfect.*
In fact, we have that if $H = [H, H]$ then $\mathbf{B}_H(G) = [H, G]$.

Proof. We have $H = [H, H]$, and we need to check $\mathbf{B}_H(G) = [H, G]$.

If $\mathbf{U}(\mathcal{G}) = (G \xrightarrow{\partial} H, \cdot)$ is perfect, then $[H, G] = G$. Since $[H, G] \subset \mathbf{B}_H(G) \subset G$, we have that $\mathbf{B}_H(G) = G$ and $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is **B**-perfect.

On the other hand, since $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is **B**-perfect, we have that $\mathbf{B}_H(G) = G$. So, we only need to prove that $\mathbf{B}_H(G) \subset [H, G]$.

If $\{h, h'\}$ is a generator of $\mathbf{B}_H(G)$, and since $H = [H, H]$ by being **B**-perfect, we have that on generators $\{h, h'\} = \{[h_1, h_2], [h'_1, h'_2]\}$,

$$\{h, h'\} = \{[h_1, h_2], [h'_1, h'_2]\} = \big\{[h_1, h_2], \partial\{h'_1, h'_2\}\big\} = {}^{[h_1,h_2]}\{h'_1, h'_2\}\,\{h'_1, h'_2\}^{-1} \in [H, G],$$

where we have used (BG3). □

Lemma 4.2. *Let $(X_1 \xrightarrow{\delta} X_2, \circledast, (\!(-,-)\!)) \xrightarrow{\gamma=(\gamma_1,\gamma_2)} (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be an extension of braided crossed modules with $(X_1 \xrightarrow{\delta} X_2, \circledast, (\!(-,-)\!))$* **B***-perfect. Then, γ is a* **B***-central extension if and only if γ is a* **U***-central extension.*
In fact, if $X_2 = [X_2, X_2]$ then $\mathrm{Z}_{\mathbf{B}}(X_2) = Z(X_2) \cap \mathrm{st}_{X_2}(X_1)$.

Proof. If γ is a **U**-central extension, then $\ker(\gamma_1) \subset X_1^{X_2}$ and $\ker(\gamma_2) \subset \mathrm{st}_{X_2}(X_1) \cap \mathrm{Z}(X_2)$. We need to prove that $\ker(\gamma_2) \subset \mathrm{Z}_{\mathbf{B}}(X_2)$.

Let $z \in \mathrm{st}_{X_2}(X_1) \cap \mathrm{Z}(X_2)$ and $y = [z_1, z_2] \in X_2 = [X_2, X_2]$. We have

$$\{y, z\} = \{[z_1, z_2], z\} = \{\partial\{z_1, z_2\}, z\} = \{z_1, z_2\}\ {}^{z}\{z_1, z_2\}^{-1} = \{z_1, z_2\}\ \{z_1, z_2\}^{-1} = 1,$$
$$\{z, y\} = \{z, [z_1, z_2]\} = \big\{z, \partial\{z_1, z_2\}\big\} = {}^{z}\{z_1, z_2\}\ \{z_1, z_2\}^{-1} = \{z_1, z_2\}\ \{z_1, z_2\}^{-1} = 1,$$

where we have used (BG2) and (BG3).

So, $\ker(\gamma_2) \subset \mathrm{Z}_B(X_2)$ and γ is a **B**-central extension. □

Theorem 4.3. *Let $\mathcal{G} = (G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a* **B***-perfect braided crossed module. Then its universal* **B***-central extension $\mathcal{B} \overset{\beta}{\twoheadrightarrow} \mathcal{G}$ and its universal* **U***-central extension $\mathcal{U} \overset{\upsilon}{\twoheadrightarrow} \mathcal{G}$ are isomorphic.*

Proof. Since $\mathcal{U} \overset{\upsilon}{\twoheadrightarrow} \mathcal{G}$ is a **U**-central extension, we know using Lemmas 4.1 and 4.2 (by hypothesis $\mathcal{G}$ is **B**-perfect) that it is a **B**-central extension, and using the universality of $\mathcal{B}$, there is a unique morphism $\mathcal{B} \xrightarrow{\varphi} \mathcal{U}$ such that $\beta = \upsilon \circ \varphi$.

Since $\mathcal{B} \overset{\beta}{\twoheadrightarrow} \mathcal{G}$ is a $\mathbf{B}$-central extension is also a $\mathbf{U}$-central extension, and so by the universality of $\mathcal{U}$, there exists a unique morphism $\mathcal{U} \xrightarrow{\psi} \mathcal{B}$ such that $\upsilon = \beta \circ \psi$.

Using the universality of $\mathcal{B}$, since $\beta \circ (\psi \circ \varphi) = \upsilon \circ \varphi = \beta$, we get that $\psi \circ \varphi = \mathrm{Id}_{\mathcal{B}}$.

By the same arguments using the universality of $\mathcal{U}$, we have that $\varphi \circ \psi = \mathrm{Id}_{\mathcal{U}}$. □

Corollary 4.4.

(i) Let $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ be a $\mathbf{B}$-perfect braided crossed module. Then $H \otimes H \simeq G \otimes H$.
(ii) If G is a perfect group, then $G \otimes G \simeq (G \otimes G) \otimes G$.

Proof. (i) By Theorem 4.3, (UBCE) and (UUCE) are isomorphic. Therefore $G \otimes H \simeq H \otimes H$.

The isomorphism can be described explicitly using Proposition 2.4 and Proposition 3.4, and it is given by $\varphi\colon H \otimes H \to G \otimes H$, $h \otimes h' \mapsto \{h_1, h_2\} \otimes h'$, with $h = [h_1, h_2]$, and $\varphi^{-1}\colon G \otimes H \to H \otimes H$, $g \otimes h \mapsto \partial(g) \otimes h$.

(ii) If G is perfect, then the braiding crossed module $(G \otimes G \xrightarrow{\partial} G, \cdot, \{-,-\})$ is $\mathbf{B}$-perfect (see Example 1.8 (2)), since $G \otimes G$ is generated by $g_1 \otimes g_2 = \{g_1, g_2\}$. By (i), we have $G \otimes G \simeq (G \otimes G) \otimes G$.

In this case, the isomorphism is described by $\varphi\colon G \otimes G \to (G \otimes G) \otimes G$, $g \otimes g' \mapsto (g_1 \otimes g_2) \otimes g'$, with $g = [g_1, g_2]$, and $\varphi^{-1}\colon (G \otimes G) \otimes G \to G \otimes G$, $(g \otimes g') \otimes g'' \mapsto [g, g'] \otimes g''$. □

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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